LECTURE 9

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M will in this lecture denote a 3 manifold and in our applications a Seifert fibred space. We wish to obtain a description for the connected essential (2-sided) surfaces embedded in a compact irreducible Seifert fibred space. More precisely, we show that such a surface is isotopic to either a vertical or horizontal surface. Σ will throughout denote a connected surface embedded in M. Recall from last time two lemmas verging on a description of incompressible (2-sided) surfaces $\Sigma \subset M^3$ with $\partial\Sigma$ contained in the boundary tori of ∂M.

Lemma 0.1. *The only connected essential surfaces in a solid torus $D^2 \times S^1$ are meridian disks $D = D^2 \times \{z\}$, $z \in S^1$.*

Lemma 0.2. *Let $\Sigma \subset M$ be an incompressible, inessential surface such that $\partial\Sigma$ is contained in a union of torus components of ∂M, then Σ is a boundary parallel torus.*

The following theorem in fact generalizes lemma 0.1 to compact irreducible Seifert fibred spaces.

Theorem 0.3. *If M is a compact irreducible Seifert fibred space, $\Sigma \subseteq M$ an essential surface in M, then Σ is isotopic to either a horizontal or a vertical surface.*

Proof. Since M is compact, it has only finitely many critical fibres, let $C_1, \dots C_m$ with $m \geqslant 1$ be a collection of fibres containing all critical fibres and for each $1 \leqslant i \leqslant m$ let N_i be a regular fibred neighbourhood about C_i. Let $M_0 := M/\{\cup_{i=1}^m N_i\}$, then $\pi : M_0 \to B$ is a fibre-bundle over its space of fibres B (the topology on B is the quotient topology). B is a compact connected surface with boundary, such a surface can be further cut by finitely many disjoint arcs $\alpha_1, \dots, \alpha_r$ into a disk (a genus g surface without boundary can be cut along $2g$ non-intersecting loops into a disk, a surface with boundary can be cut into a disk with holes which can be further cut into a disk). Let $A_i := \pi^{-1}(\alpha_i)$ be the pre-image of the arc α_i, which being a fibre-bundle over an arc is an annulus in M_0. $A = \cup_i A_i$ is a union of disjoint annuli which cuts M_0 so that its interior is an S^1 bundle over a disk, so M_0/A is a solid torus with $2r$ annuli on the boundary with each annulus A_i splitting into two. Now isotope Σ such that $\Sigma \cap N_i$ are horizontal, each component of $\partial\Sigma$ is vertical or horizontal. Let us also assume that Σ is also of minimal complexity

$$c(\Sigma) = (\mid \cup_i \Sigma \cap C_i \mid, \mid \Sigma \cap A \mid)$$

among all surfaces in the isotopy class which satisfy the aforementioned conditions. Let us take a closer look at the intersections $\Sigma \cap A_i$, these may be arcs with endpoints on the same boundary circle of ∂A_i, loops in the interior of A_i or arcs which join the two boundary circles of A_i. We are to rule out 3 cases,

case 1. There is an annulus A_i such that $\Sigma \cap A_i$ has a component in A_i which is an interior circle. As Σ is incompressible, an innermost such circle bounds two disks $D \subseteq A_i$

and $D' \subseteq \Sigma$ the union of which is a sphere in M which bounds a ball in M since M is irreducible. Thereby an isotopy yields a new Σ for which the intersection multiplicites of Σ with the fibres C_i are the same but $\mid \Sigma \cap A \mid$ decreases by 1, and so the complexity decreases, which cannot be since Σ was of minimal complexity.

case 2. We rule out the case in which there is an arc in $\Sigma \cap A_i$ for some A_i with endpoints in the same component of ∂A_i such that this component happens to be on the boundary of N_j, the fibred neighbourhood of C_j. To demonstrate this, we choose an innermost such arc with endpoints joined to distinct points on C_j. This arc along with the arc along C_j joining the two endpoints cuts out a disk in A_i which can once again be isotoped so as to reduce the number of intersections with the fibre C_j by 2 thus reducing the complexity. Therefore, the aforementioned arrangement is not possible for otherwise the isotopy described will reduce the complexity of the surface Σ.

case 3. Finally we are to rule out the case in which there are arcs in $\Sigma \cap A_i$ for some A_i with endpoints in the same component of ∂A_i which is also on the boundary of M. Once again choose an innermost such arc α which now bounds a ∂ compressing disk D in A_i. Since Σ is ∂ compressible, there is another disk in Σ, say D' for which $\partial D / \partial D'$ is an arc γ in ∂M joining the endpoints of α. But this is impossible since we assumed that each component of $\partial\Sigma$ is either vertical or horizontal. In greater detail, note that since the component of ∂A_i meeting $\partial\Sigma$ in a single fibre with the two endpoints of γ on it and γ cannot be vertical without passing through ∂A_i. This cannot be as γ is disjoint from ∂D. The endpoints of γ are on the same fibre. γ cannot therefore be horizontal since a horizontal arc must passes through the fibres monotonically and thus doesn't visit the same fibre once more.

Let M_1 be the solid torus with $2r$ boundary annuli obtained from M_0 from cutting M_0 along the annuli A_i. Let Σ_1 the surface in M_1 obtained from Σ in M_1. We may isotope σ_1 so that it has vertical or horizontal boundary. We would like to show that Σ_1 is incompressible and reduce to the case where Σ_1 is either isotopic to a union of meridian disks or isotopic to vertical surfaces. We shall complete the proof of this theorem in the next lecture. $\square$