MATH 2930 - Extra Credit 2

Wednesday, February 10, 2010
7:07 PM

3, 5: Courtesy of Anthony Basile

3.) $y' + \frac{3}{x}y = x^3 \qquad P(x) = \frac{3}{x}, \; Q(x) = x^3$

$p(x) = e^{\int P(x)\,dx}$

$= e^{\int \frac{3}{x}} = x^3$

$x^3y' + x^3\left(\frac{3}{x}\right)y = x^6$

$x^3y' + 3x^2y = x^6$

$\int (x^3y)' = \int x^6$

$x^3y = \frac{1}{7}x^7 + C$

$\boxed{y = \frac{1}{7}x^4 + C}$

Check

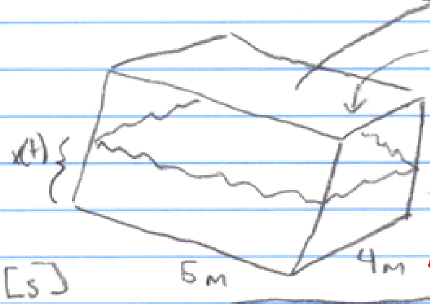
$\frac{4}{7}x^3 + \frac{3}{x}\left(\frac{1}{7}x^4\right) = x^3$

$\frac{4}{7}x^3 + \frac{3}{7}x^3 = x^3$

$x^3 = x^3$ ✓

Recognize that this is a linear 1st-order ODE

5.)



Pump H_2O out $.09 + .03\cos(\omega t)$ $\left[\frac{m^3}{s}\right]$

Pump H_2O in $.15\,[m^3/s]$

period = 2 hrs

$2\,\text{hrs}\left(\frac{60\,\text{min}}{1\,\text{hr}}\right)\left(\frac{60\,s}{1\,\text{min}}\right) = 7200$

$7200 = \frac{2\pi}{\omega}$

Those of you who took physics: ω = "angular frequency" $\rightarrow$ $\boxed{\omega = \frac{2\pi}{7200}}$

$$\boxed{\frac{dx}{dt} = \frac{.15 - \left[.09 + .03\cos\left(\frac{2\pi t}{7200}\right)\right]}{20}}$$

8: Courtesy of George Lu & Pam Snodgrass

8. $y' = xy^3 - xy \rightarrow y' = x(y^3 - y) \rightarrow \frac{1}{y^3 - y}\,dy = x\,dx \rightarrow$ ← Use separation of var's

$\frac{1}{y(y^2-1)} = \frac{1}{y(y-1)(y+1)} = \frac{A}{y} + \frac{B}{y-1} + \frac{C}{y+1}$

$1 = A(y^2-1) + B(y^2+y) + C(y^2-1)$

$1 = Ay^2 - A + By^2 + By + Cy^2 - Cy$

$y^2: A + B + C = 0 \rightarrow B + C = 1$

$y: B - C = 0 \rightarrow B = C$

$C: -A = 1 \rightarrow A = -1$

$\Big\}\; B = 1/2, \; C = 1/2$

$\int -\frac{1}{y} + \frac{1/2}{y-1} + \frac{1/2}{y+1}\,dy = x\,dx \rightarrow -\ln y + 1/2\ln(y-1) + 1/2\ln(y+1) = 1/2\,x^2 + C \rightarrow$

$\ln\frac{\sqrt{y-1}\sqrt{y+1}}{y} = \frac{x^2}{2} + C \rightarrow \boxed{\frac{\sqrt{y-1}\sqrt{y+1}}{y} = Ce^{x^2/2}}$

Reduce $\Rightarrow$ $y(x) = \pm\sqrt{\frac{1}{1 - C'e^{x^2}}}$

(where $C' = C^2$)

9, 10, 11, 12: Courtesy of Caroline Chang & Mahina Wang

9. $y' = -\frac{3x^2+y^2}{4xy}$ homogeneous $v: \frac{y}{x}$, $y = vx$ $\quad \frac{1}{v} = \frac{x}{y}$

$\frac{dy}{dx} = -\left(\frac{3}{4}\left(\frac{x}{y}\right) + \frac{1}{4}\left(\frac{y}{x}\right)\right)$ $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = -\left(\frac{3v^2}{4v} + \frac{1}{4v}\right) = -\left(\frac{3v^2+1}{4v}\right)$

$x\frac{dv}{dx} = \frac{-3v^2-1}{4v} - \frac{4v^2}{4v} = \frac{-7v^2-1}{4v}$

$-\int \frac{4v}{7v^2+1}dv = \int \frac{dx}{x}$

$-\frac{2}{7}\ln|7v^2+1| = \ln|x| + \ln C$

$u = 7v^2+1 \quad (7v^2+1)^{-2/7} = C|x|$

$du = 14v\,dv \quad 7v^2+1 = (C|x|)^{-7/2}$

$7\left(\frac{y}{x}\right)^2 + 1 = (C|x|)^{-7/2}$

$7y^2 + x^2 = \cancel{(C|x|)^{-7}}$ $C\,x^{-3/2}$ $\quad y \longrightarrow v = \frac{y}{x}$

~~$y^2 = ((C|x|)^{-7} - x^2)^{1/7}$~~ correct $\boxed{y = \pm\frac{1}{\sqrt{7}}\sqrt{Cx^{-3/2} - x^2}}$

$y' + P(x)y = Q(x)$ ~~$y = \sqrt{\frac{(C|x|)^{-7} - x^2}{7}} = \sqrt{\frac{C|x|^{-7} - x^2}{7}}$~~

"Homogeneous" means the RHS is a pure fcn of $\frac{y}{x}$:

$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Trick: make a change of var.

10. $2y + (x+2)y' = 3x+3$ linear

$\frac{2y}{x+2} + y' = \frac{3x+3}{x+2}$ $\quad M(x) = e^{\int p(x)dx} = e^{\int \frac{2}{x+2}dx} = e^{2\ln|x+2|} = (x+2)^2$

$y' + \left(\frac{2}{x+2}\right)y = \frac{3x+3}{x+2}$

$(x+2)^2y' + (x+2)^2\left(\frac{2}{x+2}\right)y = \frac{3x+3}{x+2}(x+2)^2$

$D_x\left((x+2)^2y\right) = (x+2)^2\left(\frac{3x+3}{x+2}\right)$

$(x+2)^2y = \int (x+2)^2\left(\frac{3x+3}{x+2}\right)dx + C$

$= \int (x+2)(3x+3)dx + C$

$= \int 3x^2 + 9x + 6\,dx + C$

$= 3\int x^2 + 3x + 2\,dx + C$

$(x+2)^2y = 3\left(\frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\right) + C$

$(x+2)^2y = x^3 + \frac{9}{2}x^2 + 6x + C$

$y = \frac{x^3 + \frac{9}{2}x^2 + 6x}{(x+2)^2} + C$ $\quad \frac{\frac{1}{2}}{y+1} = \frac{1}{2}\times\frac{1}{y+1}$

12. $xyy' = x^2 - 3y^2$

$y' = \frac{x}{y} - \frac{3y}{x}$ Again, "Homogeneous" ODE

$v + xv' = \frac{1}{v} - 3v$ Make sub: $v = \frac{y}{x}$

$v + x\frac{dv}{dx} = \frac{1}{v} - 3v$

$x\frac{dv}{dx} = \frac{1-4v^2}{v}$ Correct integration result

$\int \frac{v}{1-4v^2}dv = \int \frac{1}{x}dx$ $\Rightarrow -\frac{1}{8}\ln|1-4v^2| = \ln x + C_1$

$\cancel{-8\ln|v^2|} = \cancel{\ln|x| + \ln C}$ Exponentiate both sides: $(1-4v^2)^{-1/8} = C'x$

$\cancel{\ln v^2 = -8\ln|x| + \ln C}$ $\Rightarrow 1 - 4v^2 = C''x^{-8}$

$\cancel{v^2 = Cx^{-8} \rightarrow \frac{y^2}{x^2} = C\frac{1}{x^8} \rightarrow \boxed{y^2 = C\frac{1}{x^6}}}$ $\Rightarrow v^2 = \frac{1}{4}[1 - C''x^{-8}]$

$v^2 = \left(\frac{y}{x}\right)^2$

So $\boxed{y = \pm\frac{x}{2}[1 - C''x^{-8}]^{1/2}}$

11. $y' = \frac{\sqrt{y} - y}{\tan x}$ Separable.

$\frac{dy}{dx} = \frac{\sqrt{y} - y}{\tan x}$

$\int \frac{1}{\sqrt{y} - y}dy = \int \frac{1}{\tan x} \cdot dx$

$\int \frac{1}{y^{1/2} - y}dy = \int \frac{\cos x}{\sin x}dx$

$\int \frac{y^{-1/2}}{1 - y^{1/2}}dy = \ln|\sin x| + \ln C$

$-2\ln|1 - y^{1/2}| = \ln|\sin x| + \ln C$

$(1 - y^{1/2})^{-2} = \cancel{Ce^{\sin x}}\ C|\sin x|$

$\Rightarrow (1 - y^{1/2}) = [C_1|\sin x|]^{-1/2}$

$\Rightarrow y^{1/2} = 1 - C'|\sin x|^{-1/2}$

$\Rightarrow \boxed{y = [1 - C'|\sin x|^{-1/2}]^2}$