

# ACYLINDRICALLY HYPERBOLIC GROUPS AND THEIR QUASI-ISOMETRICALLY EMBEDDED SUBGROUPS

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ABSTRACT. We abstract the notion of an  $A/QI$  triple from a number of examples in geometric group theory. Such a triple  $(G, X, H)$  consists of a group  $G$  acting on a Gromov hyperbolic space  $X$ , acylindrically along a finitely generated subgroup  $H$  which is quasi-isometrically embedded by the action. Examples include strongly quasi-convex subgroups of relatively hyperbolic groups, convex cocompact subgroups of mapping class groups, many known convex cocompact subgroups of  $\text{Out}(F_n)$ , and groups generated by powers of independent loxodromic WPD elements of a group acting on a Gromov hyperbolic space. We initiate the study of intersection and combination properties of  $A/QI$  triples. Under the additional hypothesis that  $G$  is finitely generated, we use a method of Sisto to show that  $H$  is stable. We apply theorems of Kapovich–Rafi and Dowdall–Taylor to analyze the Gromov boundary of an associated cone-off. We close with some examples and questions.

## 1. INTRODUCTION

There are two important threads in the study of group actions on hyperbolic spaces which we bring together in this paper. One is the topic of quasi-convex subgroups of hyperbolic groups. Gromov introduced these in [Gro87] as the geometrically natural subgroups of a hyperbolic group, and they have been intensively studied since then. Without pretending to give a full literature review, we mention in particular some previous work closely related to what we do here. Gitik–Mitra–Rips–Sageev study finiteness and intersection properties of quasi-convex subgroups and show that they have finite height and width [GMRS98]. Bowditch shows that cone-offs of hyperbolic spaces by families of quasi-convex subspaces are themselves hyperbolic [Bow12]. Other important early papers related to our current work are [Sho91, Swe01].

The other thread is that of acylindrical (and weakly properly discontinuous) actions of groups on hyperbolic spaces. Acylindrical actions on *trees* were studied first by Sela [Sel97]. Bowditch formulated the acylindricity condition for actions on hyperbolic spaces in [Bow08], showing in particular that the action of the mapping class group of a surface on the curve complex is acylindrical. Osin in [Osi16] shows that the *existence* of an acylindrical action of a group  $G$  on some hyperbolic space is equivalent to the existence of a hyperbolically embedded subgroup, or of a weakly properly discontinuous element for the action of  $G$  on some possibly different hyperbolic space, clarifying the existence of an extremely natural class of *acylindrically hyperbolic groups*. For such a group, Abbott–Balasubramanya–Osin explore the poset of acylindrical actions on hyperbolic spaces in [ABO19]; this poset is further explored in [Abb16, ABB<sup>+</sup>17, Bal19, Val20].

In this paper we are interested in understanding actions of a group on hyperbolic spaces which are not necessarily acylindrical, but satisfy the weaker condition of being acylindrical along a subgroup which is assumed to be quasi-isometrically embedded by the action.

**Definition 1.1.** Suppose  $(X, d)$  is a Gromov hyperbolic geodesic metric space, and  $Y \subseteq X$ . An isometric action  $G \curvearrowright X$  is *acylindrical along  $Y$*  if for every  $\varepsilon \geq 0$ , there are  $R = R(\varepsilon, Y) > 0, M = M(\varepsilon, Y) > 0$  so that, whenever  $x, y \in Y$  satisfy  $d(x, y) \geq R$ ,

$$(1) \quad \#\{g \in G \mid d(gx, x) \leq \varepsilon \text{ and } d(gy, y) \leq \varepsilon\} \leq M.$$

While the constants  $R, M$  are sometimes called the constants of acylindricity of the action, we reserve this terminology for the constants in an alternate characterization of acylindricity which is easier to check in our context (see Definition 3.1). If  $G \curvearrowright X$  is acylindrical along  $Y$ , then it is acylindrical along  $W$  for any  $W$  which is finite Hausdorff distance from  $Y$ . It is also acylindrical along any translate  $gY$ . This notion interpolates between an acylindrical action and an action with a loxodromic WPD element (see Definition 6.16), in the following sense. The usual notion of an acylindrical action  $G \curvearrowright X$  is one which is acylindrical along the entire space  $X$ , while an action  $G \curvearrowright X$  with a loxodromic WPD element  $g$  is acylindrical along any axis for  $g$ .

If  $H < G$ , we say that  $G \curvearrowright X$  is *acylindrical along  $H$*  if it is acylindrical along  $Hx$  for some  $x \in X$ . The main new definition of the current paper is the following:

**Definition 1.2.** An *A/QI triple*  $(G, X, H)$  consists of a Gromov hyperbolic space  $X$  and an action  $G \curvearrowright X$  which is acylindrical along the finitely generated infinite subgroup  $H$ , and so that  $x \mapsto hx$  gives a quasi-isometric embedding of  $H$  into  $X$ .

Whenever this final condition is satisfied, we say that  $H$  is *quasi-isometrically embedded by the action*.

For example, if  $G$  is hyperbolic,  $H < G$  is quasi-convex, and  $\Gamma$  is a Cayley graph of  $G$ , then  $(G, \Gamma, H)$  is an A/QI triple. More interesting motivating examples are convex cocompact subgroups of mapping class groups. In this case, if  $H$  is a convex cocompact subgroup of a mapping class group  $\text{Mod}(\Sigma)$  of a non-exceptional surface  $\Sigma$ , and  $\mathcal{C}$  is the curve complex of  $\Sigma$ , then  $(\text{Mod}(\Sigma), \mathcal{C}, H)$  is an A/QI triple. In this situation, the action of  $G$  on  $\mathcal{C}$  is a genuine acylindrical action. Other motivating examples include certain convex cocompact subgroups  $H$  of  $\text{Out}(F_n)$ , the outer automorphism group of the free group of rank  $n$ . Indeed for many subgroups  $H$ , if  $X$  is the free factor complex, the cyclic splitting complex, or the free splitting complex, then  $(\text{Out}(F_n), X, H)$  is an A/QI triple (see Example 7.1). None of these actions  $\text{Out}(F_n) \curvearrowright X$  are known to be acylindrical, and in the case where  $X$  is the free splitting complex, it is known that the action is not acylindrical. That these examples are A/QI triples is an application of a general construction of A/QI triples from an action of a group on a hyperbolic space with a loxodromic WPD element (Corollary 4.5). We use this construction to give additional examples of A/QI triples in the Cremona group and FC-type Artin groups (see Examples 7.2 and 7.3), neither of which have known *natural* acylindrical actions on hyperbolic spaces.

**Remark 1.3.** In [Sis16], Sisto gives a more flexible definition of an action being acylindrical along a subspace  $Y$ , omitting the number  $M$  and replacing the inequality “ $\leq M$ ” in (1) by “ $< \infty$ ”. We will refer to an action satisfying this weaker condition as being *weakly acylindrical along*  $Y$ . In general an action can be weakly acylindrical along a subspace without being acylindrical along that subspace. However if  $X$  is hyperbolic,  $Y$  is a quasi-convex subspace, and  $d_X|_Y$  is a proper metric, then any action which is weakly acylindrical along  $Y$  is acylindrical along  $Y$ .

In this paper, we study three aspects of A/QI triples  $(G, X, H)$ : intersection and combination properties, stability of the subgroup  $H$ , and the boundary of a certain coned-off space of  $X$  constructed from  $(G, X, H)$ .

Motivated by the fact that the intersection of two quasi-convex subgroups of a hyperbolic group is quasi-convex, we first consider the intersection of A/QI triples. We show that if  $(G, X, H)$  and  $(G, X, K)$  are A/QI triples, then  $(G, X, H \cap K)$  is also an A/QI triple (Proposition 4.1). We also prove the following combination theorem for A/QI triples.

**Theorem 1.4.** *Let  $G$  act on a hyperbolic space  $X$ , and suppose  $H_1, \dots, H_k$  are subgroups of  $G$  such that  $(G, X, H_i)$  is an A/QI triple for each  $i = 1, \dots, k$  and  $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$  for all  $i \neq j$ . Then there exists a finite collection of elements  $\mathcal{S} \subset \cup_{i=1}^k H_i$  such that for any quasi-isometrically embedded subgroups  $K_i \leq H_i$  satisfying  $K_i \cap \mathcal{S} = \emptyset$ , we have  $K = \langle K_1, \dots, K_k \rangle \cong K_1 * K_2 * \dots * K_k$  and  $(G, X, K)$  is an A/QI triple.*

Our first finiteness result is that for any A/QI triple  $(G, X, H)$ , the subgroup  $H$  has *finite height* in  $G$  (see Proposition 4.8). Finite height is a weakening of malnormality which has proved extremely useful in studying residual finiteness and cubulability of (relatively) hyperbolic groups. See for example [AGM09, Ago13, HW14].

Restricting our attention to A/QI triples  $(G, X, H)$  where  $G$  is finitely generated, we obtain a stronger result.

**Theorem 1.5.** *Let  $(G, X, H)$  be an A/QI triple, and suppose that  $G$  is finitely generated. Then  $H$  is stable in  $G$ .*

*Stability* is a strong quasi-convexity condition introduced by Durham–Taylor [DT15] (see Definition 5.1). By work of Antolín–Mj–Sisto–Taylor [AMST19], stability implies finite height as well as the related properties of *finite width* and *bounded packing*. Our arguments here are based on those of Sisto in [Sis16], where he proves a stability result for hyperbolically embedded subgroups. Note that in our setting, the subgroup  $H$  may not be hyperbolically embedded in  $G$ . In particular, hyperbolically embedded subgroups have height at most one, whereas subgroups that are part of an A/QI triple often have height greater than one.

For our investigation of the boundary, we return to the setting of a general A/QI triple  $(G, X, H)$ . For such a triple there is an associated action  $G \curvearrowright \hat{X}$ , where  $\hat{X}$  is a hyperbolic space obtained from  $X$  by equivariantly coning off the translates of an  $H$ -orbit. (In case  $X$  is the Cayley graph of  $G$ , this is the *coned-off* or *electrified* Cayley graph of  $(G, H)$  studied, for example, in [Bow12, Far98].) We characterize the boundary of the cone-off  $\hat{X}$  as follows.

**Theorem 1.6.** *Suppose  $(G, X, H)$  is an A/QI triple and fix  $x \in X$ . Let  $\mathcal{H} = \{gHx \mid gH \in G/H\}$ . Let  $\hat{X}$  be the cone-off of  $X$  with respect to  $\mathcal{H}$ . Then  $\partial \hat{X}$  is*

homeomorphic to the subspace of  $\partial X$  obtained by deleting the limit sets of all the conjugates of  $H$ .

In [Ham16], Hamenstädt proves an analogous result to Theorem 1.6 in the case that the hyperbolic space  $X$  is strongly hyperbolic (as a metric space) relative to a collection of subspaces  $\mathcal{H}$ , without requiring the existence of a group action.

We use this description of  $\partial \hat{X}$  to understand the geometry of the action  $G \curvearrowright \hat{X}$ . In Corollary 6.15, we classify the elliptic, loxodromic, and parabolic elements for this action. In Proposition 6.17, we describe the loxodromic WPD elements of  $G \curvearrowright \hat{X}$ ; in particular we can show (Corollary 6.18) if  $G \curvearrowright X$  is a WPD action, then so is  $G \curvearrowright \hat{X}$ .

**Outline.** In Section 2, we recall some basic facts and fix some notation. In Section 3 we give an alternative characterization of acylindricity along a subspace. The following three sections can be read independently. In Section 4, we prove some properties of A/QI triples. In particular, we give short arguments establishing certain intersection properties of A/QI triples (see Propositions 4.1 and 4.8). We also prove Theorem 1.4. In Section 5, we recall the definition of stable subgroups and prove Theorem 1.5. In Section 6, we characterize the boundary of the coned-off space  $\hat{X}$  by proving Theorem 1.6, along with the associated results about the action  $G \curvearrowright \hat{X}$ . Finally, in Section 7, we use the combination theorem from Section 4 to describe some further examples of A/QI triples. We also pose several questions in this section.

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## 2. PRELIMINARIES

In this section we recall some basic notions and fix some notation regarding Gromov products and the boundary of a hyperbolic space. For more detail see, for example, [BH99, III.H].

**Definition 2.1.** Let  $(X, d_X)$  be a hyperbolic metric space with basepoint  $x_0$ . For any  $x, y \in X$ , the *Gromov product* of  $x$  and  $y$  is

$$(x \mid y)_{x_0} = \frac{1}{2} (d_X(x_0, x) + d_X(x_0, y) - d_X(x, y)).$$

A sequence of points  $(x_k)$  in  $X$  *converges to infinity* if  $\lim_{n,k \rightarrow \infty} (x_k \mid x_n)_{x_0} = \infty$ . An equivalence relation  $\sim$  on sequences  $(x_k), (y_k)$  in  $X$  which converge to infinity is defined as follows:  $(x_k) \sim (y_k)$  if and only if  $\lim_{k \rightarrow \infty} (x_k \mid y_k)_{x_0} = \infty$  if and only if  $\lim_{n,k \rightarrow \infty} (x_k \mid y_n)_{x_0} = \infty$ .

**Definition 2.2.** The *Gromov boundary*  $\partial X$  of a hyperbolic metric space  $X$  is the set of equivalence classes of sequences in  $X$  which converge to infinity.

The Gromov product extends to  $\bar{X} = X \cup \partial X$  by the formula

$$(x | y)_{x_0} = \sup \{ \liminf_{i,j \rightarrow \infty} (x_i | y_j)_{x_0} \mid x_i \rightarrow x, y_j \rightarrow y, \text{ and } x_i, y_j \in X \},$$

for all  $x, y \in \bar{X}$ . The union  $\bar{X}$  is topologized by declaring that a sequence  $(x_n)$  in  $\bar{X}$  converges to a point  $x \in \partial X$  if and only if

$$\lim_{n \rightarrow \infty} (x_n | x)_{x_0} = \infty.$$

Although convergence to infinity, equivalence, and the Gromov boundary are defined using the basepoint  $x_0$ , they are actually independent of that basepoint.

The boundary can also be thought of in terms of quasi-geodesic rays, which are quasi-isometric embeddings of  $[0, \infty)$ . To simplify notation we will bundle together all the constants describing the quality of such a quasi-isometric embedding (and hence of such a quasi-geodesic) into a single number.

**Definition 2.3.** Let  $\tau \geq 1$ . A map  $f: (X, d_X) \rightarrow (Y, d_Y)$  of metric spaces is a  $\tau$ -quasi-isometric embedding if for all  $x, y \in X$ , we have  $\tau^{-1}d_X(x, y) - \tau \leq d_Y(f(x), f(y)) \leq \tau d_X(x, y) + \tau$ .

A  $\tau$ -quasi-geodesic in  $X$  is a  $\tau$ -quasi-isometric embedding of an interval into  $X$ , that is, a map  $\gamma: I \rightarrow X$  where  $I \subseteq \mathbb{R}$  is connected and so that for all  $s, t \in I$ ,  $\tau^{-1}|s - t| - \tau \leq d_X(\gamma(s), \gamma(t)) \leq \tau|s - t| + \tau$ .

If the map  $\gamma: I \rightarrow X$  is continuous, then we denote the length of the quasi-geodesic  $\gamma$  in  $X$  by  $\ell_X(\gamma)$ . Given a discrete path (list of points)  $\sigma = x_0, \dots, x_k$  in a metric space  $X$ , we define its length to be  $\ell^0(\sigma) = \sum_{i=0}^{k-1} d_X(x_i, x_{i+1})$ .

It is occasionally convenient to assume that a quasi-geodesic  $\gamma$  is *tame* in the sense of [BH99, III.H.1.11]. When  $\gamma$  is tame, the map  $\gamma: I \rightarrow X$  is continuous, and we additionally have  $\ell_X(\gamma|_{[s,t]}) \leq \tau d_X(\gamma(s), \gamma(t)) + \tau$  for all  $s, t \in I$ . Throughout the paper, we will *not* assume quasi-geodesics are tame unless it is explicitly stated otherwise.

The following ‘‘Morse’’ property of quasi-geodesics in hyperbolic spaces is well-known (see e.g. [BH99, III.H.1.7]).

**Theorem 2.4.** *For all  $\delta \geq 0$ ,  $\tau \geq 1$ , there is an  $M = M(\tau, \delta) \geq 0$  satisfying the following. Let  $X$  be a  $\delta$ -hyperbolic space, and let  $\gamma_1, \gamma_2$  be  $\tau$ -quasi-geodesics with the same endpoints in  $X \cup \partial X$ . Then the Hausdorff distance between  $\gamma_1$  and  $\gamma_2$  is at most  $M$ .*

**Definition 2.5.** Any number  $M$  as in Theorem 2.4 will be called a *Morse constant* for the parameters  $\tau, \delta$ .

**Definition 2.6.** Let  $\kappa \geq 0$ . A subset  $Y$  of a metric space  $X$  is  $\kappa$ -quasi-convex if every geodesic in  $X$  connecting points of  $Y$  is contained in the  $\kappa$ -neighborhood of  $Y$ .

Given a quasi-geodesic ray  $\gamma$  in  $X$  starting at  $x_0$ , we obtain an equivalence class  $[\gamma] \in \partial X$  by  $[\gamma] := [(a_i)]$  for any sequence of points  $a_i$  on  $\gamma$  with  $\lim_{i \rightarrow \infty} d_X(x_0, a_i) = \infty$ .

**Remark 2.7.** By [KB02, Remark 2.16], there is a constant  $\tau_0$  depending only on the hyperbolicity constant of  $X$  such that given any point  $\xi \in \partial X$ , there is a  $\tau_0$ -quasi-geodesic ray  $\gamma_\xi$  in  $X$  starting at  $x_0$  such that  $[\gamma_\xi] = \xi$ .

The Gromov product at infinity controls the Gromov product between points on quasi-geodesics:

**Lemma 2.8.** *For any  $\tau \geq 1$ ,  $\delta \geq 0$  there is some  $E \geq 0$  so that the following holds.*

*Let  $X$  be  $\delta$ -hyperbolic,  $x_0 \in X$ , and  $\alpha, \beta \in \partial X$ . Let  $\gamma_\alpha$  and  $\gamma_\beta$  be  $\tau$ -quasi-geodesics starting at  $x_0$  and tending to  $\alpha, \beta$  respectively. Then for all  $s, t \geq 0$ ,*

$$(\gamma_\alpha(s) \mid \gamma_\beta(t))_{x_0} \leq (\alpha \mid \beta)_{x_0} + E.$$

The following lemma states that the Gromov product of two points is approximately the distance to a geodesic between the points.

**Lemma 2.9.** *For any  $\delta$ -hyperbolic space  $X$  and any  $x, y \in X$ , we have*

$$|(x \mid y)_{x_0} - d_X(x_0, [x, y])| \leq \delta,$$

where  $[x, y]$  is any geodesic from  $x$  to  $y$  and  $x_0$  is a fixed basepoint.

Given a group acting by isometries on a  $\delta$ -hyperbolic space, we define certain subsets of the boundary which arise as limits of sequences of elements from an orbit of a fixed base point.

**Definition 2.10.** Let  $G$  be a group acting by isometries on a  $\delta$ -hyperbolic metric space  $(X, d)$  and let  $x \in X$  be a basepoint. For any subgroup  $H$  of  $G$ , the *limit set*  $\Lambda(H) \subseteq \partial X$  of  $H$  is

$$\Lambda(H) := \{p \in \partial X \mid p = \lim_{n \rightarrow \infty} h_n x \text{ for some sequence } h_n \in H\}.$$

### 3. AN ALTERNATIVE CHARACTERIZATION OF ACYLINDRICITY

When a subgroup  $H < G$  is quasi-isometrically embedded by an action  $G \curvearrowright X$  on a hyperbolic space, there is an alternate characterization of acylindricity of  $G \curvearrowright X$  along  $H$ . In practice, this alternate characterization is easier to check.

**Definition 3.1.** Let  $G$  act by isometries on a hyperbolic metric space  $X$ , and let  $H$  be a subgroup that is quasi-isometrically embedded by the action. The action of  $G$  on  $X$  is *acylindrical along  $H$*  if for some (equivalently any)  $H$ -cobounded  $Y \subset X$  the following condition holds: For any  $\varepsilon \geq 0$ , there exist constants  $D = D(\varepsilon, Y)$  and  $N = N(\varepsilon, Y)$  such that any collection of distinct cosets  $\{g_i H, \dots, g_k H\}$  such that

$$\text{diam} \left( \bigcap_{i=1}^k \mathcal{N}_\varepsilon(g_i Y) \right) > D$$

has cardinality at most  $N$ .

For a fixed  $\varepsilon > 0$  and  $H$ -cobounded  $Y \subset X$ , we call the constants  $D(\varepsilon, Y), N(\varepsilon, Y)$  the *constants of acylindricity* of the action of  $G$  on  $X$  along  $H$ .

The following theorem shows that this definition is equivalent to Definition 1.1.

**Theorem 3.2.** *Let  $G$  act by isometries on a hyperbolic space  $X$ , and let  $H$  be a subgroup that is quasi-isometrically embedded by the action. Let  $Y$  be an  $H$ -cobounded subset of  $X$ . Then the action of  $G$  is acylindrical along  $H$  in the sense of Definition 1.1 if and only if it is acylindrical along  $H$  in the sense of Definition 3.1.*

*Proof.* Let  $\{g_1H, \dots, g_kH\}$  be a collection of distinct cosets and let  $x \in X$ . We fix  $Y = Hx$ .

We prove the backward direction first and assume that the action of  $G$  on  $X$  is acylindrical along  $H$  in the sense of Definition 3.1. Fix  $\varepsilon \geq 0$ , and let  $D = D(2\varepsilon, Y)$  and  $N = N(2\varepsilon, Y)$ . Since  $H$  is quasi-isometrically embedded by the action, there exists a constant  $P > 0$  such that there are at most  $P$  elements  $h \in H$  such that  $d_X(x, hx) \leq 2\varepsilon$ .

Fix  $h_1, h_2 \in H$  satisfying  $d_X(h_1x, h_2x) > D$ , and let

$$\Delta = \{g \in G \mid d_X(h_1x, gh_1x) \leq \varepsilon \text{ and } d_X(h_2x, gh_2x) \leq \varepsilon\}.$$

We claim that  $\#\Delta \leq NP$ . Towards a contradiction, assume this is not the case, so that there exist distinct  $\{g_1, \dots, g_k\} \subset \Delta$  with  $k > NP$ . Then

$$\text{diam} \left( \bigcap_{i=1}^k \mathcal{N}_\varepsilon(g_iHx) \right) > D.$$

By assumption, the number of *distinct cosets* in  $\{g_1H, \dots, g_kH\}$  is at most  $N$ . Rearranging the list, we can therefore assume that the first  $(P+1)$  cosets are the same. In other words  $g_1^{-1}g_j \in H$  for all  $j \leq P+1$ . Since the  $g_j$  are all in  $\Delta$  we have  $d_X(h_1x, g_jh_1x) \leq \varepsilon$  for each  $j$ . Let  $\hat{g}_j = h_1^{-1}g_jh_1$ , and note that  $\hat{g}_j \in H$  for  $j \leq P+1$ . Moreover

$$(2) \quad d_X(x, \hat{g}_jx) = d_X(g_1h_1x, g_jh_1x) \leq 2\varepsilon.$$

There are only  $P$  elements of  $H$  satisfying (2), so we have  $\hat{g}_j = \hat{g}_i$  for some  $i < j \leq P+1$ . But this implies  $g_j = g_i$ , which is a contradiction. We have therefore shown the action is acylindrical along  $Y$  in the sense of Definition 1.1, with  $R = D$  and  $M = NP$ .

We now prove the forward direction and assume the action is acylindrical along  $H$  in the sense of Definition 1.1. Let  $\varepsilon > 0$ , let  $\delta$  be a hyperbolicity constant for  $X$ , and let  $\lambda$  be a constant of quasi-convexity for the orbit  $Hx$ . Let  $R = R(\varepsilon')$ ,  $M = M(\varepsilon')$ , be the constants provided by Definition 1.1, where

$$\varepsilon' = 4(\varepsilon + \delta + \lambda).$$

We will show that Definition 3.1 holds with constants

$$D = D(\varepsilon, Y) = R + \varepsilon' + \varepsilon, \quad N = N(\varepsilon, Y) = MB + 1,$$

where  $B$  is the number of elements  $h$  of  $H$  with  $d_X(hx, x) \leq R + \varepsilon'$ .

Suppose that  $g_0H, \dots, g_nH$  are distinct cosets with

$$\text{diam} \left( \bigcap_{i=0}^n \mathcal{N}_\varepsilon(g_iHx) \right) > D.$$

We may translate this intersection so that it contains points close to  $x$  and to some  $kx \in H$  which is distance at least  $D - \varepsilon$  from the identity. More specifically, we can assume the following, possibly after changing the coset representatives  $g_i$ :

- (1)  $g_0 = 1$ ;
- (2)  $d_X(g_i x, x) \leq 2\varepsilon$  for each  $i$ ; and
- (3) there is some  $k \in H$  so that  $d_X(x, kx) > D - \varepsilon$  and so that for each  $i$  there is some  $k_i \in H$  so that  $d_X(g_i k_i x, kx) \leq 2\varepsilon$ .

Let  $a$  lie on a geodesic from  $x$  to  $kx$  so that  $d_X(x, a) = R + 2(\delta + \lambda + \varepsilon)$ . By quasi-convexity of  $Hx$ , there is some  $h_0 \in H$  with  $d_X(h_0x, a) \leq \lambda$ .

Let  $i \leq n$ . By slimness of quadrilaterals and our lower bound on  $D$ , there is some  $a_i$  on a geodesic from  $g_ix$  to  $g_ik_ix$  so that  $d_X(a, a_i) \leq 2\delta$ . Again using quasi-convexity, there is some  $h_i \in H$  with  $d_X(g_ih_ix, a_i) \leq \lambda$ . For each  $i$ , we have

$$d_X(h_0x, g_ih_ix) \leq 2\lambda + 2\delta,$$

and

$$(3) \quad d_X(x, h_ix) \in [R + \lambda, R + 4\varepsilon + 4\delta + 3\lambda].$$

There are at most  $B$  elements of  $H$  satisfying (3), so it remains to fix some such element  $h_i$  and bound the number of  $g_j$  associated to it. But for any two such  $g_j, g_k$ , we have  $d_X(g_jx, g_kx) \leq \varepsilon'$  and  $d_X(g_jh_ix, g_kh_ix) \leq \varepsilon'$ . Acylindricity tells us there are at most  $M = M(\varepsilon')$  such elements, so over all possible  $h$ , we have at most  $MB$  such. Thus the action is acylindrical along  $H$  in the sense of Definition 3.1.  $\square$

**Definition 3.3.** Let  $\mathcal{H}$  be a collection of subgroups of  $G$ , and suppose  $G \curvearrowright X$  is an action on a hyperbolic space so that each  $H \in \mathcal{H}$  is quasi-isometrically embedded by the action. Suppose there is a uniformly quasi-convex collection of subspaces  $\{Y_H \mid H \in \mathcal{H}\}$  so that  $H$  acts coboundedly on  $Y_H$  for each  $H \in \mathcal{H}$ . Then the action is *acylindrical along  $\mathcal{H}$*  if for all  $\varepsilon \geq 0$  there exist constants  $D = D(\varepsilon, \{Y_H\}_{H \in \mathcal{H}})$  and  $N = N(\varepsilon, \{Y_H\}_{H \in \mathcal{H}})$  such that any collection  $\mathcal{S} \subseteq \{Y_H \mid H \in \mathcal{H}\}$  satisfying

$$\text{diam} \left( \bigcap_{C \in \mathcal{S}} \mathcal{N}_\varepsilon(C) \right) > D$$

has cardinality at most  $N$ .

Using this characterization of acylindricity along a collection of quasi-isometrically embedded subgroups, we can generalize the notion of an A/QI triple to allow a collection of subgroups.

**Definition 3.4.** An *A/QI triple*  $(G, X, \mathcal{H})$  consists of a Gromov hyperbolic space  $X$  which is acylindrical along the collection  $\mathcal{H}$  of infinite subgroups of  $G$  in the sense of Definition 3.3.

The following proposition shows that the A/QI condition for *finite* collections of subgroups reduces to the A/QI condition for a single subgroup.

**Proposition 3.5.** *Let  $\mathcal{H} = \{H_1, \dots, H_n\}$  be a finite collection of infinite subgroups of  $G$ . Then  $(G, X, \mathcal{H})$  is an A/QI triple if and only if  $(G, X, H_i)$  is an A/QI triple for all  $1 \leq i \leq n$ .*

*Proof.* This follows immediately from the definition of an A/QI triple and the fact that  $\mathcal{H}$  is a finite collection of subgroups.  $\square$

#### 4. INTERSECTIONS, COMBINATIONS, AND FINITE HEIGHT

In this section, we establish basic properties of A/QI triples. We *do not* assume in this section that  $G$  is finitely generated.

**4.1. Intersections.** The first result of this section implies, in particular, that if  $(G, X, A)$  and  $(G, X, B)$  are A/QI triples for the same action  $G \curvearrowright X$ , then  $(G, X, A \cap B)$  is an A/QI triple.

**Proposition 4.1.** *Suppose  $(G, X, A)$  is an A/QI triple and  $B < G$  is an infinite subgroup quasi-isometrically embedded by the action. Then  $(G, X, A \cap B)$  is an A/QI triple.*

*Proof.* Let  $\delta$  be a constant of hyperbolicity for  $X$ , and let  $x \in X$ . Since  $A$  and  $B$  are quasi-isometrically embedded by the action, there is some  $\lambda > 0$  so that the sets  $Ax$  and  $Bx$  are  $\lambda$ -quasi-convex. There is also a constant  $M$  so that any geodesic with endpoints in  $Bx$  lies Hausdorff distance at most  $M$  from the image of a geodesic in some fixed Cayley graph for  $B$ .

Let  $C = A \cap B$ . We must show that  $Cx$  is quasi-convex in  $X$ . We suppose by contradiction that it is not. There is therefore a sequence of elements  $\{c_i\}$  of  $C$  so that geodesics  $\gamma_i$  from  $x$  to  $c_i x$  travel arbitrarily far away from  $Cx$ . In other words, there are points  $y_i \in \gamma_i$  with  $\lim_{i \rightarrow \infty} d_X(y_i, Cx) = \infty$ . Since  $Ax$  and  $Bx$  are  $\lambda$ -quasi-convex, there are, for each  $i$ ,  $a_i \in A$  and  $b_i \in B$  so that both  $d_X(a_i x, y_i)$  and  $d_X(b_i x, y_i)$  are at most  $\lambda$ . Translating by  $b_i^{-1}$  we obtain a sequence of geodesics  $b_i^{-1} \gamma_i$  joining  $b_i^{-1} x$  to  $b_i^{-1} c_i x$ . The geodesic  $b_i^{-1} \gamma_i$  is Hausdorff distance at most  $M$  from some quasi-geodesic  $\sigma_i$  which is the image of a geodesic  $\tilde{\sigma}_i$  joining  $b_i^{-1}$  to  $b_i^{-1} c_i$  in a Cayley graph for  $B$ . Passing to a subsequence we may assume the geodesics  $\tilde{\sigma}_i$  converge to a bi-infinite geodesic  $\tilde{\sigma}$  in the Cayley graph for  $B$ , whose image in  $X$  is a bi-infinite quasi-geodesic  $\sigma$ .

Let  $\varepsilon = M + \delta + 2\lambda$ , and let  $D = \max\{\varepsilon + 1, D(\varepsilon, Ax)\}$ , where  $D(\varepsilon, Ax)$  is the constant from Theorem 3.2, applied with  $H = A$ . Fix a point  $p$  on the quasi-geodesic  $\sigma$  so that  $d_X(x, p) \geq D$  and so that  $p$  is within  $M$  of the subsegment of  $b_i^{-1} \gamma_i$  joining  $b_i^{-1} y_i$  to  $b_i^{-1} c_i x$  for all but finitely many  $i$ . See Figure 1.

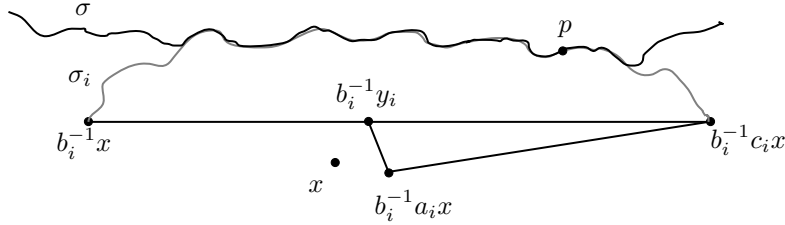


FIGURE 1. The point  $p$  is on the bi-infinite quasi-geodesic  $\sigma$ , at distance  $D$  from  $x$ . The index  $i$  is assumed to be large enough that  $\sigma_i$  contains  $p$ .

Discarding finitely many  $i$ , we may assume that  $p$  is within  $M$  of all these subsegments. Let  $p'_i$  be a nearest point to  $p$  on the subsegment of  $b_i^{-1} \gamma_i$  joining  $b_i^{-1} y_i$  to  $b_i^{-1} c_i x$ , and consider a geodesic triangle with one side equal to this subsegment, and with the third corner equal to  $b_i^{-1} a_i x$ . Let  $p_i$  be a point within  $\delta$  of  $p'_i$  on one of the other two sides of this triangle. We have  $d_X(p, p_i) \leq M + \delta$ . If  $p_i$  lay on the side joining  $b_i^{-1} a_i x$  to  $b_i^{-1} y_i$ , this would imply  $d_X(p, x) \leq M + 2\lambda + \delta$ , a contradiction, since  $d_X(p, x) \geq \varepsilon + 1$ . Thus  $p_i$  lies on a geodesic joining  $b_i^{-1} a_i x$  to  $b_i^{-1} c_i x$ . Quasi-convexity of  $Ax$  gives us an  $\hat{a}_i \in A$  so that  $b_i^{-1} \hat{a}_i x$  is at most  $\lambda$  from

$p_i$ . Thus we have  $d_X(b_i^{-1}\hat{a}_i x, p) \leq M + \delta + \lambda < \varepsilon$  and  $d_X(b_i^{-1}a_i x, x) \leq 2\lambda < \varepsilon$ . In particular,  $x$  and  $p$  are in  $N_\varepsilon(b_i^{-1}Ax)$  for all  $i$ , so we have

$$\text{diam} \left( \bigcap_i N_\varepsilon(b_i^{-1}Ax) \right) \geq D.$$

By Theorem 3.2, infinitely many of the cosets  $b_i^{-1}A$  coincide. Discarding terms and relabeling we may assume that  $b_i^{-1}A = b_0^{-1}A$  for all  $i$ . In particular we have  $b_i b_0^{-1} \in C$  for every  $i$ . But

$$d_X(b_i b_0^{-1}x, y_i) \leq d_X(b_i b_0^{-1}x, b_i x) + d_X(b_i x, y_i) \leq d_X(x, b_0 x) + \lambda,$$

so  $d_X(b_i b_0^{-1}x, y_i)$  is bounded independently of  $i$ . This contradicts our choice of  $y_i$ , showing that in fact  $Cx$  is quasi-convex.

The orbit  $Cx$  is a subset of  $Ax$ , along which  $G$  acts acylindrically, so  $G$  acts acylindrically along  $Cx$ , and  $(G, X, C)$  is an A/QI triple as desired.  $\square$

**4.2. A combination theorem.** In this subsection we prove Theorem 1.4, which we restate for the convenience of the reader.

**Theorem 1.4.** *Let  $G$  act on a hyperbolic space  $X$ , and suppose  $H_1, \dots, H_k$  are subgroups of  $G$  such that  $(G, X, H_i)$  is an A/QI triple for each  $i = 1, \dots, k$  and  $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$  for all  $i \neq j$ . Then there exists a finite collection of elements  $\mathcal{S} \subset \cup_{i=1}^k H_i$  such that for any quasi-isometrically embedded subgroups  $K_i \leq H_i$  satisfying  $K_i \cap \mathcal{S} = \emptyset$ , we have  $K = \langle K_1, \dots, K_k \rangle \cong K_1 * K_2 * \dots * K_k$  and  $(G, X, K)$  is an A/QI triple.*

We note that by Proposition 3.5, the assumption that each  $(G, X, H_i)$  is an A/QI triple is equivalent to assuming that  $(G, X, \{H_1, \dots, H_k\})$  is an A/QI triple.

Before proving the theorem, we need some preliminary lemmas. We say that a path  $q$  in a metric space  $X$  is a *piecewise quasi-geodesic* if  $q$  can be expressed as a concatenation  $q_1 \cdot \dots \cdot q_n$  where each  $q_i$  is a continuous quasi-geodesic path. The following lemma gives conditions under which a piecewise quasi-geodesic is a quasi-geodesic (with constants independent of the number of quasi-geodesic segments being concatenated).

**Lemma 4.2** ([Min05, Lemma 4.2]). *Let  $x_0, x_1, \dots, x_n$  be points in a  $\delta$ -hyperbolic space  $X$  and let  $q_i$  be a continuous  $\tau$ -quasi-geodesic from  $x_{i-1}$  to  $x_i$ . Then for any  $C_0 \geq 14\delta$  and for  $C_1 = 12(C_0 + \delta) + \tau + 1$ , if  $\ell_X(q_i) \geq \tau C_1$  and  $(x_{i-1} | x_{i+1})_{x_i} \leq C_0$ , then the concatenation  $q_1 \cdot \dots \cdot q_n$  is a  $\max\{4\tau, \frac{5}{2}M + C_1\}$ -quasi-geodesic, where  $M$  is the Morse constant for the parameters  $\tau, \delta$ .*

Let  $G, X$ , and  $H_1, \dots, H_k$  be as in the statement of Theorem 1.4. We fix the following data.

Let  $T_i$  be a finite generating set for  $H_i$ . Let  $\delta > 1$  be a hyperbolicity constant for  $X$ . We fix a basepoint  $x \in X$  and let  $\pi: G \rightarrow X$  be the orbit map defined by  $g \mapsto gx$ . Since each  $(G, X, H_i)$  is an A/QI triple, there is a constant  $\tau$  such that each  $H_i x$  is  $\tau$ -quasi-isometrically embedded in  $X$ . Let  $E$  be the constant from Lemma 2.8 with this choice of  $\tau$  and  $\delta$ .

Since  $\Lambda(H_i) \cap \Lambda(H_j) = \emptyset$  for all  $i \neq j$ , there is a constant  $\kappa$  such that

$$(4) \quad (\xi | \xi')_x < \kappa$$

for all  $\xi \in \Lambda(H_i)$  and  $\xi' \in \Lambda(H_j)$ .

Let

$$\eta = \max\{d_X(x, tx) \mid i = 1, \dots, k \text{ and } t \in T_i\},$$

which is well-defined since  $\#T_i < \infty$  for each  $i$ .

For each  $i$  and each  $t \in T_i$ , fix a geodesic  $[x, tx]$  in  $X$ . Since each  $H_i x$  is  $\tau$ -quasi-isometrically embedded in  $X$ , for any  $i$  and any geodesic word  $t_1 \dots t_\ell$  where  $t_j \in T_i$ , the concatenation

$$[x, t_1 x] \cdot [t_1 x, t_1 t_2 x] \cdot \dots \cdot [t_1 \dots t_{\ell-1} x, t_1 \dots t_\ell x]$$

is a  $\tau$ -quasi-geodesic in  $X$ . Let  $M'$  be the Morse constant for parameters  $\tau$  and  $\delta$ .

For the remainder of the subsection, the following data is fixed: the sets  $T_i$ , the basepoint  $x$ , the map  $\pi$ , the constants  $\delta$ ,  $\tau$ ,  $E$ ,  $\kappa$ ,  $\eta$ , and  $M'$ , and the geodesics  $[x, tx]$ .

Given any collection of subgroups  $K_i \leq H_i$ , define a metric graph  $\Gamma$  as follows.

**Definition 4.3.** Let  $\text{Cay}(H_i, T_i)$  be the Cayley graph of  $H_i$  with respect to the finite generating set  $T_i$ . We will build  $\Gamma$  inductively. At the base stage, we begin with a single Cayley graph  $\text{Cay}(H_1, T_1)$ , and identify  $K_1$  with the orbit of the identity under  $K_1$ . At each point in  $K_1 \subset \text{Cay}(H_1, T_1)$ , attach a disjoint copy of  $\text{Cay}(H_j, T_j)$  for each  $j \neq 1$  along the identity  $1 \in \text{Cay}(H_j, T_j)$ . That is, for each  $h \in K_1$  and each  $j \neq 1$ , we identify the identity 1 in a disjoint copy of  $\text{Cay}(H_j, T_j)$  with the vertex  $h \in \text{Cay}(H_1, T_1)$ . At each subsequent step, we repeat the same process for each new Cayley graph  $\text{Cay}(H_j, T_j)$ , by attaching a disjoint copy of  $\text{Cay}(H_i, T_i)$  for each  $i \neq j$  to each point in  $K_j \subset \text{Cay}(H_j, T_j)$  along the identity  $1 \in \text{Cay}(H_i, T_i)$ .

The resulting graph  $\Gamma$  is the smallest subgraph of  $\text{Cay}(H_1 * \dots * H_k, \cup_{i=1}^k T_i)$  which is both invariant under the action of  $K_1 * \dots * K_k \subseteq H_1 * \dots * H_k$  and contains  $\text{Cay}(H_i, T_i)$  for each  $i$ .

Vertices in  $\Gamma$  come in two types: those that lie in the orbit of the identity under  $K_i$  in some copy of  $\text{Cay}(H_i, T_i)$ , and those that do not. We call vertices of the first type *wedge vertices*.

Each edge in  $\Gamma$  is an edge of some  $\text{Cay}(H_i, T_i)$  and so is labeled by an element  $t \in T_i$ . We define the length of such an edge to be  $d_X(x, tx)$ .

Define a map  $\psi: \Gamma \rightarrow X$  by sending each edge whose initial vertex is the identity and whose label is some  $t \in T_i$  to the geodesic  $[x, tx]$  chosen above, and extending equivariantly by the action of  $K_1 * \dots * K_k$  on  $\Gamma$ . By definition,  $\psi$  restricts to an isometry on each edge. The metric graph  $\Gamma$  and the map  $\psi$  depend on the choice of subgroups  $K_i \leq H_i$ . However, we will show that for certain subgroups  $K_i$ , the map  $\psi$  is a uniform quasi-isometric embedding, independent of those choices.

We also fix constants

$$C_0 = \kappa + E + 14\delta$$

$$C_1 = 12(C_0 + \delta) + \tau + 1$$

$$\mu = \frac{5}{2}M' + 12(\kappa + E + 15\delta) + 4\tau + 1 + C_1$$

and let  $\mathcal{S}_1 = \{h \in \cup_{i=1}^k H_i \setminus \{1\} \mid d_X(x, hx) \leq C_1\}$ .

**Lemma 4.4.** *Let  $K_i \leq H_i$  be quasi-isometrically embedded subgroups of  $H_i$  satisfying*

$$K_i \cap \mathcal{S}_1 = \emptyset,$$

and let  $\Gamma$  be the graph from Definition 4.3 with this choice of subgroups  $K_i$ . The map  $\psi: \Gamma \rightarrow X$  is a  $\mu$ -quasi-isometric embedding which takes every geodesic in  $\Gamma$  to a tame  $\mu$ -quasi-geodesic.

*Proof.* A geodesic edge path in  $\Gamma$  is either completely contained in a copy of  $\text{Cay}(H_i, T_i)$  for some  $i$  or passes through multiple Cayley graphs. In the first case, the image of the path will be a  $\mu$ -quasi-geodesic because each  $\text{Cay}(H_i, T_i)$  is  $\tau$ -quasi-isometrically embedded in  $X$  and  $\mu > \tau$ .

Thus we focus on geodesic edge paths in  $\Gamma$  which pass through multiple Cayley graphs. Note that such a path  $\sigma$  can only change Cayley graphs at wedge vertices, which are necessarily in the orbit of the identity under some  $K_i$ . Thus  $\sigma$  is the concatenation of geodesic edge paths

$$(5) \quad \sigma_1 \cdot \sigma_2 \cdots \sigma_p,$$

where each  $\sigma_\ell$  is a maximal geodesic edge path contained in some  $\text{Cay}(H_{i_\ell}, T_{i_\ell})$  with  $H_{i_\ell} \neq H_{i_{\ell+1}}$ . Note that the terminal vertex of  $\sigma_1$  is a wedge vertex, the initial vertex of  $\sigma_p$  is a wedge vertex, and, if  $2 \leq \ell \leq p-1$ , then both endpoints of  $\sigma_\ell$  are wedge vertices. Moreover, for  $2 \leq \ell \leq p-1$ , the label of each  $\sigma_\ell$  is an element of  $K_{i_\ell}$  by construction.

Our main tool for understanding the image of such paths in  $X$  will be Lemma 4.2, applied with the constants  $C_0, C_1$  defined just before the statement of the lemma.

For each  $i$  and each  $h \in K_i$ , we have

$$(6) \quad d_X(x, hx) \geq C > C_1,$$

by our choice of subgroups  $K_i$ .

Let  $y_1 \in \psi(\sigma_\ell)$  and  $y_2 \in \psi(\sigma_{\ell+1})$  for some  $\ell$ . By translating by an element of  $G$ , we may assume that the image of the terminal point of  $\sigma_\ell$  (and therefore the image of the initial point of  $\sigma_{\ell+1}$ ) is  $x$ . Since  $H_{i_\ell} \neq H_{i_{\ell+1}}$ , there are elements  $\xi \in \Lambda(H_{i_\ell})$  and  $\xi' \in \Lambda(H_{i_{\ell+1}})$  so that an application of Lemma 2.8 yields

$$(7) \quad (y_1 \mid y_2)_x \leq (\xi \mid \xi')_x + E \leq \kappa + E,$$

where the second inequality comes from (4).

The inequalities (6) and (7) show that if  $\sigma_1$  and  $\sigma_p$  are empty, then  $\sigma$  is mapped by  $\psi$  to a  $(\mu - C_1)$ -quasi-geodesic (and hence a  $\mu$ -quasi-geodesic).

Suppose  $\sigma_p = \emptyset$ , while  $\sigma_1 \neq \emptyset$ , and suppose the initial vertex of  $\sigma_1$  is an arbitrary vertex  $a$  of  $\text{Cay}(H_{i_1}, T_{i_1})$  and the terminal vertex is a wedge vertex  $b$ . If  $d_X(ax, bx) > C_1$  then the argument above shows that  $\sigma$  is mapped by  $\psi$  to a  $(\mu - C_1)$ -quasi-geodesic. On the other hand, if  $d_X(ax, bx) \leq C_1$ , then  $\sigma$  is mapped by  $\psi$  to a  $\mu$ -quasi-geodesic in  $X$ . A similar argument holds when  $\sigma_2 \neq \emptyset$  and when both  $\sigma_1$  and  $\sigma_2$  are non-empty. In any case,  $\sigma$  is mapped by  $\psi$  to a  $\mu$ -quasi-geodesic in  $X$ .

Since  $\psi$  restricted to any geodesic in  $\Gamma$  gives a (tame)  $\mu$ -quasi-geodesic, the entire map  $\psi$  is a  $\mu$ -quasi-isometric embedding.  $\square$

We are now ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* We continue to use the data fixed after Lemma 4.2. Let  $M$  be the Morse constant for parameters  $\mu$  and  $\delta$ , and fix

$$\varepsilon' = 8\delta + 3M + M'.$$

The action of  $G$  of  $X$  is assumed to be acylindrical along  $\{H_1, \dots, H_k\}$ . For each  $H_i$ , let  $Y_i$  be  $\psi(\text{Cay}(H_i, T_i))$ , and let  $D = D(\varepsilon', \{Y_1, \dots, Y_k\})$  and  $N = N(\varepsilon', \{Y_1, \dots, Y_k\})$  be the constants of acylindricity of this action, as described in Definition 3.3.

Fix a constant

$$(8) \quad C \geq \max\{\mu(2D + 8\delta + 4M + 1), C_1\},$$

and let

$$\mathcal{S} = \{h \in \bigcup_{i=1}^k H_i \setminus \{1\} \mid d_X(x, hx) \leq C\}.$$

Since each  $H_i$  is uniformly quasi-isometrically embedded in  $X$  by the orbit map, there is a constant  $P$  so that for any  $i$  and any  $h \in H_i$ , the number of elements  $h' \in H_i$  so that  $d_X(hx, h'x) \leq \frac{C}{\mu} + 2M + 2M' + 8\delta - 1$  is at most  $P$ .

Let  $K_i \leq H_i$  be quasi-isometrically embedded subgroups of  $H_i$  satisfying

$$K_i \cap \mathcal{S} = \emptyset,$$

and let  $\Gamma$  be the graph from Definition 4.3 with this choice of subgroups  $K_i$ . As  $C \geq C_1$ , the graph  $\Gamma$  is  $\mu$ -quasi-isometrically embedded by Lemma 4.4.

We claim that the canonical map  $\phi : K_1 * \dots * K_k \rightarrow \langle K_1, \dots, K_k \rangle = K$  is an isomorphism. It is clearly surjective. To see that it is injective, suppose  $f$  is a non-trivial element of  $K_1 * \dots * K_k$ . If  $f \in K_i$  for some  $i$  then  $\phi(f) \neq 1$ , by our choice of  $K_i$ , so  $f$  must have a normal form involving at least two factors. By conjugating  $f$ , we may assume this normal form does not begin and end with an element of the same factor  $K_i$ , and has a geodesic axis in  $\Gamma$ . By Lemma 4.4, this bi-infinite geodesic has image a bi-infinite  $\mu$ -quasi-geodesic in  $X$ . In particular, this quasi-geodesic is unbounded, so  $\phi(f)$  is non-trivial.

It remains to show that  $G$  acts acylindrically along  $K$ . Fix  $\varepsilon > 0$ , and let

$$R = 2 \left( \frac{C}{\mu} - 1 \right) + 2\varepsilon + 2M + 4\delta.$$

Let  $\Delta = \{gK\}$  be a collection of distinct cosets satisfying

$$\text{diam} \left( \bigcap_{gK \in \Delta} \mathcal{N}_\varepsilon(gKx) \right) > R.$$

We will give a uniform bound on  $\#\Delta$ ; namely we will show  $\#\Delta \leq 2NP$ . To do so, we will show that if  $\#\Delta > 2NP$ , we contradict the acylindricity of the action of  $G$  on  $X$  along  $\{H_1, \dots, H_k\}$ .

Let  $\gamma$  be a geodesic segment in  $X$  of length at least  $R$  whose endpoints are contained in the  $\varepsilon$ -neighborhood of  $gKx$  for all  $gK \in \Delta$ . By Lemma 4.4, there are tame  $\mu$ -quasi-geodesics  $\beta_g$  in  $g\psi(\Gamma)$  with endpoints within  $\varepsilon$  of the endpoints of  $\gamma$ .

Let  $\gamma'$  be the subpath of  $\gamma$  with endpoints distance exactly  $\varepsilon$  from the endpoints of  $\gamma$ . Fix  $g$  so that  $gK \in \Delta$ . There is a subsegment  $\xi_g$  of the geodesic joining the endpoints of  $\beta_g$  which has endpoints within  $2\delta$  of those of  $\gamma'$  and which is Hausdorff distance at most  $2\delta$  from  $\gamma'$ . The segment  $\xi_g$  is Hausdorff distance at most  $2M + 2\delta$

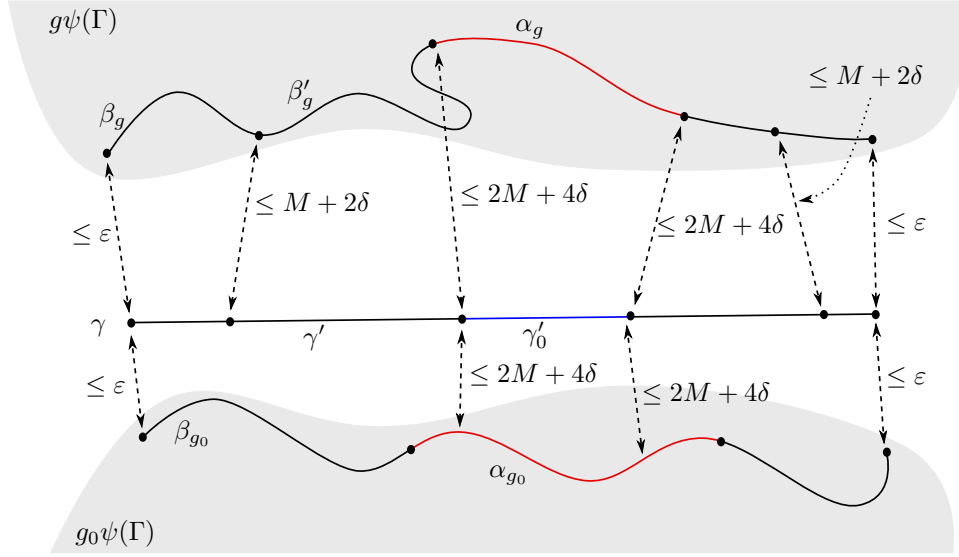


FIGURE 2. The configuration of (quasi-)geodesics in the proof of Theorem 1.4.

from a subpath  $\beta'_g$  of  $\beta_g$  whose endpoints are within  $M$  of those of  $\xi_g$ . In particular, the Hausdorff distance between  $\gamma'$  and  $\beta'_g$  is at most  $2M + 4\delta$  and the endpoints of  $\beta'_g$  are at distance at most  $M + 2\delta$  from those of  $\gamma'$ . See Figure 2. Moreover, since  $R = 2\left(\frac{C}{\mu} - 1\right) + 8\epsilon + 4M$ , we have

$$\ell_X(\gamma') = \ell_X(\gamma) - 2\epsilon > 2\left(\frac{C}{\mu} - 1\right) > \frac{C}{\mu} - 1$$

and the distance between the endpoints of  $\beta'_g$  is at least

$$\ell_X(\gamma') - 2M - 4\delta \geq R - 2\epsilon - 2M - 4\delta \geq 2\left(\frac{C}{\mu} - 1\right).$$

Each quasi-geodesic  $\beta_g$  is a translate by  $g$  of the image of a geodesic  $\sigma_g$  in  $\Gamma$  of the form

$$\sigma_g = \sigma_{g,1} \cdot \dots \cdot \sigma_{g,p_g}$$

as in (5).

Fix  $g_0K \in \Delta$ . The distance between the endpoints of  $\beta'_{g_0}$  is at least  $2\left(\frac{C}{\mu} - 1\right)$ . We will first show that  $\beta'_{g_0}$  contains a subpath  $\alpha_{g_0}$  of  $g_0\psi(\sigma_{g_0,j_0})$  for some  $j_0$  such that the distance between the endpoints of this subpath is at least  $\frac{C}{\mu} - 1$ . If  $\beta'_{g_0}$  does not contain the image of any wedge vertices, then  $\beta'_{g_0}$  itself is the desired subpath. If  $\beta'_{g_0}$  contains the image of exactly one wedge vertex, then this point divides  $\beta'_{g_0}$  into two subpaths, and we take the longer of these subpaths. Finally, if  $\beta'_{g_0}$  contains the images of at least two wedge vertices, then it contains the entire image  $g_0\psi(\sigma_{g_0,j_0})$  for some  $j_0$ , and the distance between the endpoints of  $g_0\psi(\sigma_{g_0,j_0})$  is at least  $C > \frac{C}{\mu} - 1$ .

Let  $\gamma'_0$  be a subpath of the geodesic  $\gamma'$  whose endpoints are at distance at most  $2M + 4\delta$  from  $\alpha_{g_0}$  and whose length is  $\frac{C}{\mu} - 4M - 8\delta - 1$ . For each  $gK \in \Delta \setminus \{g_0K\}$ ,

let  $\alpha_g$  be a maximal connected subpath of  $\beta'_g$  with endpoints distance at most  $2M + 4\delta$  of the endpoints of  $\gamma'_0$ . Then a geodesic connecting the endpoints of  $\alpha_g$  is at Hausdorff distance at most  $2M + 6\delta$  from  $\gamma'_0$  and at Hausdorff distance at most  $M$  from  $\alpha_g$ . Thus the Hausdorff distance between  $\alpha_g$  and  $\gamma'_0$  is at most  $3M + 6\delta$ . We have

$$\frac{C}{\mu} - 8M - 16\delta - 1 \leq \ell_X(\alpha_g) \leq \mu \left( \frac{C}{\mu} - 1 \right) + \mu \leq C.$$

There may be the image of a wedge vertex in  $\alpha_g$  for some  $g$ , but since the image of a geodesic between wedge vertices is a path of length at least  $C$ , there can be at most one such vertex in  $\alpha_g$ . Suppose there is such a vertex  $w$  in  $\alpha_g$ , and let  $z$  be a closest point to  $w$  on  $\gamma'_0$ . Then  $d_X(z, w) \leq 3M + 6\delta$ . We also have that the initial points of  $\gamma'_0$  and  $\alpha_g$  are at distance at most  $2M + 4\delta$ . Therefore the initial subgeodesic  $\gamma''_0$  of  $\gamma'_0$  ending at  $z$  and a geodesic  $\alpha'_g$  from the initial point of  $\alpha_g$  to  $w$  are at Hausdorff distance most  $3M + 8\delta$ . Moreover, the Hausdorff distance between  $\alpha'_g$  and the corresponding subpath of  $\alpha_g$  is at most  $M'$ . It follows that  $\gamma''_0$  is Hausdorff distance at most  $3M + M' + 8\delta$  from the corresponding initial subpath of  $\alpha_g$ . An analogous argument shows that the terminal subpath of  $\gamma'_0$  starting at  $w$  is at Hausdorff distance  $3M + M' + 8\delta$  from the terminal subpath of  $\alpha_g$  starting at  $z$ .

Therefore for each  $g$  there exists  $\sigma_{g,j(g)}$  such that either the initial or terminal half of  $\gamma'_0$  is contained in the  $(3M + M' + 8\delta)$ -neighborhood of  $g\psi(\sigma_{g,j(g)})$ , regardless of the number of images of wedge vertices appearing in  $\alpha_g$ . Let  $y_g$  be the initial vertex of the subpath of  $g\psi(\sigma_{g,j(g)})$  contained in  $\alpha_g$ . Then  $y_g$  is either a wedge vertex or the initial vertex of  $\alpha_g$ , and each  $y_g$  lies in some translate of  $Y_{j(g)}$ . (Recall that  $Y_i = \psi(\text{Cay}(H_i, T_i))$ .) The previous paragraph shows that

$$(9) \quad d(y_g, \gamma'_0) \leq 3M + M' + 8\delta.$$

If  $\#\Delta > 2NP$ , then there exists a subcollection  $\Delta' \subseteq \Delta$  with  $\#\Delta' > NP$  such that the  $(3M + M' + 8\delta)$ -neighborhood of  $g\psi(\sigma_{g,j(g)})$  contains (without loss of generality) the initial half of  $\gamma'_0$  for all  $gK \in \Delta'$ .

However,  $\ell_X(\gamma'_0) = \frac{C}{\mu} - 4M - 8\delta - 1 > 2D$  by (8), and so

$$(10) \quad \text{diam} \left( \bigcap_{gK \in \Delta'} \mathcal{N}_{3M+M'+8\delta}(g\psi(\sigma_{g,j(g)})) \right) > D.$$

For each  $g$ , there is some  $k_g \in K$  so that  $g\psi(\sigma_{g,j(g)})$  is contained in  $gk_g Y_{j(g)}$ . Thus the diameter of the intersection of the  $\varepsilon'$ -neighborhoods of the image of these Cayley graphs in  $X$  is larger than  $D$ . In order to contradict the acylindricity of the action of  $G$  on  $X$  along  $\{H_1, \dots, H_k\}$ , we must show that at least  $N$  of these paths  $g\psi(\sigma_{g,j(g)})$  lie in translates of these Cayley graphs corresponding to *distinct* elements of  $\bigsqcup G/H_i$ .

Since  $\ell(\gamma'_0) = \frac{C}{\mu} - 4M - 8\delta - 1$ , it follows from (9) that for distinct  $gK, g'K \in \Delta'$ , we have

$$d(y_g, y_{g'}) \leq 2(3M + M' + 8\delta) + \frac{C}{\mu} - 4M - 8\delta - 1 = \frac{C}{\mu} + 2M + 2M' + 8\delta - 1.$$

By our choice of constant  $P$ , there must be a further subcollection  $\Delta'' \subseteq \Delta'$  with  $\#\Delta'' > N$  so that for distinct  $gK, g'K \in \Delta''$ , either  $j(g) \neq j(g')$  or the vertices  $y_g$  and  $y_{g'}$ , and therefore  $g\psi(\sigma_{g,j(g)})$  and  $g'\psi(\sigma_{g',j(g)})$ , lie in translates of  $Y_{j(g)}$

corresponding to distinct elements of  $\bigsqcup G/H_i$ . In either case, this contradicts the acylindricity of the action of  $G$  on  $X$  along  $\{H_1, \dots, H_k\}$ . We conclude  $\#\Delta \leq 2NP$ , and so the action of  $G$  on  $X$  is acylindrical along  $K$ . This completes the proof of the theorem.  $\square$

Recall that loxodromic isometries of a hyperbolic space are *independent* if their fixed point sets at infinity are disjoint. If a group  $G$  acts on a hyperbolic space  $X$ , then given any finite collection  $h_1, \dots, h_k$  of independent loxodromic WPD elements with respect to this action,  $(G, X, \{h_1, \dots, h_k\})$  is an A/QI triple; see Definition 6.16 for the definition of a WPD element. The following corollary then follows immediately from Theorem 1.4.

**Corollary 4.5.** *Let  $G$  act on a hyperbolic space  $X$ , and suppose  $h_1, \dots, h_k$  are independent loxodromic WPD elements with respect to this action. There exists a constant  $D \geq 1$  such that for any  $d_1, \dots, d_k \geq D$ , if  $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$ , then  $(G, X, H)$  is an A/QI triple. In particular, every element in  $H$  is a loxodromic WPD element with respect to the action of  $G$  on  $X$ .*

**Remark 4.6.** Let  $G, h_1, \dots, h_k$  be as in the corollary. For each  $i$  let  $E_i$  be the maximal elementary subgroup containing  $h_i$ . Each  $E_i$  is hyperbolically embedded in  $G$ , as is any subcollection of  $\{E_1, \dots, E_k\}$  obtained by choosing one subgroup per conjugacy class [DGO17, Theorem 6.8]. However, the subgroup  $H$  in the conclusion of the corollary is typically not hyperbolically embedded, since it will not be almost malnormal if for some  $i$  either  $E_i \neq \langle h_i \rangle$  or  $d_i > 1$ .

**4.3. Finiteness of height.** We now turn our attention to the height of subgroups in A/QI triples. We recall the definition.

**Definition 4.7 (Height).** Let  $G$  be a group and  $H < G$ . If  $H$  is finite the height of  $H$  is 0. Otherwise the height of  $H$  in  $G$  is the largest  $n$  so that there are distinct cosets  $\{g_1H, \dots, g_nH\}$  so that the intersection  $\bigcap g_iHg_i^{-1}$  is infinite. If there is no largest  $n$  we say the height is infinite.

Quasi-convex subgroups of hyperbolic groups have finite height [GMRS98], as do full relatively quasi-convex subgroups of relatively hyperbolic groups [HW09]. Finiteness of height for convex cocompact subgroups of mapping class groups and  $\text{Out}(\mathbb{F}_n)$  was established in [DM17], and, more generally, for stable and even strongly quasi-convex subgroups of finitely generated groups in [AMST19, Tra19]. (Stable subgroups are discussed in more detail in Section 5.)

**Proposition 4.8.** *Suppose  $(G, X, H)$  is an A/QI triple. Then  $H$  has finite height.*

*Proof.* Choose  $\delta \geq 1$  so that  $X$  is  $\delta$ -hyperbolic. Then it is not hard to show that every loxodromic isometry of  $X$  has a quasi-axis which is a  $2\delta$ -quasi-geodesic.

Fix  $x_0 \in X$  and  $Q > 0$  so that  $Hx_0$  is  $Q$ -quasi-convex and so that any  $2\delta$ -quasi-geodesic with endpoints in  $Hx_0 \cup \Lambda(Hx_0)$  is contained in a  $Q$ -neighborhood of  $Hx_0$ .

Since  $G \curvearrowright X$  is acylindrical along  $H$ , Theorem 3.2 yields constants  $D, N > 0$  so that any collection of distinct cosets  $\{g_1H, \dots, g_kH\}$  satisfying

$$\text{diam} \left( \bigcap_{i=1}^k \mathcal{N}_Q(g_iHx_0) \right) > D$$

has cardinality at most  $N$ .

Suppose that  $\{g_i H\}_{i \in I}$  is a collection of distinct cosets of  $G$  so that the corresponding intersection of conjugates  $K = \bigcap_{i \in I} g_i H g_i^{-1}$  is infinite. We show  $H$  has finite height by giving a bound on the cardinality of  $I$  in terms of the quantities already specified.

The intersection  $K$  is an infinite subgroup of the hyperbolic group  $H$ , so it contains an infinite order element  $w$ . Since  $H$  is quasi-isometrically embedded by the action,  $w$  acts loxodromically on  $X$ . Replacing  $w$  by a power if necessary,  $w$  has a  $2\delta$ -quasi-geodesic axis  $\gamma$  with endpoints in  $\Lambda(g_i H)$  for each  $i$ .

By our assumptions on  $Q$ , the axis  $\gamma$  is contained in the  $Q$ -neighborhood of  $g_i H x_0$  for each  $i$ . In particular, this implies that

$$\text{diam} \left( \bigcap_{i \in I} \mathcal{N}_Q(g_i H x_0) \right) > D.$$

By acylindricity, we therefore have  $\#I \leq N$ . □

## 5. STABILITY

In this section we prove Theorem 1.5. We first recall the definition of a *stable subgroup* and then restate the theorem for convenience.

**Definition 5.1.** Let  $G$  be a finitely generated group. A finitely generated subgroup  $H$  of  $G$  is *stable* if it is undistorted and for some (equivalently, any) finite generating set  $S$  of  $G$  and for every  $\tau \geq 1$ , there exists  $L = L(S, \tau)$  such that whenever  $\gamma_1, \gamma_2$  are  $\tau$ -quasi-geodesics in  $G$  with the same endpoints in  $H$ , the Hausdorff distance between  $\gamma_1$  and  $\gamma_2$  is at most  $L$ .

**Theorem 1.5.** *Let  $(G, X, H)$  be an A/QI triple, and suppose that  $G$  is finitely generated. Then  $H$  is stable in  $G$ .*

Known examples of stable subgroups include convex cocompact subgroups of mapping class groups [DT15] and  $\text{Out}(F_n)$  [ADT17], cyclic subgroups generated by a loxodromic WPD element in an acylindrically hyperbolic group [Sis16], subgroups of relatively hyperbolic groups which are quasi-isometrically embedded into the coned-off Cayley graph by the orbit map [ADT17], and subgroups of hierarchically hyperbolic groups which are quasi-isometrically embedded into the top-level curve by the orbit map [ABB<sup>+</sup>17].

**Definition 5.2.** Let  $G$  be a finitely generated group. A finitely generated subgroup  $H$  of  $G$  is *strongly quasi-convex* if for some (equivalently, any) finite generating set  $S$  of  $G$  and for every  $\tau \geq 1$ , there exists  $L = L(S, \tau)$  such that every  $\tau$ -quasi-geodesic in  $G$  with endpoints in  $H$  is contained in the  $L$ -neighborhood of  $H$ .

We will use the following observation of Tran to prove stability.

**Theorem 5.3** ([Tra19, Theorem 4.8]). *A subgroup of a finitely generated group is stable if and only if it is hyperbolic and strongly quasi-convex in  $G$ .*

In [Sis16], Sisto investigates quasi-convexity properties of hyperbolically embedded subgroups of acylindrically hyperbolic groups and shows that they are strongly quasi-convex. The arguments in our proof of Theorem 1.5 are based on those of Sisto. However, we work in a slightly different setting. Sisto works under the assumption that  $X$  is a Cayley graph of  $G$  with respect to an infinite generating set.

By a Milnor–Schwartz argument, if  $G$  acts coboundedly on  $X$ , then this is equivalent to our situation where  $X$  is a metric space. However, we do not assume that the action of  $G$  on  $X$  is cobounded. Also, Sisto does not assume that  $X$  is a hyperbolic metric space, rather assuming that  $X$  is hyperbolic relative to the translates of  $\pi(H)$ . (This assumption is neither weaker nor stronger than ours.) Some subtle modifications need to be made to Sisto’s strategy in order to deal with our slightly different situation. We provide a complete proof, taking the opportunity to fill in some details and clarify some ambiguities in Sisto’s proof.

For the rest of the section, fix an A/QI triple  $(G, X, H)$ , a finite generating set  $S$  of  $H$ , and a finite generating set  $T \supseteq S$  of  $G$ . We consider the Cayley graph  $\text{Cay}(H, S)$  as a subset of the Cayley graph  $\text{Cay}(G, T)$  in the natural way. We use  $d_G$  and  $d_X$  to denote the metrics on  $\text{Cay}(G, T)$  and  $X$ , respectively, and  $\ell_G(\alpha)$  and  $\ell_X(\beta)$  to denote the length of a path  $\alpha$  in  $G$  (more precisely, in  $\text{Cay}(G, T)$ ) and a path  $\beta$  in  $X$ , respectively. We use  $B_r^X(x)$  to denote the ball in  $X$  about  $x \in X$  with radius  $r$  (that is,  $r$  is measured with respect to  $d_X$ ). We use  $N_K^G(Y)$  and  $N_K^X(Z)$  to denote the closed  $K$ -neighborhood in  $G$  of  $Y \subset \text{Cay}(G, T)$  and the closed  $K$ -neighborhood in  $X$  of  $Z \subset X$ , respectively.

The following lemma is [Sis16, Lemma 3.3], slightly corrected. It shows that preimages of balls of  $X$  which are far apart are *geometrically separated* in  $X$ . Two subsets  $A, B$  of a metric space  $Y$  are said to be geometrically separated if, for all  $D > 0$ , the diameter of  $N_D(A) \cap B$  is finite.

**Lemma 5.4** ([Sis16, Lemma 3.3]). *Let  $G$  act on the geodesic space  $X$ , weakly acylindrically along the subspace  $Y$ . Let  $x_0 \in X$ , and let  $\pi$  denote the orbit map  $g \mapsto gx_0$ . For each  $r \geq 0$ , there is an  $R_1 \geq 0$  so that if  $x, y \in Y$  satisfy  $d_X(x, y) \geq R_1$ , then  $\pi^{-1}(B_r^X(x))$  and  $\pi^{-1}(B_r^X(y))$  are geometrically separated.*

*Proof.* Let  $r \geq 0$ . Since  $G$  acts weakly acylindrically along  $Y$ , there is some  $R_0$  so that for any  $x, y \in Y$  with  $d_X(x, y) \geq R_0$  we have

$$\#\{g \mid d_X(gx, x), d_X(gy, y) \leq 2r\} < \infty.$$

We will prove the lemma with  $R_1 = R_0 + 2r$ . We therefore fix some  $x, y \in Y$  so that  $d_X(x, y) \geq R_1$ , and let  $A = \pi^{-1}(B_r^X(x))$  and  $B = \pi^{-1}(B_r^X(y))$ .

**Claim.** *For a fixed  $g \in G$ ,  $\#\{a \in A \mid ag \in B\} < \infty$ .*

Given the claim, we argue by contradiction. Suppose for some  $D > 0$ , the intersection  $N_D^G(A) \cap B$  is unbounded. There are thus infinitely many pairs of vertices  $(a_i, b_i)$  with  $a_i \in A$ ,  $b_i \in B$ , and  $d_G(a_i, b_i) \leq D$ . Thus all the elements  $a_i^{-1}b_i$  lie in  $B_D^G(1)$ , which is finite. Passing to a subsequence, the difference  $a_i^{-1}b_i$  is a constant element  $g$ . For this  $g$ , the set  $\{a \in A \mid ag \in B\}$  is infinite, contradicting the claim.

*Proof of Claim.* Let  $\Lambda = \{a \in A \mid ag \in B\}$ , and fix some  $a_0 \in \Lambda$ . We want to show  $\Lambda$  is finite.

For any  $a \in \Lambda$ , we have

$$d_X(x, aa_0^{-1}x) \leq d_X(x, ax_0) + d_X(ax_0, aa_0^{-1}x) = d_X(x, ax_0) + d_X(a_0x_0, x) \leq 2r.$$

Similarly we have

$$d_X(y, aa_0^{-1}y) \leq d_X(y, agx_0) + d_X(agx_0, aa_0^{-1}y) = d_X(y, agx_0) + d_X(a_0gx_0, y) \leq 2r.$$

Since  $d_X(x, y) \geq R_1 \geq R_0$  and  $x, y \in Y$ , the number of possibilities for  $aa_0^{-1}$  (and hence for  $a$ ) is finite.  $\square$

$\square$

The following lemma is adapted from [Sis16, Lemma 3.4] to our situation.

**Lemma 5.5.** *Let  $(G, X, H)$  be an A/QI triple, let  $x_0 \in X$ , and let  $\pi$  be the orbit map  $g \mapsto gx_0$ . For each  $r \geq 0$ , there is an  $R > 0$  and a function  $B: \mathbb{R}^+ \rightarrow \mathbb{R}^+$  so that for any  $x, y \in H$  with  $d_X(\pi(x), \pi(y)) \geq R$  we have, for any  $D \geq 0$ ,*

$$N_D^G(\pi^{-1}(B_r^X(x))) \cap \pi^{-1}(B_r^X(y)) \subseteq N_{B(D)}^G(H).$$

*Proof.* We are given an A/QI triple  $(G, X, H)$ , so in particular the action  $G \curvearrowright X$  is (weakly) acylindrical along  $\pi(H) = Hx_0$ . Choose  $M \geq 1$  so the orbit map  $\pi: G \rightarrow X$  is  $M$ -Lipschitz. Fix some  $r \geq 0$ .

By Lemma 5.4, there is an  $R = R_1(2r) \geq 0$  so that if  $h \in H$  satisfies  $d_X(x_0, \pi(h)) \geq R$ , then the sets  $\pi^{-1}(B_{2r}^X(x_0))$  and  $\pi^{-1}(B_{2r}^X(\pi(h)))$  are geometrically separated.

Now let  $D > 0$ . If  $d_X(x_0, \pi(h)) > MD + 2r$ , then the sets  $N_D^G(\pi^{-1}(B_r^X(x_0)))$  and  $\pi^{-1}(B_r^X(\pi(h)))$  are disjoint. Since  $\pi|_H$  is proper, the set

$$\Psi = \{h \in H \mid R \leq d_X(x_0, \pi(h)) \leq MD + 2r\}$$

is finite. For each  $h \in \Psi$ , Lemma 5.4 tells us that the diameter

$$d(h) = \text{diam } N_D^G(\pi^{-1}(B_{2r}^X(x_0))) \cap \pi^{-1}(B_{2r}^X(\pi(h)))$$

is finite. We let  $B(D) = \max\{d(h) \mid h \in \Psi\}$ , and note that this  $B(D)$  suffices for  $x = 1, y = h$ . We conclude by equivariance.  $\square$

Let  $\alpha$  be a tame  $\tau$ -quasi-geodesic edge-path in the Cayley graph of  $G$  with respect to  $T$ . The path  $\alpha$  passes through a sequence of group elements  $a_1, \dots, a_n$ . By  $\pi(\alpha)$ , we mean the discrete path  $\pi(a_1), \dots, \pi(a_n)$ . Given a discrete path  $\gamma = x_0, x_1, \dots, x_n$  in  $X$ , recall that we define the length of  $\gamma$  to be

$$\ell^0(\gamma) := \sum_{i=0}^{n-1} d_X(x_i, x_{i+1}).$$

For  $r > 0$ , if all the summands  $d_X(x_i, x_{i+1})$  are at most  $r$ , we say that  $\gamma$  is an  $r$ -coarse path. Since the generating set  $T$  of  $G$  is finite,  $\pi(\alpha)$  is an  $r$ -coarse path for  $r = \max\{d_X(x_0, tx_0) \mid t \in T\}$ .

The orbit map  $\pi$  is a Lipschitz map, and thus there exists a constant  $\kappa$  such that for any such  $\alpha$  we have  $\ell_G(\alpha) \geq \ell^0(\pi(\alpha))/\kappa$ .

Let  $\rho: X \rightarrow \pi(H)$  be a nearest point projection. Such a map exists because  $\pi(H)$  is a locally compact subspace of  $X$  by assumption. Since  $\pi(H)$  is quasi-convex, there is a  $\nu > 0$  so this map is  $\nu$ -coarsely Lipschitz; that is for any  $x, y \in X$ , we have  $d_X(\rho(x), \rho(y)) \leq d_X(x, y) + \nu$ .

**Remark 5.6.** The image of an  $r$ -coarse path under a  $\nu$ -coarsely Lipschitz map is an  $(r + 2\nu)$ -coarse path.

We fix the constants  $\kappa$  and  $\nu$  just defined for the remainder of the section.

We leave the (straightforward) proof of the following to the reader.

**Lemma 5.7.** *Let  $(Z, d_Z)$  be a hyperbolic metric space and let  $Y \subset Z$  be a quasi-convex subset so that  $d_Z$  restricts to a proper metric on  $Y$ . Let  $\rho_Y$  be a closest point projection to  $Y$ . Then there is a constant  $C$  so that if  $d_Z(\rho_Y(x), \rho_Y(y)) \geq C$ , then  $d_Z(x, y) \geq d_Z(x, Y)$ .*

By choosing geodesics in  $X$  between  $x_0$  and  $sx_0$  for each  $s \in S$ , we extend the map  $\pi|_H$  to an  $H$ -equivariant map

$$\hat{\pi}: \text{Cay}(H, S) \rightarrow X,$$

The image  $\Gamma = \hat{\pi}(\text{Cay}(H, S))$  is an  $H$ -cocompact, embedded, connected subset which contains the discrete set  $\pi(H)$ . Let  $\varkappa$  be the maximum distance between  $x_0$  and  $sx_0$  as  $s$  ranges over  $S$ . Fix  $\lambda \geq 1$  so that  $\hat{\pi}$  is a  $\lambda$ -quasi-isometric embedding.

The map  $\hat{\pi}$  and the constants  $\varkappa$  and  $\lambda$  will be fixed for the rest of the section.

For  $h \in H$  and  $0 \leq r_1 \leq r_2$ , we define the following subspaces of  $X$ , which we refer to as *annuli* and *spheres*, respectively:

$$A(h, r_1, r_2) = \Gamma \cap \left( B_{r_2}^X(hx_0) \setminus \overset{\circ}{B}_{r_1}^X(hx_0) \right)$$

and

$$S(h, r_1) = A(h, r_1, r_1).$$

Since  $\Gamma$  is connected,  $A(h, r_1, r_2), S(h, r_1) \neq \emptyset$  whenever  $r_1 > 0$ .

**Proposition 5.8** ([Sis16, Proposition 4.1]). *For any  $L \geq 0$  there exists a  $K \geq 0$  such that the following holds. For any  $h, h' \in H$  with  $d_G(h, h') > K$  and any  $0 < r_1 < r_2 < d_X(hx_0, h'x_0)$ , if  $\alpha$  is a path in  $G$  from  $h$  to  $h'$  and  $\alpha \cap N_K^G(\hat{\pi}^{-1}(A(h, r_1, r_2))) = \emptyset$ , then  $\ell_G(\alpha) \geq L(r_2 - r_1)$ .*

*Proof.* We may as well suppose  $L \geq 1$ .

By equivariance, we may assume without loss of generality that  $\alpha$  is a path in  $G$  from 1 to  $h$  for some  $h \in H$ , and we assume that

$$\alpha \cap N_K^G(\hat{\pi}^{-1}(A(1, r_1, r_2))) = \emptyset.$$

Let  $\rho: X \rightarrow \pi(H)$  be some closest point projection map. Then  $\rho$  is coarsely Lipschitz. Since  $\pi(\alpha)$  is an  $a_0$ -coarse path for some  $a_0$ , the projection of  $\pi(\alpha)$  to  $\pi(H)$  is an  $a$ -coarse path for some  $a$  depending only on  $\delta$ , the generating set  $T$ , and the quasi-convexity constant for  $\pi(H)$ . In particular, for any  $t \in [0, d_X(x_0, hx_0)]$ , there is some  $g$  on  $\alpha$  so that  $|d_X(x_0, \rho(\pi(g))) - t| \leq \frac{a}{2}$ .

Let  $D = C + 2a$ , where  $C$  is the constant from Lemma 5.7 applied with  $Y = \pi(H)$  and  $Z = X$ . Recall that  $\kappa$  is the Lipschitz constant of the orbit map  $\pi$ .

Fix  $C_2 \geq 4000LD\kappa$ . Also fix an integer  $R \geq 1$  so that Lemma 5.5 holds with  $r = C_2$  and let  $B = B(D)$  be as in the conclusion of that lemma. Let

$$B' = \lambda(B\kappa + C_2 + \varkappa/2) + \lambda^2$$

and let

$$K = \max\{1000C_2RL, B + B'\}.$$

Choose  $r_1 < r_2$  between 0 and  $d_X(x_0, hx_0)$ . There are two cases, depending on whether  $r_2 - r_1 \geq 1000C_2R$  or not. In the case  $r_2 - r_1 < 1000C_2R$ , we argue as follows. By assumption,  $d_G(1, h) > K$ , and so  $\ell_G(\alpha) > K$ . Since  $K \geq 1000C_2RL$ , it follows that  $\ell_G(\alpha) \geq 1000C_2RL > (r_2 - r_1)L$ , completing the proof in this case.

Now suppose  $r_2 - r_1 \geq 1000C_2R$ .

Let  $n = \lceil (r_2 - r_1)/(10C_2) \rceil$ . Let  $r_1 = d_0, d_1, \dots, d_n = r_2$  be evenly spaced real numbers, and notice that  $d_{i+1} - d_i \geq 5C_2$ . Consider the following neighborhoods of spheres

$$T_i = N_{C_2}^X(S(1, d_i)).$$

**Claim 1.** *If the set of indices  $i$  such that  $\pi(\alpha) \cap T_i = \emptyset$  is greater than  $(r_2 - r_1)/(1000C_2)$ , then  $\ell_G(\alpha) \geq L(r_2 - r_1)$ .*

*Proof of claim.* For each index  $i$  such that  $\pi(\alpha) \cap T_i = \emptyset$ , we construct a subpath  $\beta_i$  of  $\pi(\alpha)$  as follows. For each such  $i$ , let  $x_i$  be the last point on  $\pi(\alpha)$  so that  $d_X(x_0, \rho(x_i)) < d_i - C_2$ , and let  $y_i$  be the first point on  $\pi(\alpha)$  so that  $d_X(x_0, \rho(y_i)) > d_i + C_2$ . Let  $\beta_i = \pi(\alpha)|_{[x_i, y_i]}$ .

By the definition of the constant  $a$ , there are points  $q_1, \dots, q_m$  on  $\beta_i$  satisfying:

- (a)  $m \geq C_2/D - 1$
- (b)  $C \leq d_X(\rho(q_j), \rho(q_{j+1})) \leq C + 2a$
- (c)  $\rho(q_j) \in N_{C_2/2}^X(S(1, d_i))$ .

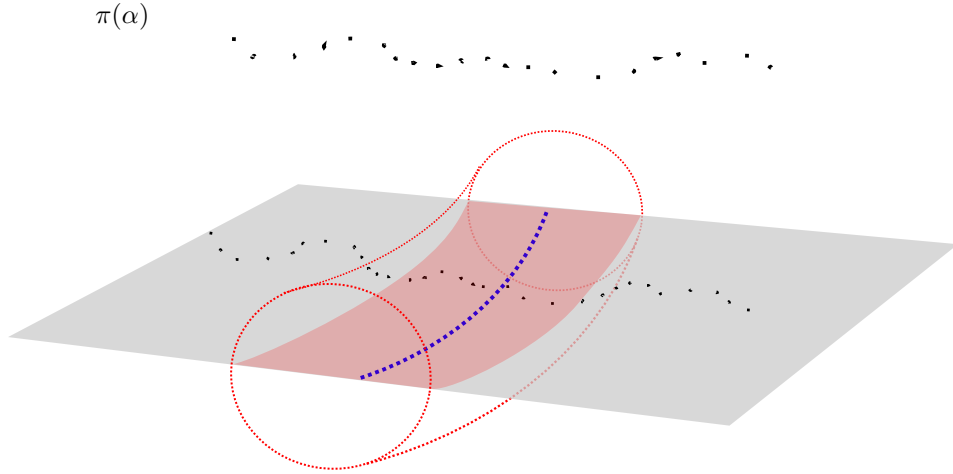


FIGURE 3. The points  $\rho(q_j)$  lie in the pink shaded region around the sphere  $S(1, d_i)$  (shown in blue).

It follows from (b) that  $d_X(q_j, q_{j+1}) \geq d_X(q_j, \pi(H))$ , and by (c) we have  $d_X(q_j, \pi(H)) \geq C_2/2$ . Combining this with (a), we conclude that

$$\ell^0(\beta_i) \geq \sum_{j=1}^m d_X(q_j, q_{j+1}) \geq \frac{C_2}{2} \left( \frac{C_2}{D} - 1 \right) \geq \frac{C_2^2}{4D} \geq 1000C_2L\kappa.$$

Moreover, we may choose these subpaths  $\beta_i$  to be disjoint for distinct  $i$ , since  $d_{i+1} - d_i \geq 5C_2 > 2C_2 + 2a$  for all  $0 \leq i \leq n - 1$ . In particular, if the set of  $i$  such that  $\pi(\alpha) \cap T_i = \emptyset$  is greater than  $(r_2 - r_1)/(1000C_2)$ , then

$$\ell^0(\pi(\alpha)) \geq \frac{r_2 - r_1}{1000C_2} \cdot 1000C_2L\kappa \geq L\kappa(r_2 - r_1).$$

Thus,  $\ell_G(\alpha) \geq L(r_2 - r_1)$ , as desired.  $\square$

Thus we may suppose that there are at least  $(r_2 - r_1)/(10C_2) - (r_2 - r_1)/(1000C_2) \geq (1 - 1/100)n$  indices  $i$  such that  $\pi(\alpha) \cap T_i \neq \emptyset$ . Subdivide the integers modulo  $R$ . Then there exists  $0 \leq s \leq R - 1$  such that the residue class  $\{s, R + s, 2R + s, \dots, \frac{n}{R} \cdot (R - 1) + s\}$  contains at least  $(1 - \frac{1}{100})\frac{n}{R}$  elements. A counting argument shows that there are at least  $(1 - \frac{2}{100})\frac{n}{R} \geq \frac{r_2 - r_1}{100C_2R}$  indices  $j$  such that  $\pi(\alpha) \cap T_{jR+s} \neq \emptyset$  and  $\pi(\alpha) \cap T_{(j+1)R+s} \neq \emptyset$ .

In such a case, pick points

$$p_j \in \pi(\alpha) \cap T_{jR+s} \quad \text{and} \quad p_{j+1} \in \pi(\alpha) \cap T_{(j+1)R+s}.$$

We may choose the pairs  $p_j, p_{j+1}$  so that the subpaths  $\pi(\alpha)|_{(p_j, p_{j+1})}$  are disjoint. Moreover, we can choose, for each such  $j$ , some

$$a_j \in \pi^{-1}(p_j) \cap \alpha \quad \text{and} \quad a_{j+1} \in \pi^{-1}(p_{j+1}) \cap \alpha,$$

so that for different  $j$  the subpaths  $\alpha|_{(a_j, a_{j+1})}$  are disjoint.

**Claim 2.**  $d_G(a_j, a_{j+1}) \geq 100C_2RL$ .

Assume the claim holds. Then as there are at least  $(r_2 - r_1)/(100C_2R)$  such pairs of points  $a_j, a_{j+1}$  and  $\ell_G(\alpha) \geq \sum_j d_G(a_j, a_{j+1})$ , we have  $\ell_G(\alpha) \geq L(r_2 - r_1)$ , as desired. Thus it remains to prove the claim.

*Proof of claim.* Suppose towards a contradiction that  $d_G(a_j, a_{j+1}) < 100C_2RL$ . Choose  $x \in H \cap \pi^{-1}(\rho(p_j))$  and  $y \in H \cap \pi^{-1}(\rho(p_{j+1}))$ . Then

$$d_X(\pi(x), \pi(y)) = d_X(\rho(p_j), \rho(p_{j+1})) \geq 3C_2R - 2C_2 \geq R.$$

Thus the assumptions of Lemma 5.5 are satisfied with this choice of  $x$  and  $y$ ,  $r = C_2$ , and  $D = 100C_2RL$ . Since

$$a_j \in \pi^{-1}(B_{C_2}^X(x)) \quad \text{and} \quad a_{j+1} \in \pi^{-1}(B_{C_2}^X(y)),$$

the conclusion of Lemma 5.5 shows that there exists  $h'' \in H$  with  $d_G(h'', a_j) \leq B = B(D)$ . Since the orbit map  $\pi$  is  $\kappa$ -Lipschitz, it follows that  $d_X(\pi(h''), p_j) \leq B\kappa$ . Recall that  $\varkappa$  is the maximum distance between  $x_0$  and  $sx_0$  as  $s$  ranges over  $S$ . By the triangle inequality

$$(11) \quad d_X(\pi(h''), \rho(p_j)) \leq B\kappa + C_2 + \varkappa/2,$$

as  $p_j \in N_{C_2}^X(\Gamma)$  and  $\rho$  is a nearest-point projection onto  $\pi(H)$ . Since  $\hat{\pi}$  is a  $\lambda$ -quasi-isometric embedding, (11) implies that

$$d_G(h'', \hat{\pi}^{-1}(\rho(p_j))) \leq \lambda(B\kappa + C_2 + \varkappa/2) + \lambda^2.$$

Therefore,

$$d_G(a_j, \hat{\pi}^{-1}(\rho(p_j))) \leq d_G(a_j, h'') + d_G(h'', \hat{\pi}^{-1}(\rho(p_j))) \leq B + B' \leq K.$$

Since  $\rho(p_j) \in A(1, r_1, r_2)$ , we have  $a_j \in N_K^G(\hat{\pi}^{-1}(A(1, r_1, r_2)))$ . As  $a_j \in \alpha$ , this contradicts our assumption that  $\alpha \cap N_K^G(\hat{\pi}^{-1}(A(1, r_1, r_2))) = \emptyset$ .  $\square$

$\square$

We now use Proposition 5.8 to prove Theorem 1.5.

*Proof of Theorem 1.5.* Since  $H$  quasi-isometrically embeds into the hyperbolic space  $X$ , it must be hyperbolic. By Tran's Theorem 5.3, it therefore suffices to show that  $H$  is strongly quasi-convex.

In order to show strong quasi-convexity it suffices to consider tame quasi-geodesics (see the discussion after Definition 2.3); in a graph, these tame quasi-geodesics can be taken to be edge-paths. So we assume  $\alpha$  is a tame  $\tau$ -quasi-geodesic edge-path in  $\text{Cay}(G, T)$  from  $h_1 \in H$  to  $h_2 \in H$ . We will find a neighborhood of  $H$  containing  $\alpha$  and only depending on the value of  $\tau$ . Recall that  $\hat{\pi}$  is a  $\lambda$ -quasi-isometric embedding.

Fix  $L = \lambda(\tau + 1)$ , and let  $K$  be as in Proposition 5.8 for this choice of  $L$ . Suppose that  $\alpha$  is not contained in  $N_K^G(H)$ . Let  $x, y$  be vertices of  $\alpha$  so that the subpath  $\alpha'$  of  $\alpha$  from  $x$  to  $y$  intersects  $N_K^G(H)$  only at its endpoints. Since  $\alpha$  is a tame  $\tau$ -quasi-geodesic, it follows that

$$(12) \quad \ell_G(\alpha') \leq \tau d_G(x, y) + \tau.$$

We will bound  $\ell_G(\alpha')$  by a constant  $M$  depending only on  $\lambda$  and  $\tau$ , which will imply that  $\alpha \subset N_{K+M}^G(H)$ , as desired. Appealing to (12), it suffices to bound  $d_G(x, y)$ . We claim:

$$(13) \quad d_G(x, y) \leq \max \{4\lambda^2 K + 3\lambda^2 + 2K, \tau + 4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K\}.$$

Consider the path  $\beta'$  obtained by concatenating (in the appropriate order)  $\alpha'$  with geodesics of length at most  $K$  connecting  $x$  and  $y$  to points  $h, h' \in H$ , respectively.

We divide into two cases, depending on whether or not

$$(14) \quad d_G(h, h') > 4\lambda^2 K + 3\lambda^2.$$

If (14) does *not* hold, then since  $x, y$  are distance at most  $K$  from  $h, h'$  respectively,

$$d_G(x, y) \leq 4\lambda^2 K + 3\lambda^2 + 2K,$$

establishing (13) in this case.

So we suppose that (14) holds. Set  $r_1 = 2\lambda K + \lambda$  and  $r_2 = d_X(\pi(h), \pi(h')) - (2\lambda K + \lambda)$ . From (14) we obtain  $r_1 < r_2$ , so  $A = A(h, r_1, r_2)$  is nonempty. Moreover,  $r_1$  and  $r_2$  are chosen so that  $d_G(\{h, h'\}, \hat{\pi}^{-1}(A)) \geq 2K$ . In particular the initial and terminal segments of  $\beta'$  connecting  $\alpha'$  to  $H$  lie outside  $N_K^G(\hat{\pi}^{-1}(A))$ . Since  $\hat{\pi}^{-1}(A)$  is contained entirely within  $\text{Cay}(H, S)$ , the path  $\alpha'$  also lies outside  $N_K^G(\hat{\pi}^{-1}(A))$ . Thus we have

$$\beta' \cap N_K^G(\hat{\pi}^{-1}(A(h, r_1, r_2))) = \emptyset.$$

Now Proposition 5.8 yields  $\ell_G(\beta') \geq L(r_1 - r_2)$ . We have

$$\begin{aligned} \ell_G(\alpha') &\geq L(r_2 - r_1) - 2K \\ &\geq L(d_X(\pi(h), \pi(h')) - (4\lambda K + 2\lambda)) - 2K \\ &\geq L(\lambda^{-1}d_G(h, h') - \lambda) - (4L\lambda K + 2L\lambda + 2K) \\ &\geq (\tau + 1)d_G(x, y) - (4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K). \end{aligned}$$

Now (12) tells us that  $\ell_G(\alpha')$  is at most  $\tau d_G(x, y) + \tau$ , so (subtracting  $\tau d_G(x, y)$  from both sides and rearranging),

$$d_G(x, y) \leq \tau + 4L\lambda K + 3L\lambda + 2K + 2L\lambda^{-1}K,$$

verifying (13) in this case.  $\square$

The work of Antolín–Mj–Sisto–Taylor [AMST19, Theorem 1.1] (cf. [Tra19, Theorem 1.2(3)]) gives the following corollary:

**Corollary 5.9.** *Let  $(G, X, H)$  be an  $A/QI$  triple, and suppose that  $G$  is finitely generated. Then  $H$  has finite height, finite width, and bounded packing.*

## 6. THE BOUNDARY OF A CONE-OFF

Let  $X$  be a graph, and let  $\mathcal{H}$  be a collection of subgraphs. The *cone-off* of  $X$  with respect to  $\mathcal{H}$  is the space  $\hat{X}_{\mathcal{H}}$  obtained from  $X$  by adding an edge between each pair of distinct  $x, y$ , so that  $\{x, y\} \subset Y^{(0)}$  for  $Y \in \mathcal{H}$ . These new edges are called *electric edges*. If there is no ambiguity about  $\mathcal{H}$ , we may use  $\hat{X}$  to denote the cone-off.

**Convention 6.1.** In this section we will restrict to continuous quasi-geodesics which are tame; see the discussion after Definition 2.3.

**Convention 6.2.** In this section, we consider two hyperbolic spaces,  $X$  and  $\hat{X} = \hat{X}_{\mathcal{H}}$ . Fix a basepoint  $x_0 \in X$ ; by definition,  $x_0 \in \hat{X}$ , as well. We will frequently calculate Gromov products in each space, and so to avoid confusion, we will use  $(\cdot | \cdot)_{x_0}$  to denote the Gromov product in  $X$  and  $\langle \cdot | \cdot \rangle_{x_0}$  to denote the Gromov product in  $\hat{X}$ .

There are various other possibilities for a definition of a cone-off of  $X$ . It is clear, for example, that if  $\hat{X}$  is the cone-off of  $X$  with respect to  $\mathcal{H}$ , then it is quasi-isometric to the space obtained by adjoining, for each  $Y \in \mathcal{H}$ , a new vertex  $v_Y$  connected by an edge to every vertex of  $Y$ . When  $X$  is the Cayley graph of a group, this construction is due to Farb [Far98].

In case the collection  $\mathcal{H}$  is *quasi-dense*, in the sense that every point is within bounded distance of some element of  $\mathcal{H}$ , there is another possibility, explored by Bowditch in [Bow12]. Namely, one can let  $\Gamma_{\mathcal{H}, R}$  be a graph with vertex set  $\mathcal{H}$ , and connect two vertices if the corresponding sets have distance at most  $R$ . For suitably chosen  $R$ , the space  $\Gamma_{\mathcal{H}, R}$  will be quasi-isometric to  $\hat{X}$ .

We will use work of Kapovich–Rafi, Bowditch, and Dowdall–Taylor to understand  $\partial\hat{X}$ . We start with a theorem of Bowditch (in the case  $\mathcal{H}$  is quasi-dense) and Kapovich–Rafi.

**Theorem 6.3** ([KR14, Proposition 2.6], [Bow12, 7.12]). *Let  $X$  be a Gromov hyperbolic graph, and let  $\mathcal{H}$  be a collection of uniformly quasi-convex subgraphs. Then the cone-off of  $X$  with respect to  $\mathcal{H}$  is Gromov hyperbolic. Moreover, geodesics in  $X$  can be reparametrized to be uniform quasi-geodesics in  $\hat{X}$ .*

Combining Theorem 6.3 with quasi-geodesic stability (Theorem 2.4) yields the following.

**Corollary 6.4.** *Let  $X$ ,  $\mathcal{H}$ ,  $\hat{X}$  be as above, and suppose that  $\gamma$  is a quasi-geodesic ray in  $X$ . Then the image of  $\gamma$  in  $\hat{X}$  is either bounded or has a unique limit point in  $\partial\hat{X}$ .*

Since the cone-off is hyperbolic, it is natural to ask about its Gromov boundary.

**Question 6.5.** *If  $\hat{X}$  is a cone-off of  $X$ , how is the Gromov boundary of  $\hat{X}$  related to that of  $X$ ?*

Note that neither space quasi-isometrically embeds in the other, so it is not immediately clear that there should be any relation. Dowdall and Taylor provide the following definition and theorem [DT17].

**Definition 6.6.** Let  $\partial_u X \subseteq \partial X$  consist of the points represented by quasi-geodesic rays whose projection to  $\hat{X}$  is unbounded.<sup>1</sup>

Theorem 6.3 implies that the projection of a quasi-geodesic ray is unbounded if and only if it is a quasi-geodesic ray (after possibly reparametrizing).

**Theorem 6.7** ([DT17, Theorem 3.2]). *The map  $X \rightarrow \hat{X}$  extends continuously to  $\partial_u X$ , and the restriction  $\psi: \partial_u X \rightarrow \partial \hat{X}$  is a homeomorphism.*

**Remark 6.8.** Dowdall–Taylor’s result is in the slightly more general setting of coarsely surjective *alignment preserving* maps. It follows from the “Moreover” statement in Kapovich–Rafi’s Theorem 6.3 that the cone-off map  $X \rightarrow \hat{X}$  is alignment preserving.

Let  $\text{Isom}(X, \mathcal{H})$  be the group of isometries of  $X$  which preserve the collection  $\mathcal{H}$ . The homeomorphism  $\psi$  from Theorem 6.7 is  $\text{Isom}(X, \mathcal{H})$ -equivariant.

Recall that given  $Y \in \mathcal{H}$ , we denote by  $\Lambda(Y) \subset \partial X$  the limit set of  $Y$  in  $\partial X$ . It is clear that  $\Lambda(Y)$  is in the complement of  $\partial_u X$ , so

$$(\ddagger) \quad \partial_u X \subseteq \partial X \setminus \left( \bigcup_{Y \in \mathcal{H}} \Lambda(Y) \right).$$

In general this can be a proper inclusion, as the following example shows.

**Example 6.9.** Let  $X$  be the Bass-Serre tree for  $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ , so that  $\partial X$  is a Cantor set. Then  $t$  acts loxodromically on  $X$  with axis  $\gamma$ . Let  $\mathcal{H}$  be the collection of translates of  $\gamma$ . The union  $\bigcup_{Y \in \mathcal{H}} \Lambda(Y)$  is a countable subset of  $\partial X$ , so  $\partial X \setminus \bigcup_{Y \in \mathcal{H}} \Lambda(Y)$  is nonempty. However the cone-off  $\hat{X}$  of  $X$  with respect to  $\mathcal{H}$  is bounded, so  $\partial \hat{X} = \emptyset$ .

We will show that if  $(G, X, H)$  is an A/QI triple, then the inclusion  $(\ddagger)$  is an equality.

**6.1. De-electrification.** We need to be able to “lift” paths from  $\hat{X}$  to  $X$  in a consistent way. There are several notions of this in the literature, usually called “de-electrification.” The following definition is from [Spr17], but it is related to definitions in Bowditch [Bow12] and Dahmani–Mj [DM17]. What we call *de-electrification* here is what Spriano calls *embedded de-electrification*.

**Definition 6.10.** Let  $\hat{X}$  be a cone-off of a graph  $X$  with respect to a family of uniformly quasi-isometrically embedded subgraphs  $\mathcal{H}$ . An  $\mathcal{H}$ -geodesic is a path in  $X$  which is completely contained in some  $Y \in \mathcal{H}$  and is geodesic in  $Y$ . Let  $\gamma = u_1 \cdot e_1 \cdot \dots \cdot e_n \cdot u_{n+1}$  be a concatenation of geodesics in  $\hat{X}$ , where each  $e_i$  is an electric edge and the  $u_i$  are (possibly trivial) segments of  $X$ . A *de-electrification* of  $\gamma$  is a concatenation  $u_1 \cdot \eta_1 \cdot \dots \cdot \eta_n \cdot u_{n+1}$  where each  $\eta_i$  is an  $\mathcal{H}$ -geodesic joining the endpoints of  $e_i$ .

<sup>1</sup>Dowdall and Taylor use the notation  $\partial_{\hat{X}} X$  in [DT17].

If the subgraphs in  $\mathcal{H}$  are not geometrically separated, then a de-electrification of a quasi-geodesic in  $\hat{X}$  can be arbitrarily far from being geodesic in  $X$ . In particular we cannot expect every de-electrification to be quasi-geodesic. However, Spriano shows that points of  $\hat{X}$  are always connected by quasi-geodesics with nice de-electrifications. Say that  $K$  is a *constant of quasi-isometric embeddedness* for  $\mathcal{H}$  if every element of  $\mathcal{H}$  is  $(K, K)$ -quasi-isometrically embedded, with  $K$ -quasi-convex image.

**Lemma 6.11** ([Spr17, Corollary 2.29]). *For any  $\delta \geq 0$ ,  $K \geq 1$ , there is a constant  $\omega \geq 1$  so that the following holds. Let  $X$  be a  $\delta$ -hyperbolic graph and  $\mathcal{H}$  a family of uniformly quasi-isometrically embedded subgraphs with constant of quasi-isometric embeddedness  $K$ . Let  $\hat{X}$  be the cone-off of  $X$  with respect to  $\mathcal{H}$ . Then for each pair of vertices  $x, y \in X$  there exists an  $\omega$ -quasi-geodesic  $\gamma$  of  $\hat{X}$  from  $x$  to  $y$  so that every de-electrification of  $\gamma$  is an  $\omega$ -quasi-geodesic.*

We fix the following data for the rest of the section:  $(G, X, H)$  is an A/QI triple where  $X$  is a  $\delta$ -hyperbolic graph containing a basepoint  $x_0$  and  $Y_0$  is an  $H$ -cocompact graph in  $X$  containing  $x_0$  with constant of quasi-isometric embeddedness  $K$ . The space  $\hat{X}$  is the cone-off of  $X$  with respect to the collection  $\mathcal{H}$  of  $G$ -translates of  $Y_0$ . The constant  $\omega$  is obtained by applying Lemma 6.11 to  $\delta$  and  $K$ .

## 6.2. Characterizing rays with bounded image and proof of Theorem 1.6.

**Lemma 6.12.** *Suppose  $\gamma$  is a quasi-geodesic ray in  $X$ . If  $\gamma$  is bounded in  $\hat{X}$  then  $[\gamma] \in \bigcup_{g \in G} \Lambda(gHg^{-1})$ .*

*Proof.* Fix  $\tau_0$  so that  $\gamma$  is a  $\tau_0$ -quasi-geodesic ray. We can suppose that  $\gamma$  begins at  $x_0$ . Choose a sequence  $\{x_i\}$  on  $\gamma$  tending to infinity in  $X$ . We will pass to subsequences several times but use the same notation for the subsequences.

For each  $i$ , let  $\sigma_i$  be the  $\omega$ -quasi-geodesic from  $x_0$  to  $x_i$  in  $\hat{X}$  provided by Lemma 6.11, and let  $\tilde{\sigma}_i$  be any de-electrification. Then  $\tilde{\sigma}_i$  is contained in an  $M$ -neighborhood of  $\gamma$ , where  $M$  is the Morse constant for parameters  $\max\{\tau_0, \omega\}, \delta$ .

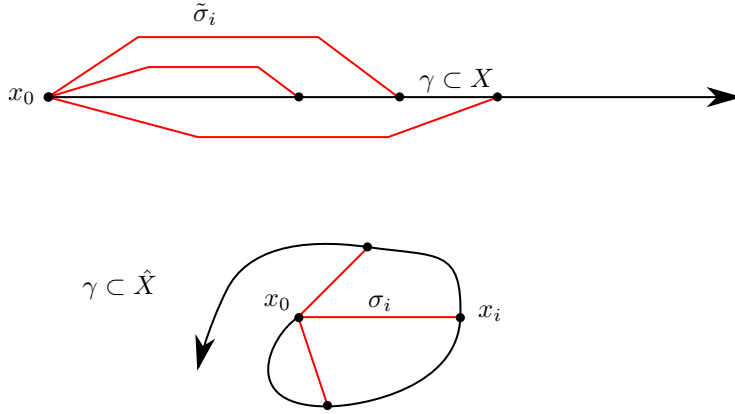


FIGURE 4. De-electrifications of the paths  $\sigma_i$  must have some subsegments whose lengths go to infinity.

Since  $\gamma$  is bounded in  $\hat{X}$ , there is some  $C \geq 0$  so that  $\ell_{\hat{X}}(\sigma_i) \leq C$  for all  $i$ . The quasi-geodesics  $\tilde{\sigma}_i$  are piecewise quasi-geodesics of the form  $\tilde{\sigma}_i = u_{i,1} \cdot \eta_{i,1} \cdot \dots \cdot \eta_{i,k_i} \cdot u_{i,k_i+1}$ , where the  $u_{i,j}$  are (possibly trivial) paths in  $X$ , each  $\eta_{i,j}$  is a geodesic in some  $Y \in \mathcal{H}$ , and  $k_i \leq C$  for all  $i$ . Since  $\ell_X(\tilde{\sigma}_i) \rightarrow \infty$ , some of the  $\eta_{i,j}$  must be unbounded.

Passing to a subsequence, we may assume that for a fixed  $j$ , the lengths  $\ell_X(\eta_{i,j})$  tend to infinity, but the lengths of the prefixes  $u_{i,1} \cdot \eta_{i,1} \cdot \dots \cdot u_{i,j}$  are uniformly bounded by a constant  $L$ . Set  $\varepsilon = 3M + 2L$ , and let  $D = D(\varepsilon, Y_0)$ ,  $N = N(\varepsilon, Y_0)$  be the constants of acylindricity of the action. Discarding the first few terms of the sequence, we may assume that the  $X$ -distance between the endpoints of  $\eta_{i,j}$  exceeds  $D + 2M$  for all  $i$ . Let  $\gamma_{i,j}$  be the maximal subpath of  $\gamma$  which is contained in the  $M$ -neighborhood of  $\eta_{i,j}$ . Then the length of  $\gamma_{i,j}$  is at least  $D$ . Moreover, for all  $i$ , the initial points of  $\gamma_{i,j}$  are at distance at most  $2L + 2M$ . Let  $y$  be a point on  $\gamma$  closest to  $x_0$  that is an endpoint of  $\gamma_{i,j}$  for some  $j$ , and let  $\gamma'$  be the subpath of  $\gamma$  beginning at  $y$  having length  $D$ .

Let  $\mathcal{Y} \subseteq \mathcal{H}$  be the subset of  $\mathcal{H}$  such that each  $\eta_{i,j} \subset Y$  for some  $Y \in \mathcal{Y}$ . Then  $\gamma' \subset \mathcal{N}_\varepsilon(Y)$  for each  $Y \in \mathcal{Y}$ .

Therefore we have

$$\text{diam} \left( \bigcap_{Y \in \mathcal{Y}} \mathcal{N}_\varepsilon(Y) \right) \geq D.$$

As  $\mathcal{Y}$  is a collection of translates of the  $H$ -cocompact subset  $Y_0$ , Theorem 3.2 implies that  $\mathcal{Y}$  can contain at most  $N$  distinct elements. In particular the paths  $\eta_{i,j}$  lie in a finite union  $\bigcup_{k=1}^N Y_k$  of elements of  $\mathcal{H}$ . The paths  $\eta_{i,j}$  fellow travel  $\gamma$  for longer and longer intervals, so the ray  $\gamma$  must limit to a point in  $\Lambda(\bigcup_{k=1}^N Y_k)$ . The limit set of a finite collection of quasi-convex sets is the union of their limit sets, so  $[\gamma] \in \Lambda(Y_k)$  for some fixed  $Y_k \in \mathcal{H}$ .  $\square$

For the readers convenience we restate the main theorem of this section.

**Theorem 1.6.** *Suppose  $(G, X, H)$  is an A/QI triple and fix  $x \in X$ . Let  $\mathcal{H} = \{gHx \mid gH \in G/H\}$ . Let  $\hat{X}$  be the cone-off of  $X$  with respect to  $\mathcal{H}$ . Then  $\partial\hat{X}$  is homeomorphic to the subspace of  $\partial X$  obtained by deleting the limit sets of all the conjugates of  $H$ .*

*Proof.* Dowdall–Taylor’s Theorem 6.7 tells us that  $\partial\hat{X}$  is homeomorphic to the subspace  $\partial_u X$  of  $\partial X$  consisting of points represented by quasi-geodesic rays whose projection to  $\hat{X}$  is unbounded. But Lemma 6.12 tells us that the only way for the image of a quasi-geodesic ray to be bounded is for it to limit to a point in  $\Lambda(gHg^{-1})$  for some  $g \in G$ .  $\square$

### 6.3. Corollaries.

**Corollary 6.13.** *Let  $(G, X, H)$  be an A/QI triple, let  $K \leq G$  be quasi-isometrically embedded by the action of  $G$  on  $X$ , and suppose  $\Lambda(K) \subseteq \partial X - \bigcup_{g \in G} \Lambda(gH)$ . Then  $K$  is quasi-isometrically embedded by the action of  $G$  on  $\hat{X}$ .*

The proof of this corollary will make use of a version of the Milnor–Schwartz Lemma which applies to quasi-geodesic spaces. A metric space is *quasi-geodesic* if there exists some  $\tau \geq 1$  so that any two points are joined by a  $\tau$ -quasi-geodesic.

**Lemma 6.14** ([Löh17, Proposition 5.4.1]). *If a group  $G$  acts coboundedly and metrically properly on a quasi-geodesic metric space  $Z$ , then  $G$  is finitely generated and for any  $z \in Z$ , the orbit map  $g \mapsto gz$  is a quasi-isometry.*

*Proof of Corollary 6.13.* We will apply the Milnor–Schwartz Lemma with  $Z = Kx_0 \subseteq \hat{X}$ , from which the result is immediate. It is clear that  $K$  acts cobounded on  $Kx_0$ , so it remains to show that  $(Kx_0, d_{\hat{X}})$  is quasi-geodesic and that  $K$  acts metrically properly.

By Theorem 6.3, geodesics in  $X$  can be reparametrized to be  $\tau$ -quasi-geodesics in  $\hat{X}$  for some fixed  $\tau \geq 1$ . Let  $M$  be the Morse constant for  $\tau$ -quasi-geodesics in  $\hat{X}$ , and let  $\lambda$  be the quasi-convexity constant of  $Kx_0$  in  $X$ .

We first show that  $(Kx_0, d_{\hat{X}})$  is a quasi-geodesic metric space. As the reader may verify, it suffices to show that  $Kx_0$  is quasi-convex in  $\hat{X}$ . Let  $x, y$  be points of  $Kx_0$ , and let  $\beta$  be a geodesic of  $\hat{X}$  joining them. Let  $z \in \beta$ , and let  $\alpha$  be an  $X$ -geodesic joining  $x$  to  $y$ . Since  $\alpha$  is (after reparametrizing) a  $\tau$ -quasi-geodesic in  $\hat{X}$ , there is a point  $z'$  of  $\alpha$  so that  $d_{\hat{X}}(z, z') \leq M$ . Since  $\alpha$  is a geodesic in  $X$ , there is a point  $z'' \in Kx_0$  so that  $d_X(z', z'') \leq \lambda$ . So

$$d_{\hat{X}}(z, Kx_0) \leq d_{\hat{X}}(z, z') + d_{\hat{X}}(z', z'') \leq d_{\hat{X}}(z, z') + d_X(z', z'') \leq M + \lambda,$$

and so  $Kx_0$  is  $(M + \lambda)$ -quasi-convex in  $\hat{X}$ .

To show that  $K$  acts metrically properly on  $Kx_0 \subseteq \hat{X}$ , let  $B \subset Kx_0$  be a finite diameter subset, and suppose toward a contradiction that there exists an infinite collection of elements  $k_i \in K$  such that  $k_i B \cap B \neq \emptyset$ . Up to passing to a subsequence, the points  $k_i x_0 \in X$  converge to a point  $\xi \in \Lambda(K) \subseteq \partial X$ . The set  $\{k_i x_0\}$  is bounded in  $\hat{X}$ , so the point  $\xi$  cannot persist in  $\partial \hat{X}$ . However, this contradicts Theorem 1.6, as  $\Lambda(K) \subseteq \partial X - \bigcup_{g \in G} \Lambda(gH)$ .  $\square$

We now use the description of the boundary of  $\hat{X}$  given in Theorem 1.6 to understand the action of  $G$  on  $\hat{X}$ .

If a group  $G$  acts by isometries on a Gromov hyperbolic metric space  $Y$  with basepoint  $y_0$ , then  $g \in G$  is *elliptic* if some (equivalently any) orbit in  $Y$  is bounded, *loxodromic* if the map  $\mathbb{Z} \rightarrow Y$  defined by  $n \mapsto g^n y_0$  is a quasi-isometric embedding, and *parabolic* if it is neither elliptic nor loxodromic. One may also use limit sets in  $\partial Y$  to distinguish between these types of isometries: an element  $g \in G$  is elliptic, parabolic, or loxodromic if  $\#\Lambda(\langle g \rangle) = 0, 1$ , or  $2$ , respectively. If  $\#\Lambda(\langle g \rangle) = 2$ , then we let  $\Lambda(\langle g \rangle) = \{g^{\pm\infty}\}$ . In this case, there is a quasi-geodesic axis in  $Y$  limiting to  $g^{\pm\infty}$ , and  $g$  acts as translation along this axis.

As  $X$  is a hyperbolic space, every element of  $G$  is either elliptic, loxodromic, or parabolic with respect to  $G \curvearrowright X$ . Since the canonical map from  $X$  to  $\hat{X}$  is 1-Lipschitz and  $G$ -equivariant, if  $g \in G$  is elliptic with respect to  $G \curvearrowright X$ , then  $g$  is also elliptic with respect to  $G \curvearrowright \hat{X}$ . It remains to consider elements of  $G$  that are parabolic and loxodromic with respect to  $G \curvearrowright X$ .

**Corollary 6.15.** *Suppose  $(G, X, H)$  is an  $A/QI$  triple, and let  $\mathcal{H}, \hat{X}$  be as in Theorem 1.6.*

- (1) *If  $g \in G$  acts loxodromically on  $X$ , then either it acts loxodromically on  $\hat{X}$  or a power of  $g$  stabilizes some  $Y \in \mathcal{H}$ , and so  $g$  acts elliptically on  $\hat{X}$ .*
- (2) *An element  $g \in G$  acts parabolically on  $X$  if and only if it acts parabolically on  $\hat{X}$ .*

*Proof.* Suppose that  $g$  acts loxodromically on  $X$ . Then  $\langle g \rangle$  is quasi-isometrically embedded by the action of  $G$  on  $X$ . If each of two fixed points of  $g$  in  $\partial X$  avoids  $\Lambda(Y)$  for all  $Y \in \mathcal{H}$ , then  $\langle g \rangle$  is quasi-isometrically embedded by the action of  $G$  on  $\hat{X}$  by Corollary 6.13, hence  $g$  acts loxodromically on  $\hat{X}$ .

Thus we may assume that at least one of the fixed points of  $g$  lies in the limit set of an element  $Y \in \mathcal{H}$ . Up to replacing  $g$  by  $g^{-1}$ , we can assume that  $\Lambda(Y)$  contains the repelling fixed point  $g^{-\infty} \in \partial X$ . The following claim implies statement (1) of the corollary.

**Claim 3.** *Some power of  $g$  stabilizes  $Y$ .*

*Proof.* Let  $y_0 \in Y$  be a closest vertex to  $\alpha_g(0)$ , and let  $\gamma_0$  be a quasi-geodesic ray lying in  $Y$  and limiting to  $g^{-\infty}$ . By quasi-geodesic stability, there is some  $B > 0$  so that  $d_X(\alpha_g(0), y_0) \leq B$  and the quasi-geodesic rays  $\gamma_0$  and  $\alpha_g|_{(-\infty, 0]}$  are Hausdorff distance at most  $B$  from one another.

Fix  $\varepsilon = B$ , and let  $D = D(\varepsilon, Y)$  and  $N = N(\varepsilon, Y)$  be the constants of acylindricity of the action.

For any integer  $i \geq 0$ , there is a  $T_i \geq 0$  so that  $\alpha_g(T_i) = g^i \alpha_g(0)$ , and we may assume the  $T_i$  tend monotonically to infinity. By equivariance,  $\alpha_g|_{(-\infty, T_i]}$  is Hausdorff distance at most  $B$  from  $\gamma_i = g^i \gamma_0 \subseteq g^i Y$ . All these rays tend to  $g^{-\infty}$  and lie in the  $B$ -neighborhood of  $\alpha_g$ .

In particular,  $\alpha_g|_{(-\infty, T_0]} \subset \mathcal{N}_B(\gamma_i) \subseteq \mathcal{N}_B(g^i Y)$  for each  $i \geq 0$ . Therefore,

$$\text{diam} \left( \bigcap_{i=1}^{\infty} \mathcal{N}_B(g^i Y) \right) > D.$$

By Theorem 3.2, the collection  $\{g^i Y\}$  contains at most  $N$  distinct elements.

This implies that there is some  $j' \neq j$  such that  $g^{j'} Y = g^j Y$ . Therefore,  $g^{j'-j}$  stabilizes the subgraph  $Y$ .  $\square$

We next deal with parabolic elements. If  $g \in G$  acts parabolically on  $\hat{X}$ , then it has unbounded orbits in  $X$ , so it cannot act elliptically on  $X$ . But it also cannot act loxodromically, by the first statement. The only remaining possibility is that it acts parabolically.

Conversely, suppose that  $g$  acts parabolically on  $X$ . Then  $g$  fixes a single point  $p \in \partial X$ .

**Claim 4.** *The fixed point  $p$  of  $g$  lies in  $\partial_u X$ .*

*Proof.* We argue by contradiction, supposing that  $p$  is not in  $\partial_u X$ . By Theorem 1.6, this means that  $p \in \Lambda(Y)$  for some  $Y \in \mathcal{H}$ . The proof proceeds similarly to the proof of Claim 3 with the following changes: we no longer have a quasi-axis  $\alpha_g$ , but we instead simply choose some  $y_0 \in Y$  and consider a  $K$ -quasi-geodesic ray  $\gamma_0$  in  $Y$  from  $y_0$  to  $p$ . For some infinite subset  $J \subseteq \mathbb{Z}_{\geq 0}$  and some fixed  $B > 0$  as before, there are numbers  $T_j$  tending monotonically to infinity so that for all  $t \geq T_j$ ,  $g^j \gamma_0(t) = \gamma_j(t)$  is contained in the  $B$ -neighborhood of  $\gamma_0$ .

Arguing as in the proof of Claim 3, we conclude that there is some  $j' \neq j$  such that  $g^{j'-j}$  stabilizes the subgraph  $Y$ . However, since  $Y$  is a uniformly locally finite graph, it cannot be stabilized by a parabolic isometry. Thus we reach a contradiction, and we conclude that  $p$  lies in  $\partial_u X$ .  $\square$

Let  $q = h(p)$  for some loxodromic element  $h \in H$ , and note that  $q$  is in  $\partial_u X$ . The points  $\{g^i q \mid q \in \mathbb{Z}\}$  accumulate on  $p$  in  $\partial \hat{X}$ . If  $g$  had a bounded orbit in  $\hat{X}$ , this would be impossible. (To see this, consider the image of a  $\tau_0$ -quasi-geodesic joining  $p$  to  $q$ .) Thus  $g$  acts parabolically on  $\hat{X}$ .  $\square$

There are several similar results in the literature for cone-offs, with varying conditions on  $X$  and the subgroup  $H$ , for example [Osi06, Theorem 1.14] and [ABO19, Proposition 6.5], which both conclude that if  $g$  is loxodromic on  $X$  then it is either loxodromic on the cone-off or conjugate into  $H$ . In both of these results, there is some hypothesis of properness, which allows the slightly stronger conclusion.

We now consider actions  $G \curvearrowright X$  which contain loxodromic WPD elements.

**Definition 6.16.** Let  $G$  act on a hyperbolic space  $X$ , and let  $g \in G$  act as a loxodromic isometry of  $X$ . We say  $g$  is *WPD element of  $G$  (with respect to  $X$ )* if for every  $x \in X$  and every  $\varepsilon \geq 0$ , there exists  $N > 0$  such that the set

$$\{\gamma \in G \mid d(x, \gamma x) \leq \varepsilon \text{ and } d_X(g^N x, \gamma g^N x) \leq \varepsilon\}$$

is finite. The action of  $G$  on  $X$  is a *WPD action* if  $G$  is not virtually cyclic,  $G$  contains at least one element which is loxodromic with respect to the action on  $X$ , and every loxodromic element is a WPD element.

We show that if such an element is also loxodromic with respect to  $G \curvearrowright \hat{X}$ , then it is also still WPD with respect to the action on the cone-off. The proof will use the notion of a WWP element, introduced by Bestvina–Bromberg–Fujiwara in [BBF16]. The definition we use here is due to Handel–Mosher in [HM19b]: a loxodromic element  $g \in G$  is *WWPD* if the  $G$ -orbit of  $(g^\infty, g^{-\infty})$  is a discrete subset of  $\partial X \times \partial X \setminus \Delta$ , where  $\Delta$  is the diagonal subset. A loxodromic element is WPD if and only if it is WWP and its centralizer in  $G$  is virtually cyclic [HM19b, Corollary 2.4].

The following proposition is related to [MMS20, Theorem 2.4], but has slightly different hypotheses.

**Proposition 6.17.** *Let  $(G, X, H)$  be an A/QI triple. If  $g \in G$  is a loxodromic WPD element with respect to the action of  $G$  on  $X$  and no power of  $g$  stabilizes any  $Y \in \mathcal{H}$ , then  $g$  is a loxodromic WPD element with respect to the action on  $\hat{X}$ .*

*Proof.* By Corollary 6.15, the element  $g$  acts loxodromically on  $\hat{X}$  and the fixed points of  $g$  in  $\partial X$  lie in  $\partial_u X$ . Since  $g$  is WWP, the  $G$ -orbit of  $(g^\infty, g^{-\infty})$  is a discrete subset of  $\partial X \times \partial X \setminus \Delta$ . This orbit is therefore discrete in  $\partial \hat{X} \times \partial \hat{X} \setminus \Delta \cong \partial_u X \times \partial_u X \setminus \Delta$ . Therefore  $g$  is a WWP element with respect to the action of  $G$  on  $\hat{X}$ . Since  $g$  is WPD on  $X$ , its centralizer is virtually cyclic, and so  $g$  is WPD with respect to the action on  $\hat{X}$ , as desired.  $\square$

**Corollary 6.18.** *Let  $G, X, \mathcal{H}$ , and  $\hat{X}$  be as in Theorem 1.6, and suppose additionally that  $G \curvearrowright X$  is a WPD action. If there is a loxodromic isometry in the action of  $G$  on  $\hat{X}$ , then  $G \curvearrowright \hat{X}$  is a WPD action.*

## 7. EXAMPLES AND QUESTIONS

In this section, we use the combination theorem presented in Section 4 to give examples of A/QI triples in  $\text{Out}(F_n)$ , the Cremona group, and FC-type Artin groups (see Examples 7.1, 7.2, and 7.3). All of these groups are acylindrically hyperbolic,

but while we have natural hyperbolic spaces on which each group acts with loxodromic WPD elements, these actions are not known to be acylindrical (and in several cases are known not to be acylindrical). Constructing A/QI triples in these situations provides subspaces along which the action of the group is acylindrical.

We first consider the group  $\text{Out}(F_n)$ , which acts by isometries on several hyperbolic complexes, including the free factor complex, the cyclic splitting complex, and the free splitting complex. We refer the reader to [HM19a, KR14, Man14] for details on these complexes and the actions. Fully irreducible elements are loxodromic WPD elements with respect to the action on all three complexes. These are the only loxodromic isometries of the action on the free factor complex, and so this is a WPD action. However, it is not known if this action is acylindrical. It is also not known if the action on the cyclic splitting complex is WPD or acylindrical. On the other hand, Handel and Mosher [HM19a] show that there are loxodromic isometries of the free splitting complex which are not WPD elements, and so this action is neither WPD nor acylindrical. However, in all cases, we are able to construct A/QI triples using Corollary 4.5. As noted in Remark 4.6, the subgroups constructed will typically *not* be hyperbolically embedded.

**Example 7.1** ( $\text{Out}(F_n)$ ). Let  $G = \text{Out}(F_n)$ , let  $h_1, \dots, h_k \in G$  be independent fully irreducible elements, and let  $X$  be the free factor complex, the cyclic splitting complex, or the free splitting complex. Then  $h_1, \dots, h_k$  are independent loxodromic WPD elements with respect to the action on  $X$ . By Corollary 4.5, there is a constant  $D$  such that for any  $d_1, \dots, d_m \geq D$ , if  $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$ , then  $(G, X, H)$  is an A/QI triple.

A subgroup of  $\text{Out}(F_n)$  is called *convex cocompact* if it is quasi-isometrically embedded in the free factor complex via the orbit map. Thus all the subgroups in Example 7.1 are convex cocompact. These subgroups are known to be stable by [ADT17].

We now turn our attention to the Cremona group.

**Example 7.2** (Cremona group). The 2-dimensional Cremona group  $\text{Bir}(\mathbb{P}_k^2)$  over an algebraically closed field  $k$  is the group of birational transformations of the projective space  $\mathbb{P}_k^2$ . We note that  $\text{Bir}(\mathbb{P}_k^2)$  is not a finitely generated group; in fact, it is uncountable. In [CL13], Cantat and Lamy construct an action of  $\text{Bir}(\mathbb{P}_k^2)$  on an infinite-dimensional hyperbolic space  $\mathbb{H}_{\overline{\mathbb{Z}}}$ , which is inspired by work of Manin [Man86] and Zariski [Zar58]. Cantat and Lamy also introduce the notion of a *tight* loxodromic isometry of  $\mathbb{H}_{\overline{\mathbb{Z}}}$ ; see [CL13, Section 2.3.3]. An element  $g \in \text{Bir}(\mathbb{P}_k^2)$  is WPD if and only if some positive iterate  $g^m$  of  $g$  is a tight loxodromic isometry [Can18, Remark 7.5]. (Lonjou also constructs WPD elements in the case that  $k$  is not algebraically closed [Lon16].)

Let  $G = \text{Bir}(\mathbb{P}_k^2)$ , and let  $h_1, \dots, h_n \in G$  be independent tight elements. Then  $h_1, \dots, h_n$  are independent loxodromic WPD isometries for the action on  $\mathbb{H}_{\overline{\mathbb{Z}}}$ . By Corollary 4.5, there is a constant  $D$  such that for any  $d_1, \dots, d_n \geq D$  and  $H = \langle h_1^{d_1}, \dots, h_n^{d_n} \rangle$ , we have that  $(G, \mathbb{H}_{\overline{\mathbb{Z}}}, H)$  is an A/QI triple. Since  $G$  is not finitely generated, there is not a notion of a stable subgroup, but Corollary 4.8 implies that  $H$  has finite height.

For our final example, which is more involved than the previous examples, we consider FC-type Artin groups.

**Example 7.3** (FC-type Artin groups). Let  $\Gamma$  be a finite graph with vertex set  $V(\Gamma)$  and edge set  $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$ . Suppose that each edge  $(s, t) \in E(\Gamma)$  is labeled with a positive integer  $m(s, t)$ . The *Artin group*  $A(\Gamma)$  associated to the labeled graph  $\Gamma$  is defined by

$$A_\Gamma = \left\langle V(\Gamma) \mid \underbrace{st \cdots}_{m(s,t)} = \underbrace{ts \cdots}_{m(s,t)} \text{ for all } (s, t) \in E(\Gamma) \right\rangle.$$

Let  $S = V(\Gamma)$ . Given a collection of vertices  $T \subset S$ , the subgroup  $A_T$  of  $A_\Gamma$  generated by  $T$  is isomorphic to the Artin group associated to the subgraph of  $\Gamma$  spanned by  $T$ . An Artin group is *spherical* if the associated Coxeter group (which is formed by adding the relation  $s^2 = 1$  for all  $s \in S$ ) is finite. An important larger class of Artin group are the FC-type Artin groups, which were first introduced by Charney–Davis [CD95]. An Artin group is *FC-type* if  $A_T$  is spherical if and only if  $T$  is a complete subgraph.

An Artin group  $A_\Gamma$  acts by isometries on its *Deligne complex*  $\mathcal{D}(A_\Gamma)$ . The Deligne complex (called the *modified Deligne complex* by Charney–Davis) is a cubical complex whose vertices are cosets  $aA_T$  where  $A_T$  is spherical. Whenever  $T \subset T'$  and  $\#(T' - T) = 1$ , an edge is added between the cosets  $aA_T$  and  $aA_{T'}$ ; higher-dimensional cubes are filled in as their 1-skeleta appear. Charney–Davis show in [CD95] that the Deligne complex  $\mathcal{D}(A_\Gamma)$  is CAT(0) exactly when  $A_\Gamma$  is FC-type.

Chatterji–Martin show that FC-type Artin groups are acylindrically hyperbolic by applying the following general result about groups acting on CAT(0) cube complexes, whose proof relies on [Mar16, Theorem 1.2]. We refer the reader to [CM19] for the definitions of the terms appearing in theorem. The action of  $A_\Gamma \curvearrowright \mathcal{D}(A_\Gamma)$  satisfies the conditions in the first sentence by [CM19, Proposition 5.1].

**Theorem 7.4** ([CM19, Theorem 1.1]). *Let  $G$  be a group acting essentially and non-elementarily on an irreducible finite-dimensional CAT(0) cube complex. If there exist two hyperplanes whose stabilizers intersect along a finite subgroup, then  $G$  admits an acylindrical action on a hyperbolic space.*

The proof of this theorem has two steps. The first is to show that there exists a loxodromic WPD isometry  $g \in G$  with respect to the action on the CAT(0) cube complex whose axis in  $G$  is strongly contracting. Since the CAT(0) cube complex is not required to be hyperbolic, this is not sufficient to prove that  $G$  is acylindrically hyperbolic. The second step is to apply the projection complex machinery of Bestvina–Bromberg–Fujiwara from [BBF15] to conclude that  $g$  is a loxodromic WPD isometry in an action of  $G$  on a hyperbolic space. Osin shows that this is equivalent to  $G$  acting acylindrically on a (possibly different) hyperbolic space [Osi16, Theorem 1.4]. The projection complex is not the only hyperbolic space that could be used as the input for Osin’s theorem. Genevois constructs a family of hyperbolic spaces on which  $G$  acts by isometries, and the element  $g$  will be a loxodromic WPD isometry of infinitely many of these spaces [Gen19]. (Genevois’s construction generalizes Hagen’s construction of the contact graph [Hag14].)

A graph *has no empty squares* if every circuit of length four has at least one diagonal pair of vertices spanning an edge. Charney–Crisp [CC07, Theorem 4.2] show that if  $A_\Gamma$  is an FC-type Artin group and  $\Gamma$  has no empty squares, then  $\mathcal{D}(A_\Gamma)$  is actually CAT(−1); in particular, it is hyperbolic. Thus to prove Theorem 7.4

in the special case of an FC-type Artin group whose defining graph has no empty squares, the Deligne complex can be used directly as the input to Osin's theorem.

In this setting, the following statement can be extracted from the proof of [CM19, Theorem 1.1]. Given disjoint hyperplanes  $\hat{H}_1, \hat{H}_2$  in an irreducible CAT(0) cube complex  $X$ , an automorphism  $g$  of  $X$  *double-skewers*  $\hat{H}_1$  and  $\hat{H}_2$  if there is a choice of corresponding nested half spaces  $H_1 \subset H_2$  such that  $H_1 \subset H_2 \subset gH_1$ .

**Proposition 7.5** ([CM19, Theorem 1.1]). *Let  $A_\Gamma$  be an Artin group of FC type whose defining graph has no empty squares. If there exist two disjoint hyperplanes in the Deligne complex  $\mathcal{D}(A_\Gamma)$  whose stabilizers intersect along a finite subgroup, then any element which double-skewers these hyperplanes is a loxodromic WPD element with respect to the action of  $A_\Gamma$  on  $\mathcal{D}(A_\Gamma)$ .*

Chatterji–Martin use the Sector Lemma [CS11, Lemma 5.2] and the Double-Skewering Lemma [CS11, Double-Skewering Lemma] to show that such elements must always exist. However, the proof does not explicitly construct such elements. We now explicitly construct a family of examples.

The construction involves understanding the intersection pattern of hyperplanes in the Deligne complex. As explained in [MW19, Remark 2.2], every hyperplane in  $\mathcal{D}(A_\Gamma)$  is a translate of a hyperplane that crosses an edge  $(A_\emptyset, A_v)$  for some vertex  $v$  in  $\Gamma$ . Such a hyperplane is called a *hyperplane of type  $v$* . The action of  $A_\Gamma$  on the Deligne complex preserves hyperplane types, and a hyperplane of type  $v$  intersects some hyperplane of type  $w$  exactly when  $w$  is in the link of  $v$  in  $\Gamma$ .

**Lemma 7.6.** *Let  $A_\Gamma$  be an FC-type Artin group whose defining graph  $\Gamma$  has diameter at least 3 and no empty squares. For any two vertices  $s, t \in V(\Gamma)$  at distance at least 3 in  $\Gamma$  and any integers  $m, n \in \mathbb{Z} \setminus \{0\}$ , the product  $s^m t^n$  is a loxodromic WPD element with respect to the action of  $A_\Gamma$  on the Deligne complex. Moreover, if  $m', n' \in \mathbb{Z} \setminus \{0\}$  and  $(m', n') \neq (m, n)$ , then the elements  $s^{m'} t^{n'}$  and  $s^m t^n$  are independent.*

*Proof.* If  $\Gamma$  is not connected, then  $A_\Gamma$  splits as a free product and the Deligne complex  $\mathcal{D}(A_\Gamma)$  has the structure of a tree of spaces. If  $s, t$  are in different connected components of  $\Gamma$ , the result is clear. Thus we may assume that  $\Gamma$  is connected.

Let  $\hat{H}_s, \hat{H}_t$  be the hyperplanes of  $\mathcal{D}(A_\Gamma)$  labeled by  $s$  and  $t$ , respectively. As noted in the proof of [CM19, Theorem 1.2], since  $s$  and  $t$  are at distance at least 3 in  $\Gamma$ , we have  $\text{Stab}(\hat{H}_s) \cap \text{Stab}(\hat{H}_t) = \emptyset$ . Moreover,  $s^m t^n$  is a loxodromic isometry of  $\mathcal{D}(A_\Gamma)$  which has a geodesic axis formed by the union of the images of the segment shown in Figure 5 under  $\langle s^m t^n \rangle$  by [CM19, Lemma 5.5]. (Strictly speaking, [CM19, Lemma 5.5] only applies to the element  $st$ , but the same proof will give the result for  $s^m t^n$ ; this is explicitly stated for the element  $s^2 t$  in the proof of [CM19, Lemma 5.4].)

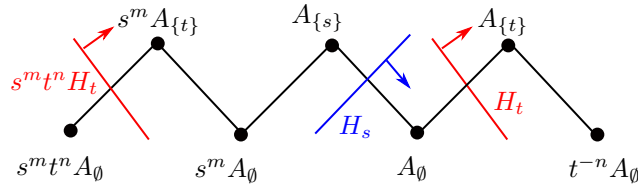


FIGURE 5. A geodesic axis of  $s^m t^n$  in  $\mathcal{D}(A_\Gamma)$ , with the relevant half spaces.

Fix  $m, n \in \mathbb{Z} \setminus \{0\}$ . Since  $t$  is not contained in the link of  $s$ , we have  $\hat{H}_s \cap \hat{H}_t = \emptyset$ . Thus there exist half spaces  $H_s$  and  $H_t$  satisfying  $H_t \subset H_s$  (see Figure 5). We will show that  $s^m t^n$  double skewers  $H_s$  and  $H_t$ , that is, we will show that  $H_t \subset H_s \subset s^m t^n H_t$ . First, notice that since  $s$  is not contained in the link of  $t$ , the hyperplanes  $\hat{H}_t$  and  $s^m t^n \hat{H}_s$  are disjoint. Thus  $H_s \subset s^m t^n H_t$  or  $s^m t^n H_t \subset H_s$ . If  $s^m t^n H_t \subset H_s$ , then  $\hat{H}_s$  must intersect the path  $A_{\{s\}}, s^m A_\emptyset, s^m A_{\{t\}}$  exactly once. However, since  $\hat{H}_s$  is of type  $s$ , it does not cross the edge  $(s^m A_\emptyset, s^m A_{\{t\}})$ . Moreover, since hyperplanes in a CAT(0) cube complex do not self-osculate and  $\hat{H}_s$  crosses the edge  $(A_\emptyset, A_{\{s\}})$ , we see that  $\hat{H}_s$  cannot cross  $(A_{\{s\}}, s^m A_\emptyset)$ . Thus  $H_s \subset s^m t^n H_t$ , and we conclude that  $s^m t^n$  double-skewers  $H_s$  and  $H_t$ .

Therefore  $s^m t^n$  is a loxodromic WPD element with respect to the action of  $A_\Gamma$  on the Deligne complex by [CM19, Theorem 1.1].

The proof of [CM19, Lemma 5.4] shows that  $st$  and  $s^2 t$  are independent, but the proof goes through as written for elements  $s^{m'} t^{n'}$  and  $s^m t^n$  whenever  $(m', n') \neq (m, n)$ .  $\square$

Let  $A_\Gamma$  be an FC-type Artin group whose defining graph  $\Gamma$  has diameter at least 3 and no empty squares. Let  $s, t \in V(\Gamma)$  be at distance at least 3, and fix distinct elements  $h_1, \dots, h_k \in \{s^m t^n \mid m, n \in \mathbb{Z} \setminus \{0\}\}$ . By Corollary 4.5 there is a constant  $D$  such that for any  $d_1, \dots, d_k \geq D$  and  $H = \langle h_1^{d_1}, \dots, h_k^{d_k} \rangle$ , we have that  $(A_\Gamma, \mathcal{D}(A_\Gamma), H)$  is an A/QI triple. In particular, the subgroup  $H$  is stable.

**7.1. Questions.** The first question we pose asks for a generalization of Proposition 4.1.

**Question 7.7.** *Suppose  $(G, X, H)$  and  $(G, Y, K)$  are A/QI triples and  $X \neq Y$ . Does there exist a hyperbolic space  $Z$  such that  $(G, Z, H \cap K)$  is an A/QI triple?*

The next question asks if there is a more general combination theorem than the one presented in Theorem 1.4.

**Question 7.8.** *Suppose  $(G, X, H)$  and  $(G, X, K)$  are A/QI triples and  $H_0 \leq H$  and  $K_0 \leq K$  are quasi-convex subgroups such that every element of  $H_0 \setminus K$  and every element of  $K_0 \setminus H$  translates the basepoint of  $X$  a large distance. Let  $P = \langle H_0, K_0 \rangle$ . Is  $(G, X, P)$  an A/QI triple?*

Corollary 6.18 shows that if  $(G, X, H)$  is an A/QI triple and  $G \curvearrowright X$  is WPD, then  $G \curvearrowright \hat{X}$  is also WPD. Since an acylindrical action is a WPD action, we can conclude that if  $G \curvearrowright X$  is acylindrical, then  $G \curvearrowright \hat{X}$  is WPD. This motivates the following question.

**Question 7.9.** *Suppose that  $(G, X, H)$  is an A/QI triple where  $G \curvearrowright X$  is actually acylindrical. Form the cone-off  $\hat{X}$  as in Section 6. Is the action  $G \curvearrowright \hat{X}$  also acylindrical?*

Example 7.1 shows that many known convex cocompact subgroups of  $\text{Out}(F_n)$  give A/QI triples. Moreover, every convex cocompact subgroup of  $\text{Out}(F_n)$  quasi-isometrically embeds in the free factor complex.

**Question 7.10.** *Does  $\text{Out}(F_n)$  always act acylindrically along the image of a convex cocompact subgroup in the free factor complex?*

As well as finite height, Antolín–Mj–Sisto–Taylor show finite width and bounded packing for stable subgroups [AMST19]. We would like to know if these properties hold for  $H$  in an A/QI triple  $(G, X, H)$  even when  $G$  is not finitely generated.

**Question 7.11.** *When  $G$  is not finitely generated, but there is an A/QI triple  $(G, X, H)$ , does  $H$  have finite width? If  $G$  is countable, does  $H$  have bounded packing?*

Part of the usefulness of finite height in context of quasi-convex subgroups of hyperbolic groups is that one can pass to an *almost malnormal core* of a finite collection of quasi-convex groups. This core is a collection of quasi-convex subgroups of height 1, so the ambient group is hyperbolic relative to this collection, and one can attempt to run various inductive arguments on height using Dehn filling (as for example in [AGM09] and the appendix to [Ago13]). Rather than aiming to produce a relatively hyperbolic structure on  $G$ , one might hope to produce a hyperbolically embedded collection of subgroups. Such a collection would necessarily have height one. As we have mentioned, the  $H$  in an A/QI triple  $(G, X, H)$  is not necessarily hyperbolically embedded as its height may be strictly bigger than one. In order to pass to a well-behaved almost malnormal core, one would like to know the answer to the following question.

**Question 7.12.** *Consider the collection of infinite subgroups of  $H$  of the form  $\bigcap_{i=1}^k g_i H g_i^{-1}$ . Do these subgroups fall in finitely many  $H$ -conjugacy classes? Do the minimal such subgroups give rise to a hyperbolically embedded collection in  $G$  (after possibly replacing each by a finite index supergroup and choosing one per  $G$ -conjugacy class)?*

Using the machinery of hierarchically hyperbolic groups, Spriano in [Spr17] proves Theorem 1.6 in the special case where  $\mathcal{H}$  is a collection of infinite quasi-convex subspaces which can be extended to form a *weak factor system*. Indeed Spriano gives finer information about the boundary in this case, decomposing it into the boundaries of the various elements of the factor system, one of which is  $\partial\hat{X}$ . When  $X$  is the Cayley graph of a hyperbolic group and  $\mathcal{H}$  is the set of cosets of a finite collection of infinite quasi-convex subgroups, Spriano shows that  $\mathcal{H}$  extends to a weak factor system [Spr17, Section 5.2]. Spriano’s argument seems to strongly use properness. In general,  $X$  may not be a proper metric space, and thus we ask the following question:

**Question 7.13.** *Under the assumptions of Theorem 1.6, can the set of translates  $gHx$  can be extended to form a weak factor system?*

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