MATH 2930 - Differential Equations - Spring 2010

Worksheet 14 - May 6 & 7, 2010

Brief recap: 1D Waves & Boundary Conditions

At this stage you should know, from either last week's material or this week's assignment, that the general solution to the 1D wave equation $y_{xx} = (1/a^2)y_{tt}$ without any initial/boundary condition is

$$y(x,t) = F(x+at) + G(x-at).$$

Now let's restrict the solution to the interval $0 \leq x \leq L$ (think finite string). Suppose the boundary condition dictates that y vanish at the endpoints $x = 0$ and $x = L$ (*i.e.,* string is fixed at both endpoints). Describe explicitly how the function F is related to the function G, and whether F (or G) possesses certain symmetry or periodic properties.

[*Home exercise:* Consider instead the boundary condition where y' vanishes at both endpoints. What's the relationship between F and G now?]

[*Important terminology:* We say that a BVP has ________________ boundary condition (named after a 19th century French mathematician) if the value of u is specified on the boundary of the domain of interest. On the other hand, if instead the *normal derivative* of u, *i.e.,* the derivative of u along the unit outward vector normal to the boundary, is specified, then we say the BVP has ________________ boundary condition (named after a 19th/20th century German mathematician).]

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What is the Laplacian? Symbol: _______

- Laplacian of $u(x)$ (1D): ______________
- Laplacian of $u(x, y)$ (2D, in Cartesian coords): __________________
- Laplacian of $u(r, \theta)$ (2D, in polar coords): __________________

What is Laplace's Equation? ____________________ w/ boundary conditions

Nomenclature: A function which satisfies Laplace's eqn is called a ___________ function.

Example Problem: Polar Laplace Equation
Solve the following 2D Laplace equation exterior to the disk of radius a:

$$\begin{cases} \Delta u(r,\theta) = 0 & (r > a, 0 \le \theta < 2\pi) \\ u(a,\theta) = f(\theta) & (0 \le \theta < 2\pi) \\ \lim_{r\to\infty} u(r,\theta) \text{ is bounded} & (0 \le \theta < 2\pi) \end{cases}$$

This is a ______________ BVP (in DiffEq terminology).

(a). As before, first attempt separation of variables $u(r,\theta) = R(r)\Theta(\theta)$. Verify that from Line 1 you get two coupled ODEs

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0 \quad \text{and} \quad \Theta''(\theta) + \lambda\Theta(\theta) = 0.$$

We call the ODE for R the **radial equation**, and the ODE for Θ the **angular** or **azimuthal equation**. However, we need some boundary conditions and an implicit periodic condition for Θ to proceed to the next step. State these conditions carefully.

(b). Next, solve the ODE for Θ subject to the correct periodic condition. This is an eigenvalue problem, *i.e.,* λ can only take on discrete values $\{\lambda_1, \lambda_2, \cdots\}$. Why?

(c). Now that you've got the eigenfunction $\Theta_n(\theta)$ associated with eigenvalue λ_n, use the same n and λ_n to solve the ODE for $R(r)$. This is a 2nd-order linear homogeneous ODE with *non*-constant coefficients, so how would you go about solving it? For clarity, denote your solution by $R_n(r)$ to indicate that this is the n-th solution to the radial equation.

(d). From $\Theta_n(\theta)$ and $R_n(r)$, build the n-th solution $u_n(r,\theta)$ to Laplace's equation, satisfying Line 1.

(e). Since Laplace's equation is linear (in u), how would you get the general solution from the individual $u_n(r,\theta)$?

(f). Line 3 tells us that the solution shouldn't blow up at infinity. Impose this condition to eliminate some terms in the general solution.

(g). Finally, impose Line 2. This step looks familiar, doesn't it?

In what physical contexts does Laplace's equation appear?

- ***Irrotational, incompressible and inviscid fluid flow***

  We say that a **flow** or **velocity field** $\vec{v}$ is *irrotational* if its curl vanishes everywhere:

  $$\nabla \times \vec{v} = 0$$

  In particular this means that $\vec{v}$ must be the gradient of some scalar function u (recall the identity $\mathrm{curl}(\mathrm{grad}(u)) \equiv 0$), which we write as

  $$\vec{v} = \nabla u,$$

  where u is the **velocity potential**. To say that a flow is *incompressible* turns out to be equivalent to the condition that the divergence of the flow vanishes everywhere:[1]

  $$\nabla \cdot \vec{v} = 0$$

  For an inviscid (*i.e.,* non-viscous) flow which is both irrotational and incompressible, we can combine the aforementioned definitions to show that the velocity potential u satisfies

  $$\nabla \cdot (\nabla u) = \Delta u = 0$$

  which is Laplace's equation.

  Now let's specialize to the 2D case. We define the **stream function** ψ of a 2D flow by

  $$\vec{v} = \nabla \times \vec{\psi}$$

  where $\vec{v} := v_1\hat{x} + v_2\hat{y}$ and $\vec{\psi} := \psi\hat{z}$. Explicitly,

  $$v_1 = \frac{\partial \psi}{\partial y}\ , \quad v_2 = -\frac{\partial \psi}{\partial x}.$$

  It's an easy exercise in vector calculus to show that ψ also satisfies its own Laplace's equation: $\Delta\psi = 0$. The level curves of the stream function ψ give the streamlines of the flow. Intuitively, the mass particle flows along some streamline, so the velocity field at point (x, y) is the tangent vector to the corresponding streamline at (x, y). An equivalent way to say this is that $\vec{v}$ must be perpendicular to the gradient of the streamline:

  $$\vec{v} \cdot \nabla\psi = 0 \quad \text{or} \quad \nabla u \cdot \nabla \psi = 0.$$

[1] For those of you who have had honors-level physics, here's the proof. Recall that the **continuity equation** for mass flow is

$$\rho_t + \nabla \cdot (\rho\vec{v}) = 0$$

where ρ is the mass density of the material. [*Translation:* the outward flux of "mass current density" $\rho\vec{v}$ must match the rate of mass density decrease $-\rho_t$.] Plugging the $\nabla \cdot \vec{v} = 0$ condition into the continuity equation gives

$$\rho_t + \rho(\cancel{\nabla \cdot \vec{v}}) + \nabla\rho \cdot \vec{v} = 0.$$

But recognize that the LHS is the *total* time derivative, or *convective derivative*, of the mass density ρ:

$$D_t\rho := \partial_t\rho + \nabla\rho \cdot \vec{v} = 0.$$

This means that ρ at some fixed point must remain constant forever, *i.e.,* the material is incompressible.

- ***Electrostatics***

Recall Gauss' Law in integral form: Given any closed surface S, let Q_{enc} be the charge enclosed by S. Then the electric flux through S satisfies

$$\int_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon} \qquad (\epsilon : \text{permittivity})$$

Denote by U the volume bounded by the surface S. It's clear that

$$Q_{\text{enc}} = \int_U \rho dV \quad (\rho : \text{charge density}).$$

On the other hand, the divergence theorem says that

$$\int_S \vec{E} \cdot d\vec{A} = \int_U \left(\nabla \cdot \vec{E}\right) dV.$$

Therefore

$$\int_U \left(\nabla \cdot \vec{E}\right) dV = \int_U \left(\frac{\rho}{\epsilon}\right) dV.$$

Since this identity holds for any bounded volume U, it must be that

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}.$$

This is Gauss' Law in differential form.

A lesser known fact is that the electrostatic field is irrotational,[2] *i.e.*,

$$\nabla \times \vec{E} = 0$$

This implies that $\vec{E}$ must be the gradient of some scalar function ϕ (recall the identity curl(grad(ϕ)) $\equiv 0$), which we write as

$$\vec{E} = -\nabla\phi,$$

where ϕ is (as you surely know) the **electrostatic potential**. (Why the minus sign?) Plugging this into Gauss' Law we have

$$\nabla \cdot (-\nabla\phi) = -\Delta\phi = \frac{\rho}{\epsilon} \quad \text{ or } \quad \Delta\phi = -\frac{\rho}{\epsilon}.$$

If $\rho = 0$, we get **Laplace's equation** $\Delta\phi = 0$. In general, when $\rho \neq 0$, we have **Poisson's equation**.

[2]That $\vec{E}$ is irrotational is true only for electrostatics. If the electric field $\vec{E}$ varies with time, then it creates a time-varying magnetic field $\vec{B}$ (according to Faraday), which in turn feeds back to the time-varying $\vec{E}$ field (according to Ampère). The correct field equations are

$$\begin{aligned} \nabla \times \vec{B} &= -\vec{E}_t && \text{(Faraday's Law)} \\ \nabla \times \vec{E} &= \epsilon\mu\vec{B}_t + \mu\vec{J} && \text{(Ampère-Maxwell Law)} \end{aligned}$$

These, along with Gauss' Law ($\nabla \cdot \vec{E} = \rho/\epsilon$) and the "no magnetic monopole" law ($\nabla \cdot \vec{B} = 0$), combine to give **Maxwell's Equations**, which describe all of electromagnetism.

Extra Credit: *Two important properties of harmonic functions*

Let U be a bounded domain, and denote by ∂U its boundary. In 1D U is typically an interval. In 2D, U can be a circular disk, a rectangle, an ellipse, etc.

Let's focus on Laplace's equation with Dirichlet boundary condition

$$\begin{cases} \Delta u(\vec{x}) = 0 & \text{in } U \\ u(\vec{x}) = f(\vec{x}) & \text{on } \partial U \end{cases}$$

There are two important properties satisfied by the solution u.

1. **Mean value property**

   (a) First take the 1D Laplace's equation on the interval $U = (0, 1)$. You can specify the boundary values of u to your heart's content, say, $u(0) = a$ and $u(1) = b$. Then what should the solution $u(x)$ be? Sketch a graph of $u(x)$. **Check** that for any c with $0 < x - c < x + c < 1$,

   $$u(x) = \frac{1}{2}[u(x-c) + u(x+c)],$$

   and

   $$u(x) = \frac{1}{2c}\int_{x-c}^{x+c} u(y)dy.$$

   This says that the value of a harmonic function at point x must be the average of u on a (1D) sphere centered at x, as well as the average of u over a (1D) ball centered at x. Is this fact obvious from your graph of $u(x)$?

   (b) Now do the 2D Laplace's equation, with U being a disk of radius a centered at the origin. For simplicity, you may take the boundary value of u to be a (nonzero) constant function, although any $u(a, \theta) = f(\theta)$ will do. This example was done in lecture and is also covered in the text. Use the solution $u(r, \theta)$ to show that

   $$u(0,0) = \frac{1}{2\pi r}\int_0^{2\pi} u(r,\theta)d\theta \quad (0 < r < a)$$

   and

   $$u(0,0) = \frac{1}{\pi r^2}\int_0^r \int_0^{2\pi} u(\rho,\theta)\rho d\theta d\rho \quad (0 < r < a)$$

   This says that the value of a harmonic function at the origin must be the average of u on a (2D) sphere centered at the origin, as well as the average of u over a (2D) ball centered at the origin.

   If you're adventurous, then verify the mean value property on any point within the disk.

2. **Maximum principle**

   Here we are going to assume that the boundary value of u equals identically 0 ($f \equiv 0$).

   (a) Solve the 1D Laplace's equation on the unit interval, with u vanishing at the endpoints. What is the solution u?

   (b) Next, solve the 2D Laplace's equation on a circular disk with radius a, with u vanishing on the disk's boundary. What is the solution u?

   (c) Your answers suggest that u cannot reach a maximum (or minimum) in the interior of the domain U. Actually, the maximum (or minimum) must occur somewhere on the boundary. Here's a way to think about this: If u were to reach a maximum (or minimum) in the interior, could u still satisfy Laplace's equation $\Delta u = 0$? Use anything you've learned from calculus to explain this curious fact.