3-MANIFOLDS

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1. Haken Finiteness

Last time: If M is compact, irreducible, and triangulated, then any embedded union of closed incompressible surfaces is isotopic to a normal surface.

Lemma. *If $F_1, \ldots, F_k$ is a collection of disjoint, properly embedded surfaces in M then $[F_1], \ldots, [F_k]$ are independent in $H_2(M, \partial M; \mathbb{Z}_2)$ if and only $M \backslash (F_1 \cup \cdots \cup F_k)$ is connected.*

Proof idea: Triangulate M with the union of the F_i in the 2-skeleton and think about chains.

Haken Finiteness: If M is compact and irreducible then there is a number $N = N(M)$ so that any embedded union of closed, incompressible, non-parallel surfaces in M has fewer than N components.

Proof. Choose a triangulation of M. Let $N = v + f + \beta$ where v is the number of vertices, f is the number of 2-cells, and $\beta = \dim_{\mathbb{Z}_2}(H_2(M, \partial M; \mathbb{Z}_2))$. Let $F_1, \ldots, F_n$ be a disjoint collection of closed, non-parallel, incompressible surfaces. From last time, we assume the $\bigcup F_i$ is normal with respect to the triangulation. By reordering, assume that $F_1, \ldots, F_k$ is a maximal linearly independent subcollection in $H_2(M, \partial M; \mathbb{Z}_2)$. (i.e. each $[F_r]$ for $r > k$ is in the span of $S = \{[F_1], \ldots, [F_k]\}$ and so $\dim_{\mathbb{Z}_2} S = k$.)

From the lemma, the number of components of $M \backslash \bigcup F_i = 1 + n - k$. Further, by looking at the normal surface within the triangulation we get that the maximum number of non-I-bundle components of $M \backslash \bigcup F_i$ is $v + f$ because there is potentially a non-rectangular region in the center of each 2-cell and around each vertex of the triangulation.

Consider an I-bundle component C of $M \backslash \bigcup F_i$. Since none of the F_i are parallel, C must be a non-trivial I-bundle over some non-separating surface Σ which is the 0-section of the I-bundle.

Let $\Sigma_1, \ldots, \Sigma_m$ be the 0-sections of the I-bundle pieces of $M \backslash \bigcup F_i$. Since the Σ_i are non-separating, we have that $M \backslash (F_1 \cup \cdots \cup F_k \cup \Sigma_1 \cup \cdots \cup \Sigma_m)$ is connected. By the lemma, we know that $k + m \leqslant \beta$. Therefore the number of components of $M \backslash \bigcup F_1$ is less than $v + f + m$ and so $1 + n - k \leqslant v + f + m$ which gives that $n \leqslant v + f + \beta - 1$, as desired. □

Remark: The same argument gives a bound on the number of seminormal surfaces in a fixed triangulation.

Corollary. *(Existence of a torus decomposition) If M is compact and irreducible then there is a disjoint union Σ of incompressible tori in M so that every component of $M \backslash \Sigma$ is atoroidal.*

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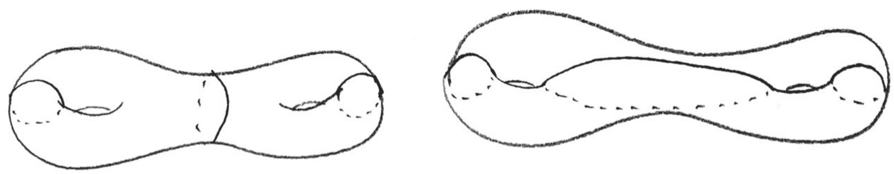
Definition. M a compact 3-manifold is **atoroidal** is every incompressible torus in M is boundary-parallel.

Definition. $\Sigma^2 \subset M^3$ is **boundary-parallel** if it is isotopic rel $\partial\Sigma$ to ∂M.

Remark: Often **atoroidal** means: Any π_1-injective immersion $T^2 \hookrightarrow M$ is homotopic into ∂M.

Note: The "torus decomposition" from the corollary is not unique. For example, let $M = F^2 \times S^1$ where F is the genus 2 handlebody. Any simple closed curve α on F gives an embedded torus $T_\alpha = \alpha \times S^1 \hookrightarrow M$. We see that T_α will be incompressible if and only if α is essential in F. Thus the following two systems of curves give different torus decompositions of M:



Fact: (Pair of pants) $\times S^1$ is atoroidal in the weak sense (our definition) but not the strong sense (the other definition).

Looking ahead:

Theorem (JSJ-decomposition)**.** *If M is compact, orientable, and irreducible then there is a disjoint union Σ of incompressible tori so that every component of $M \backslash \Sigma$ is either atoroidal or Seifert fibered. A minimal such collection is unique up to isotopy.*

2. Seifert Fibered Spaces

Intuitively, a *Seifert fibered space* is a 3-manifold together with a decomposition into circles. A formal definition will be given shortly.

The following are **model Seifert fibered spaces:**

1. Fibered solid tori $T_{p,q} = (D^2 \times I)/\sim$ where $(z, 0) \sim \left(e^{2\pi ip/q}z, 1\right)$. Thus we are gluing $D^2 \times \{0\}$ to $D^2 \times \{1\}$ via a rotation of $2\pi p/q$.
2. Fibered solid Klein bottle: $K = (D^2 \times I)/\sim$ where $(z, 0) \sim (\overline{z}, 1)$.

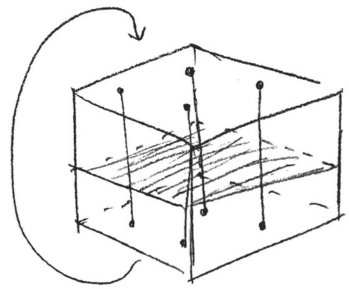
Definition. M is a **Seifert fibered space** if it has a decomposition into S^1's so that each S^1 has a neighborhood which is fiber-preservingly homeomorphic to one of the two model Seifert fibered spaces.

Example.

1. S^3 with the Hopf fibration is a Seifert fibered space.
2. $S^3 \backslash K$ where K is the trefoil knot can be fibered by copies of K along with a copy of S^1 for the cores of each of the two solid tori which glue together into S^3. These special S^1 are called **critical fibers**.
   Exercise: What are the models around these critical fibers?

Definition. If M is a Seifert fibered space then a surface $\Sigma \subset M$ is called **vertical** if it is a union of fibers. Σ is **horizontal** if it is transverse to the fibers.

Example. $\varphi : T^2 \to T^2$ is rotation of order 4. $M = M_\varphi$ the mapping torus.



The middle shaded surface is horizontal.