THREE MANIFOLDS NOTES

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1. Incompressible/Compressible Surfaces

Definition 1.1. Let M be a 3-manifold and let $\Sigma \neq D^2, S^2$ be an embedded surface. Σ is *incompressible* if, for every embedded disk D with $\partial D = \Sigma \cap D$, there is an embedded disk in Σ with boundary equal to ∂D. If $\Sigma \neq D^2, S^2$ is not incompressible, then Σ is *compressible* and there is a *compressing disk* (i.e. an embedded disk $D \hookrightarrow M$ with $\partial D = \Sigma \cap D$ such that ∂D does not bound a disk in Σ)

Lemma 1.2. *Suppose M is irreducible and let T be a compressible torus in M. Then, either T bounds $D^2 \times S^1$ or $T \subset B^3$ where B^3 is an embedded ball.*

Proof. Let D be a compressing disk and let N be a regular neighborhood of $D \cup T$. S^2 is a boundary component of N. By irreducibility of M, this S^2 bounds a ball B where either

(1) $T \subset B$
(2) $T \cap B = \emptyset$ and we can attach B to N to get a $S^2 \times S^1$ (figure 1)

□

Theorem 1.3. *Let M be an irreducible, triangulated, compact 3-manifold. Let Σ_0 be an embedded union of incompressible surfaces in M. Then, Σ is isotopic to a normal surface*

Definition 1.4. Let M be a triangulated 3-manifold and let $\Sigma \subset M$ be a surface. Then, Σ is normal if

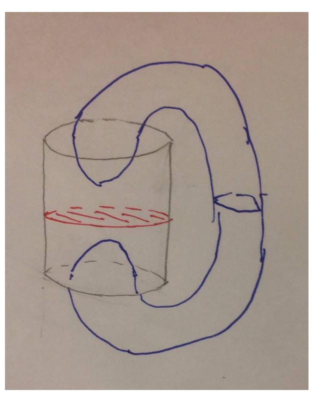

Figure 1

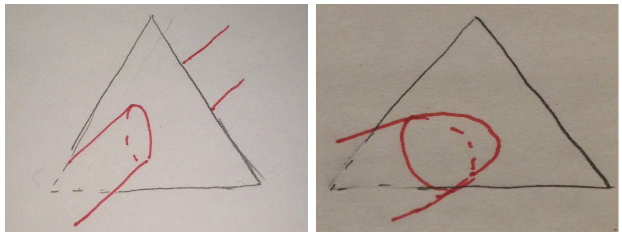

FIGURE 2

(1) Σ is transverse to the triangulation
(2) The arcs of intersection with a 2-simplex σ go between distinct edges (in particular, there are no loops on the intersection)
(3) For each 3-simplex τ, $\Sigma \cap \tau$ is the union of triangles and quadrilaterals

Lemma 1.5. *Let S be an embedded union of incompressible surfaces in a 3-manifold M. Suppose T is a surface in $M \setminus S$. Then, T is incompressible in M if and only if T is incompressible in $M \setminus S$.*

Proof. If T is incompressible in M then it is obviously incompressible in $M \setminus S$.

We may assume that $T \neq D^2, S^2$. Suppose that T is compressible in M. Choose a compressing disk D which is transverse to S and has minimal number of intersections with S over disks with the same boundary. Let α be an innermost curve on $S \cap D$ on D. Since S is the union of incompressible surfaces, α bounds a disk on S. We can use this disk to surger D and decrease the number of intersections between D and S. So, $D \cap S = \emptyset$ which implies that D is a compressing disk for T in $M \setminus S$. Therefore, T is compressible in $M \setminus S$. □

Proof of theorem. Choose Σ in isotopy class of Σ_0 transverse to the triangulation of M and minimizing the complexity $c(\Sigma) = (\#\Sigma \cap M^{(1)}, \sum_{\sigma \in A} \Sigma \cap \sigma)$ where $M^{(1)}$ denotes the 1-skeleton of M and A is the set of 2-simplices of M.

The idea is the same as that of a previous proof but we need to ensure that reducing complexity is realized by isotopies.

First, we show that $\Sigma \cap \sigma$ contains no loops for any 2-simplex σ. Suppose there is a loop in the intersection. Let α be an innermost loop. Then, α bounds embedded disks in both σ and T. Since M is irreducible, the union of these two disks bounds a ball. We can use these balls to isotope Σ to have a lower complexity (figure 2).

We now show that $\Sigma \cap \sigma$ does not contain arcs from an edge of σ to itself. Suppose otherwise. Let α be an outermost arc of $\Sigma \cap \sigma$ whose endpoints lie on the same edge of σ. Then, α cuts off a disk in σ which does not intersect Σ except in α. A neighborhood of that disk gives a ball across which we can isotope Σ to decrease complexity (figure 3).

Eliminating non-disk components of $\Sigma \cap \tau$, where τ is a 2-simplex, is the same as before but with isotopy instead of surgery. Also, if there are long disks, complexity can be decreased by an isotopy. If there is a long disk, its boundary his some edge of the simplex twice. A carefully chosen such pair of edges gives a disk whose boundary is a subarc of the union of an edge and an arc in Σ. A neighborhood of this disk gives an isotopy of Σ that reduces complexity. □

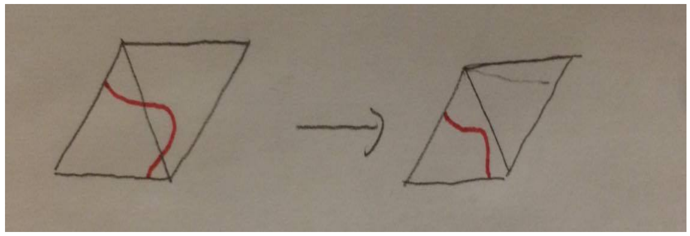

FIGURE 3

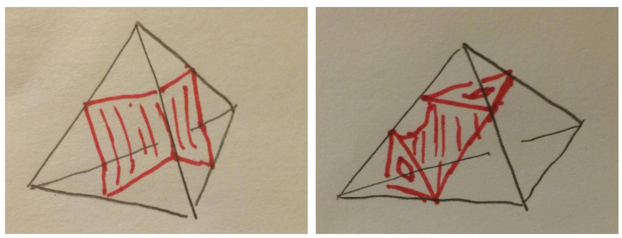

FIGURE 4. non-disk components and long disks in 2-simplex

Corollary 1.6 (Haken Finiteness). *Let M be a compact, connected, irreducible 3-manifold. There is a number $N = N(M)$ so that any embedded collection $\mathcal{S}$ of closed nonparallel incompressible surfaces in M has cardinality $\#\mathcal{S} < N$.*

Definition 1.7. A and B are parallel if they bound a component $A \times I \cong B \times I$.

Proof. Choose a triangulation of M. Let $N = v + f + \dim_{\mathbb{Z}/2\mathbb{Z}}(H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z}))$ where v is the number of vertices and f is the number of faces.

Lemma 1.8. *If $F_1, ..., F_k$ is a collection of disjoint properly embedded surfaces in M, $[F_1], ..., [F_k]$ are independent in $H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ if and only if $M \backslash (F_1 \cup ... \cup F_k)$ is connected.*

□