THREE-MANIFOLDS NOTES (LECTURE 8)

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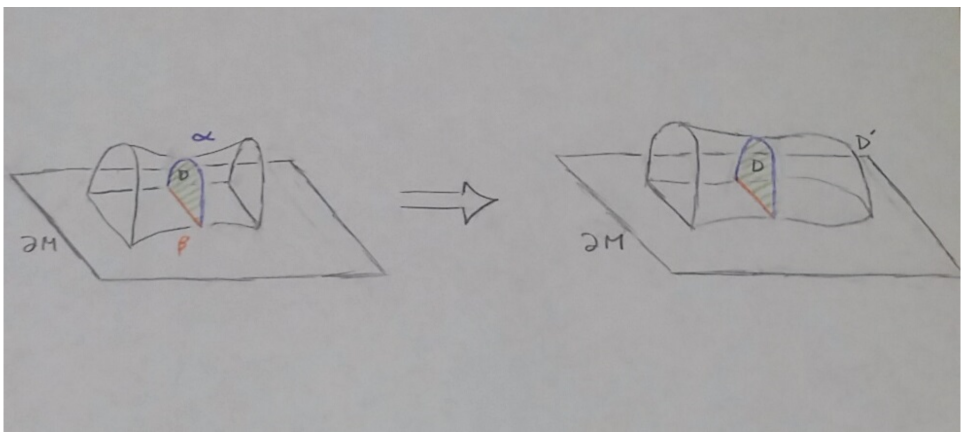
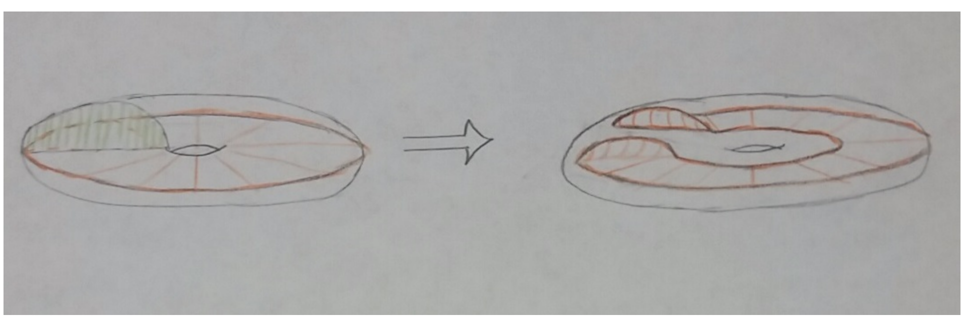
Some Definitions

Definition. A properly embedded two-sided surface $\Sigma^2 \subset M^3$ is **boundary incompressible** if whenever there is an embedded disk $D \subset M$ with $\partial D \cap \Sigma$ an arc α such that $\partial D \setminus \Sigma \subset \partial M$, then there is a disk $D' \subset \Sigma$ with $\partial D' \setminus M = \alpha$.



If $\Sigma \neq D^2, S^2$ and not boundary incompressible, we say that Σ is **boundary compressible**. In this case, there exists a **boundary compressing disk** $D \subset M$.



Definition. Let $\Sigma^2 \subset M^3$ be a compact properly embedded surface not equal to D^2 or S^2. Then Σ is **essential** if it is both incompressible and boundary incompressible.

Definition. A 2-disk $D \subset M$ is essential if it is not boundary-parallel. A 2-sphere is essential if it does not bound a 3-ball. A disconnected surface is essential if all its components are essential and non-parallel.

Remark. Let Σ be a properly embedded surface in a 3-manifold M not equal to D^2 or S^3. If

$$\pi_1(\Sigma, \partial\Sigma, p) \to \pi_1(M, \partial M, p)$$

is injective for all choices of $p \in \partial\Sigma$, then Σ is boundary incompressible.

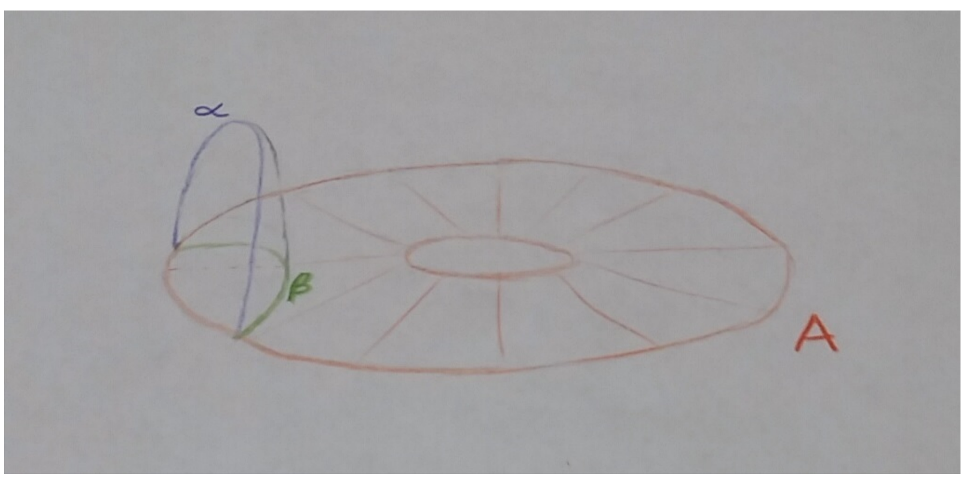
Essential Surfaces in Seifert Fibered Spaces

Lemma 1. *Let M be irreducible, and let $\Sigma \subset M$ be incompressible but not essential. Suppose $\partial\Sigma$ is contained in a union of tori inside of ∂M. Then Σ is a boundary-parallel annulus.*

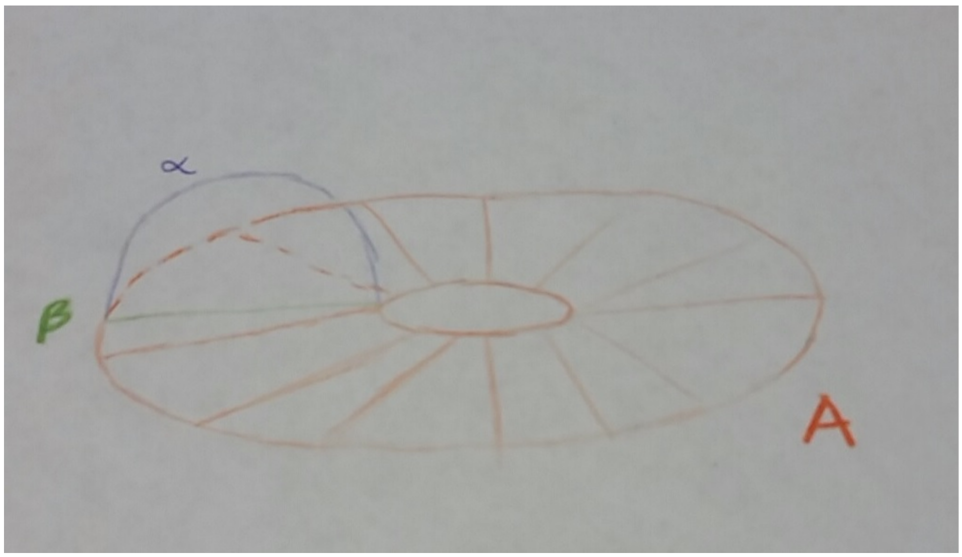
Proof. Since Σ is not essential there exists a boundary compressing disk $D \subset M$ with $\partial D = \alpha \cup \beta$ where α is an essential arc in Σ, and β is an arc lying in a torus component $T \subseteq \partial M$. Now suppose a circle in $\partial S \cap T$ bounded a disk in T. Then since S is incompressible, such a circle would bound a disk in S, implying that S is itself a disk. But this contradicts incompressibility of S, since by definition disks cannot be incompressible. Hence components of $\partial S \cap T$ must be essential simple closed curves. This implies that β is contained in an annulus $A \subset T$ with $\partial A \subseteq \partial S \cap T$.

We now have the following two cases:

Case 1: Suppose β is not an essential arc in A. Then it is homotopic to a curve $\beta' \subset \partial A$ with $\partial\beta' = \partial\alpha$. But this contradicts incompressibility of Σ, since the loop $\alpha \cup \beta' \subset \Sigma$ bounds a disk in M, but not in Σ.



Case 2: Suppose β is an essential arc in A. Then the curve α runs between two boundary components c_1, c_2 of Σ. Let N be a regular neighborhood of $\alpha \cup c_1 \cup c_2$ in Σ, diffeomorphic to a 2-sphere with three punctures. Then $\partial N \setminus \partial\Sigma$ is a loop in σ which bounds a disk D_1 in M, and by incompressibility of Σ it must also bound a disk D_2 in Σ. This implies Σ is homeomorphic to an annulus, since $D_2 \approx \Sigma \setminus \alpha$. Surgering $\Sigma \cup A$ along D_2, we obtain a sphere. And since M is irreducible, this sphere bounds a 3-ball. This implies that $\Sigma \cup A$ bounds a solid torus, and thus Σ is a boundary-parallel annulus, isotopic to A relative to $\partial\Sigma$.



□

Lemma 2. *Let $M = D^2 \times S^1$ be a solid torus. Then the only connected essential surfaces in M are meridian disks, up to isotopy.*

Proof. Let $\Sigma \subset M$ be a connected, essential surface. By Alexander's Theorem, $\partial\Sigma$ must be nonempty and essential in $\partial M \approx \mathbb{T}^2$. Components of $\partial\Sigma$ cut T into a finite set of annuli. By an isotopy, we can arrange $\partial\Sigma$ to be either a union of meridian disks homeomorphic to $\partial D^2 \times \{\text{pt}\}$ or transverse to every meridian.

Suppose the latter. Fix a meridian disk D transverse to Σ. Since Σ is incompressible and M is irreducible, we can assume that $\Sigma \cup D$ is a union of arcs. Consider an outermost arc $\alpha \subset \Sigma \cup D$. Since Σ is boundary incompressible, α is actually an arc in $\partial\Sigma$. But this contradicts the fact that $\partial\Sigma$ is transverse to every meridian. Hence $\Sigma \cap D = \emptyset$, and $\Sigma \subset M \setminus D$. Now since Σ is essential, $\partial\Sigma$ consists of a single meridian. We can "cap off" with D to get a closed surface Σ' in a ball. Since Σ' cannot have positive genus, Σ' must be a sphere which bounds a ball. This gives us an isotopy from Σ to D. □

Essential Surfaces in Seifert Fibered Spaces

Proposition 1. *Every reducible Seifert fibered space is diffeomorphic to either* $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, *or* $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Exercise. Determine the covering space action $\mathbb{Z} \curvearrowright \mathbb{R}P^3 \# \mathbb{R}P^3$.