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A SOLUTION TO THE L SPACE PROBLEM

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1. Introduction

In [23], Todorcevic gives a survey of basis problems in combinatorial set theory, listing nine theorems and six working conjectures[1] — all in the presence of PFA — including the following three of interest to us here:

Conjecture 1. *(Todorcevic; [23]) If R is a binary relation, then either $R \leq \aleph_0 \cdot \omega_1$ or $[\omega_1]^{<\aleph_0} \leq R$.*

Conjecture 2. *(Hajnal, Juhasz; [5]) If X is a regular Hausdorff space, then the following are equivalent:*

(i) *X is hereditarily separable.*
(ii) *X is hereditarily Lindelöf.*
(iii) *X does not contain an uncountable discrete subspace.*

Conjecture 3. *(Gruenhage; [3]) The uncountable regular Hausdorff spaces have a three element basis consisting of a set of reals of cardinality $\aleph_1$ with the metric, the Sorgenfrey, and the discrete topology.*

According to the survey article [7], Kunen formulated the partition relation

$$\omega_1 \to (\omega_1, (\omega_1; \omega_1))^2,$$

while studying Conjecture 2 in the 1970s. He showed that it would imply positive resolution of Conjecture 2 and asked whether it is consistent. Afterward Laver showed that under $\mathrm{MA}_{\aleph_1}$ this partition relation is equivalent to the formally weaker partition relation

$$\omega_1 \to (\omega_1; \omega_1)^2_2.$$

Around the same time, Galvin showed that the positive square bracket relation $\omega_1 \to [\omega_1]^2_5$ implies $\omega_1 \to (\omega_1; \omega_1)^2_2$.

In [19] Todorcevic showed that $\omega_1 \not\to [\omega_1]^2_5$ by employing a new technique — the *method of minimal walks* — which has also proved useful in many other applications (see [24]). This technique will be employed in this paper where I will prove the following results which refute $\omega_1 \to (\omega_1; \omega_1)^2_2$ and Conjectures 1–3.

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[1] Many of these were not phrased as conjectures by their authors but rather as problems. Moreover, they were often phrased as "Is it consistent that ...?" rather than "Does PFA imply that ...?".

Theorem 1.1. *There is a function $f : [\omega_1]^2 \to \omega_1$ such that whenever $A, B \subseteq \omega_1$ are uncountable and $\xi < \omega_1$ there are α in A and β in B such that $\alpha < \beta$ and $f(\alpha, \beta) = \xi$. In particular $\omega_1 \not\to (\omega_1; \omega_1)^2_2$.*

Theorem 1.2. *There is a family $\mathscr{F}$ of binary relations which contains an antichain of cardinality $2^{\aleph_1}$, is downwards directed, and such that for all R in $\mathscr{F}$, $R \not\leq \aleph_0 \cdot \omega_1$ and $[\omega_1]^{<\aleph_0} \not\leq R$.*

Theorem 1.3. *There is a non-separable, hereditarily Lindelöf subspace $\mathscr{L}$ of $\mathbb{T}^{\omega_1}$. Moreover the closure in $\mathbb{T}^{\omega_1}$ of any uncountable subset of $\mathscr{L}$ contains a canonical homeomorphic copy of $\mathbb{T}^{\omega_1}$.*

Theorem 1.4. *Any basis for the uncountable topological spaces has cardinality strictly greater than $\aleph_1$.*

It is worth noting that it was known early on that the existence of an L space alone gives a refutation of these conjectures. The above results, however, take the failures of these conjectures a step further and require slightly different combinatorial refinements.

The theorems in this paper are consequences of an analysis of *coherent sequences of finite-to-one functions* $\langle e_\beta : \beta < \omega_1 \rangle$ and *lower trace functions* $L : [\omega_1]^2 \to [\omega_1]^{<\omega_1}$. These are combinatorial objects which can be routinely constructed using the method of minimal walks. If we let $\operatorname{osc}(\alpha, \beta)$ denote the number of changes in the relation between corresponding entries of $e_\alpha \restriction L(\alpha, \beta)$ and $e_\alpha \restriction L(\alpha, \beta)$ from $\leq$ to $>$, then the main result of the paper can be stated as follows.

Theorem 1.5. *For every $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ which are uncountable families of pairwise disjoint sets and every natural number n, there are a in $\mathscr{A}$ and b_m $(m < n)$ in $\mathscr{B}$ such that for all $i < k$, $j < l$, and $m < n$:*

$$a < b_m$$
$$\operatorname{osc}\big(a(i), b_m(j)\big) = \operatorname{osc}\big(a(i), b_0(j)\big) + m.$$

This can be likened to the following two results of Todorcevic. The first is used to draw a number of conclusions about Conjecture 2 in [21]. The second yields one of many proofs in Section 4 of [19] that $\omega_1 \not\to [\omega_1]^2_\omega$. See [19] and Section 2 below for undefined notation.[2]

Theorem 1.6. [20] *If X and Y are unbounded and countably directed in $(\omega^\omega, <^*)$ and consist of monotonic functions, then there is a natural number l such that for all n there are x and y in X and Y respectively such that*

$$\operatorname{osc}(x, y) = n + l.$$

Theorem 1.7. [19] *If X is an uncountable subset of ω_1, then the set of all values*

$$\operatorname{osc}\big(\operatorname{Tr}(\zeta, \alpha), \operatorname{Tr}(\zeta, \beta)\big)$$

for $\alpha \neq \beta$ in X with $\zeta = \Delta(\alpha, \beta)$ contains arbitrarily long intervals of integers.

This paper is organized as follows. Section 2 provides some background on the method of minimal walks and introduces the lower trace function which is used in the statement of the main theorem on oscillations. Section 3 provides

[2] What I am referring to as $\operatorname{Tr}(\alpha, \beta)$ is denoted as $F_\alpha(\beta)$ in [19]. Also, the usage of $\operatorname{osc}(\cdot, \cdot)$ in Theorem 1.7 is different as it counts the oscillations between finite subsets of ω_1 — see [19].

a review of elementary submodels. The main combinatorial results of the paper are proved in Section 4. Theorem 1.1 is deduced in Section 5 and a two place function $o : [\omega_1]^2 \to \mathbb{T}$ is introduced which harnesses most of the strength of the more technical main theorem. Theorem 1.2 is then deduced in Section 6 after some motivation is provided. The paper closes with Section 7 where Theorems 1.3 and 1.4 are proved. This final section also contains a basic analysis of the space in Theorem 1.3.

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This paper is intended to be accessible to any interested reader who is fluent in set theory. Basic background can be found in, e.g., [10]. Elementary submodels will be employed at a number of points in the argument. Ironically, this represents the only non-elementary technique used in the proofs. The essentials are reviewed in Section 3. The reader is referred to III.1 of [6] for more information on elementary submodels and stationary sets. The proofs will also employ the method of minimal walks introduced in [19]. The necessary background is presented in Section 2. The reader is referred to [24] for further reading on minimal walks.[3]

The notation is fairly standard. All ordinals are von Neumann ordinals — they are the set consisting of their predecessors. In particular, $n = \{0, 1, \ldots, n-1\}$ and the first infinite ordinal ω is the set of all finite ordinals $\{0, 1, 2, \ldots\}$ and is identified with the natural numbers. All counting starts at 0. We will also need the unit circle viewed as the set

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

equipped with the usual multiplication operation. If k is a natural number and X is a set, then $[X]^k$ is the set of all k-element subsets of X. If X has a canonical linear ordering associated with it and a is in $[X]^k$, then a will be identified with the increasing sequence which enumerates it. If a and b are finite subsets of ω_1, then $a < b$ will be used to abbreviate "$\alpha < \beta$ whenever α is in a and β is in b." Similarly one defines statements such as $\alpha < b$ and $a < \beta$ if α and β are ordinals.

This paper is dedicated to Stevo Todorcevic for teaching me how to traverse ω_1 and for his inspirational [23].

2. The trace functions

In this section I will provide the necessary background on minimal walks; see [24] for further reading. With the exception of the definition and axiomatization of L, everything discussed in this section is contained either explicitly or implicitly in [19]. Minimal walks are facilitated by a *C-sequence* which one uses to "walk" from an ordinal β down to a smaller ordinal α.

Definition 2.1. A *C-sequence* is a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that C_α is a cofinal subset of α and if $\gamma < \alpha$ then $C_\alpha \cap \gamma$ is finite. It will be useful at certain points to assume that 0 is an element of every C_α.

[3] [24] is an updated, expanded, and clearer account of the methods and results of [19]. My references to [19] are for historical accuracy.

The following two functions will be of interest to us. The upper trace will not be necessary, but is useful in making the other definitions more transparent.

Definition 2.2. [24] (*upper trace*) If $\alpha \leq \beta$, then $\mathrm{Tr}(\alpha,\beta)$ is defined recursively by

$$\mathrm{Tr}(\alpha,\alpha) = \emptyset,$$
$$\mathrm{Tr}(\alpha,\beta) = \mathrm{Tr}\left(\alpha, \min(C_\beta \setminus \alpha)\right) \cup \{\beta\}.$$

Definition 2.3. (*lower trace*) If $\alpha \leq \beta$, then $L(\alpha,\beta)$ is defined recursively by

$$L(\alpha,\alpha) = \emptyset,$$
$$L(\alpha,\beta) = L\big(\alpha, \min(C_\beta \setminus \alpha)\big) \cup \{\max(C_\beta \cap \alpha)\} \setminus \max(C_\beta \cap \alpha).$$

Hence the upper trace $\mathrm{Tr}(\alpha,\beta)$ is enumerated by the sequence $\beta = \beta_0 > \beta_1 > \cdots > \beta_{l-1}$ where $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$ and $\alpha = \beta_l$ is in $C_{\beta_{l-1}}$. It is easily verified that the lower trace $L(\alpha,\beta)$ is listed as $\xi_0 \leq \xi_1 \leq \cdots \leq \xi_{l-1}$ where

$$\xi_k = \max \bigcup_{j=0}^{k} (C_{\beta_j} \cap \alpha).$$

While the upper trace $\mathrm{Tr}(\alpha,\beta)$ and the *full lower trace* $F(\alpha,\beta)$ of a walk are well studied (see [24]), this is, to my knowledge, the first explicit analysis of the lower trace $L(\alpha,\beta) \subseteq F(\alpha,\beta)$.

For our purposes, the lower trace can be axiomatized by the following facts.

Fact 1. *If $\alpha \leq \beta \leq \gamma$ and $L(\beta,\gamma) < L(\alpha,\beta)$, then*

$$L(\alpha,\gamma) = L(\alpha,\beta) \cup L(\beta,\gamma).$$

Proof. Let α, β, and γ satisfy the hypothesis of the fact. Observe that $L(\beta,\gamma) < \alpha$ and hence $C_\zeta \cap \alpha = C_\zeta \cap \beta$ whenever ζ is in $\mathrm{Tr}(\beta,\gamma)$. Hence β is in $\mathrm{Tr}(\alpha,\gamma)$ and $\mathrm{Tr}(\alpha,\gamma) = \mathrm{Tr}(\alpha,\beta) \cup \mathrm{Tr}(\beta,\gamma)$. Let

$$\gamma = \gamma_0 > \gamma_1 > \ldots > \gamma_{l-1}$$

enumerate $\mathrm{Tr}(\alpha,\gamma)$ and let l_0 be such that $\beta = \gamma_{l_0}$. Then $L(\alpha,\gamma)$ is listed as

$$\xi_0 \leq \xi_1 \leq \cdots \leq \xi_{l-1}$$

where

$$\xi_k = \max \bigcup_{j=0}^{k} C_{\gamma_j} \cap \alpha.$$

If $k < l_0$, then $C_{\gamma_k} \cap \alpha = C_{\gamma_k} \cap \beta$ and so $L(\beta,\gamma) = \{\xi_k\}_{k=0}^{l-1}$. On the other hand,

$$\max\left(C_{\gamma_{l_0}} \cap \alpha\right) > \xi_{l_0 - 1}$$

and so if $k \geq l_0$

$$\max \bigcup_{j=0}^{k} \left(C_{\gamma_j} \cap \alpha\right) = \max \bigcup_{j=l_0}^{k} \left(C_{\gamma_j} \cap \alpha\right)$$

and hence $L(\alpha,\beta) = \{\xi_k\}_{k=l_0}^{l-1}$. □

Fact 2. *If $\alpha < \beta$, then $L(\alpha,\beta)$ is a non-empty finite set and if $0 < \beta$ is a limit ordinal, then*

$$\lim_{\alpha \to \beta} \min L(\alpha,\beta) = \beta.$$

Proof. This follows immediately from the observation that
$$\min L(\alpha,\beta) = \max(C_\beta \cap \alpha).$$
□

Remark 2.4. Notice that if $0 < \beta < \gamma$ are fixed and β is a limit ordinal, then Fact 2 implies that for all but a bounded set of $\alpha < \beta$, the inequality in Fact 1 holds.

At this point I recall the following definition.

Definition 2.5. If $\langle e_\beta : \beta < \omega_1 \rangle$ is a sequence of functions such that $e_\beta : \beta \to \omega$ for all $\beta < \omega_1$, then the sequence is *coherent* if whenever $\beta \leq \beta' < \omega_1$, $e_{\beta'} \restriction \beta$ differs from e_β only on a finite set.

For brevity, if I say that $\langle e_\beta : \beta < \omega_1 \rangle$ is a *coherent sequence of finite-to-one functions*, then it is implicit that each e_β is a function from β to ω. The main combinatorial result of the paper will be concerned with counting oscillations between pairs of elements of such sequences. The following construction provides a standard example.

Definition 2.6. [19](*maximal weight*) If $\alpha \leq \beta$, then $\varrho_1(\alpha,\beta)$ is defined recursively by
$$\varrho_1(\alpha,\alpha) = 0$$
$$\varrho_1(\alpha,\beta) = \max\Big(|C_\beta \cap \alpha|, \varrho_1\big(\alpha, \min(C_\beta \setminus \alpha)\big)\Big).$$

Alternately, $\varrho_1(\alpha,\beta)$ is the maximum value of the form $|C_\zeta \cap \alpha|$ where ζ ranges over $\mathrm{Tr}(\alpha,\beta)$. For each $\beta < \omega_1$, define $e_\beta : \beta \to \omega$ by $e_\beta(\alpha) = \varrho_1(\alpha,\beta)$.

Fact 3. [19] *$\langle e_\beta : \beta < \omega_1 \rangle$ is a coherent sequence of finite-to-one functions.*

Proof. Let $\beta \leq \beta' < \omega_1$ and $n < \omega$ be given and set D equal to the set of all $\alpha < \beta$ such that either $e_\beta(\alpha) \leq n$ or $e_\beta(\alpha) \neq e_{\beta'}(\alpha)$. It suffices to show that D has no limit points.

To this end, suppose $\delta \leq \beta$. It is easy to check that there is a $\delta_0 < \delta$ such that
$$\mathrm{Tr}(\alpha,\beta) = \mathrm{Tr}(\alpha,\delta) \cup \mathrm{Tr}(\delta,\beta),$$
$$\mathrm{Tr}(\alpha,\beta') = \mathrm{Tr}(\alpha,\delta) \cup \mathrm{Tr}(\delta,\beta'),$$
$$|C_\delta \cap \alpha| > n$$
whenever $\delta_0 < \alpha < \delta$. Notice that if δ is a successor, then this can be made vacuously true by letting $\delta_0 = \delta - 1$. If $\delta_0 < \alpha < \delta$, then
$$e_\beta(\alpha) \geq |C_\delta \cap \alpha| > n$$
$$e_\beta(\alpha) = e_\delta(\alpha) = e_{\beta'}(\delta)$$
and hence α is not in D. Consequently δ is not a limit point of D. □

Remark 2.7. It is interesting to note that while coherence and the finite-to-one property are at tension with each other, the verifications of these properties in the previous fact are virtually identical.

Remark 2.8. It follows from Fact 3 that $T(\varrho_1) = \{e_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}$ is an *Aronszajn tree* — its levels and chains are countable.

Throughout this paper we will assume that L and $\langle e_\beta : \beta < \omega_1 \rangle$ satisfy Facts 1–3. We will also need the following two place function.

Definition 2.9. If $\alpha < \beta < \omega_1$, then let $\Delta(\alpha, \beta)$ be the least ordinal ξ such that either $e_\alpha(\xi) \neq e_\beta(\xi)$ or $\xi = \alpha$.

3. Basic facts about elementary submodels

In the proof of the main theorem, we will need the following facts about elementary submodels of $H(\aleph_1^+)$. Recall that $H(\aleph_1^+)$ is the collection of all sets of hereditary cardinality at most $\aleph_1$. This structure is of interest to us since it contains all of the objects relevant to us (e.g. ω_1, L, and $\langle e_\beta : \beta < \omega_1 \rangle$), is closed under taking subsets, and satisfies all of the axioms of ZFC except the power set axiom. An *elementary submodel* M of $H(\aleph_1^+)$ is a subset of $H(\aleph_1^+)$ such that whenever ϕ is a logical formula with parameters in M, M satisfies ϕ if and only if $H(\aleph_1^+)$ satisfies ϕ. The following fact is well known.

Fact 4. *If M is a countable elementary submodel of $H(\aleph_1^+)$, then $M \cap \omega_1$ is an ordinal. Moreover, if F is a countable subset of $H(\aleph_1^+)$, then the set of all ordinals of the form $M \cap \omega_1$ such that M is a countable elementary submodel of $H(\aleph_1^+)$ with $F \subseteq M$ contains a club.*[4]

The following standard facts are very useful when working with countable elementary submodels.

Fact 5. *If M is a countable elementary submodel of $H(\aleph_1^+)$ which contains some element X, then X is countable iff $X \subseteq M$.*

Fact 6. *If M is a countable elementary submodel of $H(\aleph_1^+)$ which contains as an element some subset A of ω_1, then A is uncountable iff $A \cap M \cap \omega_1$ is unbounded in $M \cap \omega_1$.*

Fact 7. *If M is a countable elementary submodel of $H(\aleph_1^+)$, X is in $H(\aleph_1^+)$ and X is definable from a logical formula with parameters in M, then X is in M.*

Remark 3.1. The above facts remain true if $\aleph_1^+$ is replaced with any uncountable regular cardinal.

4. Oscillations on the lower trace

In this section I will prove the main combinatorial theorem of the paper. We will take the following as our definition of the oscillation function.

Definition 4.1. Suppose that s and t are two functions defined on a common finite set of ordinals F. Let $\mathrm{Osc}(s, t; F)$ be the set of all ξ in $F \setminus \{\min F\}$ such that $s(\xi^-) \leq t(\xi^-)$ and $s(\xi) > t(\xi)$ where ξ^- is the greatest element of F less than ξ.[5]

The following notation will be convenient.

Definition 4.2. If $\alpha < \beta < \omega_1$, let $\mathrm{Osc}(\alpha, \beta)$ denote

$$\mathrm{Osc}\left(e_\alpha, e_\beta; L(\alpha, \beta)\right)$$

and $\mathrm{osc}(\alpha, \beta)$ denote the cardinality of $\mathrm{Osc}(\alpha, \beta)$.

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[4]A subset of ω_1 is *club* if it is closed and unbounded.

[5]Actually, we will later see that we are really interested in counting oscillations between the relations $=$ and $\neq$ (rather than the more conventional $<$ and $>$). The above definition is a compromise between the two.

Theorem 4.3. *For every $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ which are uncountable families of pairwise disjoint sets and every $n < \omega$, there are a in $\mathscr{A}$ and b_m $(m < n)$ in $\mathscr{B}$ such that for all $i < k$, $j < l$, and $m < n$:*

$$a < b_m$$

$$\operatorname{osc}\big(a(i), b_m(j)\big) = \operatorname{osc}\big(a(i), b_0(j)\big) + m.$$

Theorem 4.3 will be derived from the following lemma.

Lemma 4.4. *Let $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ be uncountable and pairwise disjoint. There is a closed and unbounded set of $\delta < \omega_1$ such that if a is in $\mathscr{A} \setminus \delta$, b is in $\mathscr{B} \setminus \delta$, and R is in $\{=, >\}$, then there are a^+ in $\mathscr{A} \setminus \delta$, and b^+ in $\mathscr{B} \setminus \delta$ such that for all $i < k$ and $j < l$:*[6]

(i) $\max L\big(\delta, b(j)\big)$ *is less than both* $\Delta\big(a(i), a^+(i)\big)$ *and* $\Delta\big(b(j), b^+(j)\big)$.
(ii) $L\big(\delta, b(j)\big)$ *is a proper initial part of* $L\big(\delta, b^+(j)\big)$.
(iii) *If* ξ *is in* $L^+ = L\big(\delta, b^+(j)\big) \setminus L\big(\delta, b(j)\big)$, *then* $e_{a^+(i)}(\xi)\ R\ e_{b^+(j)}(\xi)$.

Proof. Let M be a countable elementary submodel of $H(\aleph_1^+)$ which contains everything relevant and let $\delta = M \cap \omega_1$. By Fact 4, it suffices to show that δ satisfies the conclusion of the theorem.

First suppose that R is $=$. Applying Fact 3, find a $\gamma_0 < \delta$ satisfying the following conditions:

(1) If $j < l$, then $L\big(\delta, b(j)\big) < \gamma_0$.
(2) If $i < k$, $j < l$, and $\gamma_0 < \xi < \delta$, then $e_{a(i)}(\xi) = e_{b(j)}(\xi)$.

Applying Fact 2, pick a $\gamma < \delta$ such that if $\gamma < \xi < \delta$, then $\gamma_0 < L(\xi, \delta)$. Consider the set $D \subseteq \omega_1$ consisting of all δ^+ such that for some a^+ in $\mathscr{A} \setminus \delta^+$ and b^+ in $\mathscr{B} \setminus \delta^+$ the following conditions are satisfied:

(3) $e_{a^+(i)} \restriction \gamma_0 = e_{a(i)} \restriction \gamma_0$ and $e_{b^+(j)} \restriction \gamma_0 = e_{b(j)} \restriction \gamma_0$ for all $i < k$ and $j < l$.
(4) $L\big(\delta^+, b^+(j)\big) = L\big(\delta, b(j)\big)$ for all $j < l$.
(5) If $\gamma < \xi < \delta^+$, then $\gamma_0 < L(\xi, \delta^+)$.
(6) If $\gamma_0 < \xi < \delta^+$, $i < k$, and $j < l$, then $e_{a^+(i)}(\xi) = e_{b^+(j)}(\xi)$.

Observe that for all $\beta \geq \gamma_0$, $e_\beta \restriction \gamma_0$ is in M since by Fact 3 it differs from e_{γ_0} on a finite set. Hence D is definable from the parameters γ_0, γ, $e_{a(i)} \restriction \gamma_0$, etc. which are all elements of M. Therefore D is in M by Fact 7. Since D has δ as a member, it is uncountable by Fact 5. Hence there is a $\delta^+ > \delta$ in D. Let $a^+ \in \mathscr{A} \setminus \delta^+$ and $b^+ \in \mathscr{B} \setminus \delta^+$ witness that δ^+ is in D.

Now let $i < k$ and $j < l$ be arbitrary. First observe that

$$\gamma_0 \leq \Delta\big(a(i), a^+(i)\big),$$

$$\gamma_0 \leq \Delta\big(b(j), b^+(j)\big).$$

Put $L^+ = L(\delta, \delta^+)$. Notice that $L\big(\delta, b(j)\big) = L\big(\delta^+, b^+(j)\big) < L^+$ and hence

$$L\big(\delta, b^+(j)\big) = L\big(\delta^+, b^+(j)\big) \cup L^+ = L\big(\delta, b(j)\big) \cup L^+$$

holds by Fact 1. Since $\delta < \delta^+$, L^+ is non-empty. If ξ is in L^+, then $\gamma_0 < \xi < \delta^+$ and so

$$e_{a^+(i)}(\xi)\ R\ e_{b^+(j)}(\xi).$$

[6] It will be convenient to let $\mathscr{A} \setminus \delta$ denote the set of all a in $\mathscr{A}$ such that $\delta \leq a$ whenever $\mathscr{A}$ is a collection of finite subsets of ω_1 and $\delta < \omega_1$.

Now suppose that R is $>$. Put E equal to the set of all limit $\nu < \omega_1$ such that for all a_0 in $\mathscr{A} \setminus \nu$, all $\nu_0 < \nu$, $\varepsilon < \omega_1$, $n < \omega$, and finite $L^+ \subseteq \omega_1 \setminus \nu$ there is an a_1 in $\mathscr{A} \setminus \varepsilon$ with

$$\nu_0 \leq \Delta\big(a_0(i), a_1(i)\big),$$
$$e_{a_1(i)}(\xi) > n$$

whenever $i < k$ and ξ is in L^+. By Fact 7, E is an element of M.

Claim 4.5. *δ is in E. In particular, E is uncountable by Fact 5.*

Proof. Let a_0, ν_0, ε, n, and L^+ be given as in the definition of E for $\nu = \delta$. By Fact 3 we may assume without loss of generality that ν_0 is an upper bound for all $\xi < \delta$ such that $e_{a_0(i)}(\xi) \leq n$ for some $i < k$. Now, applying elementarity of M, there is a δ^+ above ε, δ, and $\max L^+$ and an a_1 in $\mathscr{A} \setminus \delta^+$ such that the following conditions are satisfied:

(7) For all $i < k$, $e_{a_0(i)} \restriction \nu_0 = e_{a_1(i)} \restriction \nu_0$.
(8) If $\nu_0 < \xi < \delta^+$ and $i < k$, then $e_{a_1(i)}(\xi) > n$.

Since $L^+ \subseteq \delta^+ \setminus \delta$, this completes the proof of the claim. □

Applying elementarity of M and uncountability of E, find an element γ_0 of E such that $L\big(\delta, b(j)\big) < \gamma_0 < \delta$ for all $j < l$. Applying Fact 2, find a $\gamma < \delta$ such that if $\gamma < \xi < \delta$, then $\gamma_0 < L(\xi, \delta)$. Again using elementarity of M, select a limit $\delta^+ > \delta$ and a b^+ in $\mathscr{B} \setminus \delta^+$ so that the following conditions are satisfied:

(9) $e_{b^+(j)} \restriction \gamma_0 = e_{b(j)} \restriction \gamma_0$ for all $j < l$.
(10) $L\big(\delta^+, b^+(j)\big) = L\big(\delta, b(j)\big)$ for all $j < l$.
(11) If $\gamma < \xi < \delta^+$, then $\gamma_0 < L(\xi, \delta^+)$.

Put $L^+ = L(\delta, \delta^+)$. Applying the definition of E, find an a^+ in $\mathscr{A} \setminus \delta$ such that for all $i < k$, $j < l$, and ξ in L^+

$$L\big(\delta, b(j)\big) < \Delta\big(a(i), a^+(i)\big)$$
$$e_{a^+(i)}(\xi) > e_{b^+(j)}(\xi).$$

The rest of the verification is as in the previous case. This completes the proof of Lemma 4.4. □

Now we are ready to prove Theorem 4.3

Proof. Let $\mathscr{A}$ and $\mathscr{B}$ be given and select a countable elementary submodel M of $H(\aleph_1^+)$ containing everything relevant, setting $\delta = M \cap \omega_1$. Since M contains $\mathscr{A}$ and $\mathscr{B}$, the closed and unbounded set provided by Lemma 4.4 is in M and therefore δ is in this closed and unbounded set. Using Lemma 4.4, it is possible to select a_m $(m < \omega)$ in $\mathscr{A} \setminus \delta$, b_m $(m < \omega)$ in $\mathscr{B} \setminus \delta$, and ξ_m $(m < \omega)$ in δ so that for all $m < \omega$ the following conditions are satisfied whenever $i < k$ and $j < l$:

(12) $L\big(\delta, b_m(j)\big)$ is a proper initial part of $L\big(\delta, b_{m+1}(j)\big)$.
(13) $L\big(\delta, b_{m+1}(j)\big) \setminus L\big(\delta, b_m(j)\big)$ does not depend on j and contains ξ_m as an element.
(14) $\mathrm{Osc}\,\big(a_{m+1}(i), b_{m+1}(j); L(\delta, b_{m+1}(j))\big)$ is formed by adding ξ_m to $\mathrm{Osc}\,\big(a_m(i), b_m(j); L(\delta, b_m(j))\big)$.
(15) If $m' < m$, then $\xi_{m'}$ is strictly less than both $\Delta\big(a_m(i), a_{m+1}(i)\big)$ and $\Delta\big(b_m(j), b_{m+1}(j)\big)$.
(16) $e_{a_m(i)}\big(\max L(\delta, b_m(j))\big) > e_{b_m(j)}\big(\max L(\delta, b_m(j))\big)$.

Notice that, once a_m and b_m have been selected, Lemma 4.4 is applied twice in order to obtain a_{m+1} and b_{m+1} — first with R being $=$ and then with R being $>$. The ordinal ξ_m is the minimum of what might be described as

$$L\big(\delta, b_{m+1}(j)\big) \setminus L\big(\delta, b_{m+1/2}(j)\big).$$

Now let n be given. Pick a $\gamma_0 < \delta$ which is an upper bound for each $L\big(\delta, b_n(j)\big)$ for $j < l$ and all $\xi < \delta$ such that for some $m, m' \leq n$ and $j, j' < l$

$$e_{b_m(j)}(\xi) \neq e_{b_{m'}(j')}(\xi)$$

(the latter set is finite by Fact 3). Using elementarity of M and Fact 2, select an a in $\mathscr{A}$ such that $a < \delta$ and for all $i < k$ and $j < l$

$$L\big(\delta, b_n(j)\big) < \Delta\big(a(i), a_n(i)\big),$$
$$\gamma_0 < L(a(i), \delta).$$

Now let $i < k$, $j < l$, and $m < n$ be fixed. It follows from Fact 1 that

$$L\big(a(i), b_m(j)\big) = L\big(a(i), \delta\big) \cup L\big(\delta, b_m(j)\big).$$

Finally, $e_{b_m(j)} \restriction L(a(i), \delta)$ does not depend on m and therefore

$$\operatorname{Osc}\big(a(i), b_0(j); L(a(i), \delta)\big) = \operatorname{Osc}\big(a(i), b_m(j); L(a(i), \delta)\big).$$

By (16), $\operatorname{Osc}\big(a(i), b_m(j)\big)$ is the union

$$\operatorname{Osc}\big(a(i), b_m(j); L(a(i), \delta)\big) \cup \operatorname{Osc}\big(a(i), b_m(j); L(\delta, b_m(j))\big)$$

(i.e. no new oscillation occurs at the "seam"). Hence, by (14),

$$\operatorname{Osc}\big(a(i), b_m(j)\big) = \operatorname{Osc}\big(a(i), b_0(j)\big) \cup \{\xi_{m'} : m' < m\}.$$

This completes the proof of Theorem 4.3. □

5. A negative partition relation and the coloring o

We will now consider the following function.

Definition 5.1. If $\alpha < \beta$, define

$$o(\alpha, \beta) = z_\alpha^{\operatorname{osc}(\alpha,\beta)+1}.$$

This definition is motivated by the following theorem — commonly known as *Kronecker's Theorem* — discovered independently by Kronecker and Tchebychef (see, e.g., [1]).

Kronecker's Theorem. [8] [17] *Suppose that z_i $(i < k)$ are elements of $\mathbb{T}$ which are rationally independent. For every $\varepsilon > 0$ there is a natural number n_ε such that if u, v are in $\mathbb{T}^k$, there is an $m < n_\varepsilon$ such that for all $i < k$*

$$|u_i z_i^m - v_i| < \varepsilon.$$

Remark 5.2. The usual formulation of this theorem is that n_ε depends on u and v. The uniformity of n_ε as u and v vary follows from a standard compactness argument.

For the remainder of the paper, fix a sequence $\langle z_\alpha : \alpha < \omega_1\rangle$ of elements of $\mathbb{T}$ which are rationally independent. This is possible since, given any countable rationally independent subset I of $\mathbb{T}$, there are only countably many z for which $I \cup \{z\}$ is rationally dependent.

The following function will also be useful.

Definition 5.3. The function $* : \mathbb{T} \to \omega$ is defined by letting $*(z) = n$ iff there is a θ in $(\frac{2\pi}{n+2}, \frac{2\pi}{n+1}]$ such that $z = e^{i\theta}$. If f is a function taking values in $\mathbb{T}$, then f^* will denote the composition of f followed by $*$.

The point is that $*$ takes all values on every dense subset of $\mathbb{T}$. The notation for $*$ is chosen to mimic its usage in [24]. We will shortly see that o^* defines a partition of $[\omega_1]^2$ with strong properties.

In [18], Todorcevic considered two variations of $\omega_1 \to (\omega_1; \omega_1)^2_2$ which are closely related to (S) and (L) respectively:

$$\omega_1 \to \big(\omega_1, (\omega_1; \text{fin } \omega_1)\big)^2,$$

$$\omega_1 \to \big(\omega_1, (\text{fin } \omega_1; \omega_1)\big)^2.$$

The statement $\omega_1 \to \big(\omega_1, (\omega_1; \text{fin } \omega_1)\big)^2$ asserts that if $c : [\omega_1]^2 \to 2$, then either

(A) there is an uncountable $X \subseteq \omega_1$ such that c is constantly 0 on $[X]^2$ or
(B) there are uncountable $A \subseteq \omega_1$ and uncountable pairwise disjoint $\mathscr{B} \subseteq [\omega_1]^{<\aleph_0}$ such that for all α in A and b in $\mathscr{B}$ with $\alpha < b$, there is a β in b with $c(\alpha, \beta) = 1$.

The statement $\omega_1 \to \big(\omega_1, (\text{fin } \omega_1; \omega_1)\big)^2$ is similarly defined by replacing $A \subseteq \omega_1$ by a pairwise disjoint $\mathscr{A} \subseteq [\omega_1]^{<\aleph_0}$ and $\mathscr{B}$ by $B \subseteq \omega_1$. These imply (S) and (L) respectively (see [18]) and are each immediate consequences of $\omega_1 \to (\omega_1; (\omega_1; \omega_1))^2_2$.

In [18], Todorcevic showed that $\omega_1 \to \big(\omega_1, (\omega_1; \text{fin } \omega_1)\big)^2$ was relatively consistent with ZFC and in fact follows from PFA. We will now see that o^* witnesses a strong failure of $\omega_1 \to \big(\omega_1, (\text{fin } \omega_1; \omega_1)\big)^2$.

Theorem 5.4. *Let $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ be uncountable families of pairwise disjoint sets. For every $\chi : k \to \omega$ and $\phi : k \to l$ there are a in $\mathscr{A}$ and b in $\mathscr{B}$ such that $a < b$ and for all $i < k$,*

$$o^*\big(a(i), b(\phi(i))\big) = \chi(i).$$

By a "stretching" argument which can be found at the beginning of Section 4 of [19], Theorem 5.4 yields the following corollary.

Corollary 5.5. $\omega_1 \not\to [\omega_1; \omega_1]^2_{\omega_1}$.

Theorem 5.4 is an immediate consequence of the following theorem.

Theorem 5.6. *Let $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ be uncountable families of pairwise disjoint sets. For every sequence U_i $(i < k)$ of open neighborhoods in $\mathbb{T}$ and every $\phi : k \to l$ there are a in $\mathscr{A}$ and b in $\mathscr{B}$ such that $a < b$ and for all $i < k$,*

$$o\big(a(i), b(\phi(i))\big) \in U_i.$$

Proof. Without loss of generality, we may assume that each U_i is an ε-ball about a point v_i for some fixed $\varepsilon > 0$. By refining $\mathscr{A}$ if necessary, we may assume that the n_ε be given by Kronecker's theorem for the sequence $z_{a(i)}$ $(i < k)$ is uniform for a in $\mathscr{A}$. Let a in $\mathscr{A}$ and b_m $(m < n_\varepsilon)$ in $\mathscr{B}$ be such that for all $i < k$, $j < l$, and $m < n_\varepsilon$

$$a < b_m$$

$$\text{osc}\,\big(a(i), b_m(j)\big) = \text{osc}\,\big(a(i), b_0(j)\big) + m.$$

For each $i < k$, put $u_i = o\big(a(i), b_0(\phi(i))\big)$. By choice of n_ε, there is an $m < n_\varepsilon$ such that for all $i < k$

$$|u_i z^m_{a(i)} - v_i| < \varepsilon$$

or, equivalently, $o\big(a(i), b_m(\phi(i))\big)$ is in U_i. This finishes the proof. □

Question 5.7. *(PFA[7]) If $c : [\omega_1]^2 \to 2$, are there $A \subseteq \omega_1$ and $\mathscr{B} \subseteq [\omega_1]^2$ which are uncountable with $\mathscr{B}$ being pairwise disjoint and $\chi : 2 \to 2$ such that for all $\alpha \in A$ and $b \in \mathscr{B}$ with $\alpha < b$ there is an $i < 2$ such that $c(\alpha, b(i)) \neq \chi(i)$?*

6. A family of binary relations on ω_1

In this section we will consider the family of binary relations on ω_1 with the Tukey order.

Definition 6.1. [25] (Tukey order) If R and S are binary relations, then we write $R \leq S$ iff there are functions $f : \mathrm{dom}(R) \to \mathrm{dom}(S)$ and $g : \mathrm{ran}(S) \to \mathrm{ran}(R)$ such that

$$f(x)\ S\ y \text{ implies } x\ R\ g(y).$$

This order was first considered by Tukey in the class of transitive relations [25]. It makes sense, however, to consider this order in the more general setting of the class of binary relations (see [26]).

Our focus will be on binary relations on ω_1. Two fundamental examples are the well order ω_1 and the family $[\omega_1]^{<\aleph_0}$ of finite subsets of ω_1 ordered by inclusion. We will also need a few standard operations on relations. If R and S are relations, then $R \oplus S$ is the relation which is the disjoint union of R and S. If m is a cardinal and R is a relation, then we will let $m \cdot R$ will denote the direct sum of m copies of R. If R and S are two binary relations, then $R \wedge S$ and $R \vee S$ are the relations with domains $\mathrm{dom}(R) \times \mathrm{dom}(S)$ and ranges $\mathrm{ran}(R) \times \mathrm{ran}(S)$ such that

$$(a, b)\ R \wedge S\ (c, d) \text{ iff } a\ R\ c \text{ or } b\ S\ d,$$

$$(a, b)\ R \vee S\ (c, d) \text{ iff } a\ R\ c \text{ and } b\ S\ d.$$

The "join" $R \vee S$ is often denoted $R \times S$. It is easily verified that $\wedge$ and $\vee$ give lower and upper bounds respectively.[8]

It was observed that the binary relations on ω_1 which one knew how to construct in ZFC were either below $\aleph_0 \cdot \omega_1$ or above $[\omega_1]^{<\aleph_0}$ in the Tukey order.

Example 6.2. Suppose that r_α $(\alpha < \omega_1)$ is a sequence of distinct reals. Define $\alpha R \beta$ iff $\alpha < \beta$ and $r_\alpha < r_\beta$.

The relation R is essentially Sierpiński's partition which witnesses $\omega_1 \not\to (\omega_1)^2_2$. In this case $R \leq \aleph_0 \cdot \omega_1$ by the following Tukey maps:

$$f(\alpha) = (\min\{n : r_\alpha < q_n\}, \alpha)$$

$$g(\alpha, n) = \min\{\beta : \alpha < \beta \wedge q_n < r_\beta\}$$

where $\{q_n\}_{n<\omega}$ is an enumeration the rationals. On the other hand, the following example shows that with an additional hypothesis such as $\diamondsuit$ one can construct more complex relations.

[7] Or any appropriate hypothesis; the same comment applies to Conjecture 4 below.

[8] In general these need not be optimal — they are in the class of directed relations.

Example 6.3. Let R be a tree order on a set T. If $R \leq \aleph_0 \cdot \omega_1$, then T is the union of countably many R-chains. If $[\omega_1]^{<\aleph_0} \leq R$, then T contains an uncountable R-antichain. In particular, if (T, R) is a Suslin tree, then $R \not\leq \aleph_0 \cdot \omega_1$ and $[\omega_1]^{<\aleph_0} \not\leq R$.

Observations such as these led Todorcevic to make the following conjecture.

Conjecture. *(*PFA*) If R is a binary relation, then either $R \leq \aleph_0 \cdot \omega_1$ or else $[\omega_1]^{<\aleph_0} \leq R$.*

This can be considered a basis conjecture in the following sense. Let $\mathscr{R}$ denote the class of all binary relations R on ω_1 such that R is not reducible to $\aleph_0 \cdot \omega_1$. Then the above conjecture is just the assertion that $\mathscr{R}$ has a single element basis consisting of $[\omega_1]^{<\aleph_0}$. This conjecture was given further plausibility by the following theorem which implies that it is true for transitive relations.

Theorem 6.4. [22] *(*PFA*) Every transitive relation on ω_1 is Tukey equivalent to one of the following for some nonnegative integers n_i $(i < 5)$:*

(i) $n_0 \cdot 1 \oplus n_1 \cdot \omega \oplus n_2 \cdot \omega_1 \oplus n_3 \cdot \omega \times \omega_1 \oplus n_4 \cdot [\omega_1]^{<\aleph_0}$.
(ii) $\aleph_0 \cdot 1 \oplus n_2 \cdot \omega_1 \oplus n_3 \cdot \omega \times \omega_1 \oplus n_4 \cdot [\omega_1]^{<\aleph_0}$.
(iii) $\aleph_0 \cdot \omega_1 \oplus n_4 \cdot [\omega_1]^{<\aleph_0}$.
(iv) $\aleph_0 \cdot [\omega_1]^{<\aleph_0}$.
(v) $=$.

We will now see, however, that this conjecture is provably false.

Definition 6.5. Let R denote the relation with domain and range ω_1 defined by letting $\alpha R \beta$ iff $\alpha = \beta$ or $\alpha < \beta$ and $o^*(\alpha, \beta) > 0$. If X is an uncountable subset of ω_1, let R_X denote the relation with domain X and range ω_1 which is the restriction of R to $X \times \omega_1$.

It is easily verified that the enumeration map $\varepsilon : \omega_1 \to X$ and the identity map on ω_1 witness $\omega_1 \leq R_X$ for every uncountable X. If $X \subseteq Y$, then the inclusion and identity maps witness $R_X \leq R_Y$. The following shows that none of these relations are above $[\omega_1]^{<\aleph_0}$ in the Tukey order.

Theorem 6.6. $[\omega_1]^{<\aleph_0} \not\leq R$.

Proof. Suppose that $f : [\omega_1]^{<\aleph_0} \to \omega_1$ and $g : \omega_1 \to [\omega_1]^{<\aleph_0}$ are functions. It suffices to show that they are not Tukey reductions.

Applying the pressing down lemma, find a stationary $B \subseteq \omega_1$ and a finite $x_0 \subseteq \omega_1$ such that $g(\beta) \cap \beta = x_0$ for all β in B. Let $\mathscr{X} = \{x \in [\omega_1]^{<\aleph_0} : x \not\subseteq x_0\}$ and $A = \{f(x) : x \in \mathscr{X}\}$. By refining B if necessary, we may assume that if α is in A and β is in B with $\alpha < \beta$, then there is an x in $\mathscr{X}$ such that $f(x) = \alpha$ and $x \subseteq \beta$.

If A is countable, then we can find a α in A such that

$$\{x \in \mathscr{X} : f(x) = \alpha\}$$

is uncountable. It is then easy to find an x in $\mathscr{X}$ such that x is not contained in $g(\alpha)$ but $f(x) = \alpha$ and hence $f(x) \ R \ \beta$, witnessing that f, g are not Tukey maps.

If A is uncountable, apply Theorem 5.4 to obtain an α in A and a β in B such that $\alpha < \beta$ and $o^*(\alpha, \beta) > 0$. Now pick an x in $\mathscr{X}$ such that $f(x) = \alpha$ and $x \subseteq \beta$ and observe that x is not contained in $g(\beta)$ and yet $f(x) \ R \ \beta$. $\square$

Theorem 6.7. *For all finite sets F of uncountable subsets of ω_1,*

$$\bigwedge_{X\in F} R_X \not\leq \aleph_0 \cdot \omega_1.$$

Proof. By replacing F with a disjoint refinement if necessary, we may assume without loss of generality that F consists of pairwise disjoint sets. It suffices to show that if

$$f : \prod_{X\in F} X \to \omega \times \omega_1 \qquad \text{and} \qquad g : \omega \times \omega_1 \to \prod_{X\in F} \omega_1,$$

then f and g are not Tukey maps. For convenience, we will identify elements of $\prod_{X\in F} X$ with their range. Let S denote the relation on $\omega \times \omega_1$ given by $\aleph_0 \cdot \omega_1$. Find uncountable pairwise disjoint $\mathscr{A} \subseteq [\omega_1]^{|F|}$ and an n such that:

(17) For each element a of $\mathscr{A}$, $a \cap X$ is a singleton for all X in F.
(18) $f \restriction \mathscr{A}$ is an injection into $\{n\} \times \omega_1$.
(19) The ordering on F which is induced by an element a of $\mathscr{A}$ — X is the i^{th}-least element of F if $a(X)$ is the i^{th}-least coordinate of a — does not depend on a.

This is possible since if $\prod_{X\in F} X$ is partitioned into countably many pieces, then one piece must contain an uncountable pairwise disjoint family. If the g-image of $\{n\} \times \omega_1$ is countable, then it is not possible for f and g to be Tukey maps; this is a consequence of the following argument in which $\tilde{\mathscr{B}}$ is replaced by the trivial Δ-system with a single element.

Hence we may select an uncountable $\tilde{\mathscr{B}}$ contained in the image of $\{n\}\times\omega_1$ under g such that the collection of sets of coordinates of elements of $\tilde{\mathscr{B}}$ forms a Δ-system with root r. Let $\mathscr{B}$ be the elements of this Δ-system with r removed and let k denote the uniform cardinality of the members of $\mathscr{B}$. By refining $\tilde{\mathscr{B}}$ if necessary, we may further assume that there is a function $\phi : |F| \to k \cup \{-1\}$ such that if X is the i^{th} element in the induced enumeration of F and $\tilde{b}$ is in $\tilde{\mathscr{B}}$, then $\tilde{b}(X)$ is the $\phi(i)^{\text{th}}$-least element of $\operatorname{ran}(\tilde{b}) \setminus r$ if $\tilde{b}(X)$ is not in r and $\phi(i) = -1$ otherwise. Let χ be constantly 0 on the subset of $|F|$ on which ϕ is non-negative.

Applying Theorem 5.4, it is possible to find an a in $\mathscr{A}$, a $\tilde{b}$ in $\tilde{\mathscr{B}}$, and a ζ with $g(n,\zeta) = \tilde{b}$ such that $r < a < \min(b \setminus r)$, $f(a)$ is S-related to (n,ζ), and for all X in F for which $\tilde{b}(X)$ is not in r

$$o^*(a(X), \tilde{b}(X)) = 0$$

where b denotes the set of coordinates of $\tilde{b}$.

If X is in F, then either $\tilde{b}(X)$ is in r in which case $\tilde{b}(X) < a(X)$ or else $\tilde{b}(X)$ is not in r and $c(a(X), \tilde{b}(X)) = 0$. In either case $a(X)$ is not R_X-related to $\tilde{b}(X)$. Since this is true for arbitrary X, a is not $\bigwedge_{X\in F} R_X$-related to $g(x)$ and yet $f(a)Sx$. Consequently f and g are not Tukey maps. $\square$

The next theorem shows that the relations R_X ($X \subseteq \omega_1$) are typically incomparable.

Theorem 6.8. *If $X \setminus Y$ is uncountable, then $R_X \not\leq R_Y$.*

Proof. Suppose for contradiction that $f : X \to Y$ and $g : \omega_1 \to \omega_1$ are Tukey maps. To obtain a contradiction, it suffices to find an α in X and a β in ω_1 such that $f(\alpha) < \beta$, $\alpha < g(\beta)$, $o^*\big(f(\alpha), \beta\big) = 1$, and $o^*\big(\alpha, g(\beta)\big) = 0$. Since f and g are

Tukey reductions it is possible to find a uncountable $X_0 \subseteq X \setminus Y$ and uncountable $Z_0 \subseteq \omega_1$ such that $f \restriction X_0$ and $g \restriction Z_0$ are injections, the f-image of X_0 is disjoint from X_0, the g-image of Z_0 is disjoint from Z_0 unless g is eventually equal to the identity, and the inequalities $\alpha < f(\alpha)$, $\beta = g(\beta)$, $\beta < g(\beta)$ are uniformly true or false as α ranges over X_0 and β ranges over Z_0. Put

$$\mathscr{A} = \{\{\alpha, f(\alpha)\} : \alpha \in X_0\}, \qquad \mathscr{B} = \{\{\beta, g(\beta)\} : \beta \in Z_0\}$$

and notice that these families are uncountable and consist of pairwise disjoint sets of uniform cardinality which, in the case of $\mathscr{A}$, is two. By our uniformity assumption on the relations $<$ and $=$, it is possible to find a functions $\phi : 2 \to 2$ and $\chi : 2 \to 2$ such that:

(20) $\{a(i), b(\phi(i))\}$ $(i < 2)$ lists the pairs $\{\alpha, g(\beta)\}$ and $\{f(\alpha), \beta\}$.
(21) $\chi(i) = 1$ if $\{a(i), b(\phi(i))\} = \{f(\alpha), \beta\}$ and $\chi(i) = 0$ otherwise.

Applying Theorem 5.4 to $\mathscr{A}$, $\mathscr{B}$, ϕ, and χ, there are a in $\mathscr{A}$ and b in $\mathscr{B}$ such that $a < b$ and

$$o^*\big(a(i), b(\phi(i))\big) = \chi(i).$$

Translating the outcome, it is easily checked that $\alpha \in a \cap X_0$ and $\beta \in b \cap Z_0$ are as desired. □

I will close this section with two questions.

Question 6.9. *For which families $\mathscr{F} \subseteq [\omega_1]^{\aleph_1}$ is there an S such that $S \not\leq \aleph_0 \cdot \omega_1$ and $S \leq R_X$ for all X in $\mathscr{F}$?*

Question 6.10. *Is the collection of all S with $S \not\leq \aleph_0 \cdot \omega_1$ downwards-directed in the Tukey order?*

7. An L space and the non-existence of a small basis for the regular Hausdorff spaces.

In this section I will give an example of an *L space* — a regular Hausdorff space which is hereditarily Lindelöf but not hereditarily separable. The question of the existence of such spaces was first asked explicitly in [5], though arguably this question can be traced to Sierpinski's [14] where he constructed Hausdorff examples distinguishing HS and HL and to Suslin's [15] where he posed his famous hypothesis. For instance, an immediate consequence of Kurepa's [11] is that a Suslin line is an example of an L space. From the 1960s until the 1980s, there was a concerted effort to understand both L spaces and their "dual," the S space. I refer the reader to Juhasz's [7] and M. E. Rudin's [13] as well as Roitman's more recent [12] for more discussion on these developments. I have selected a few to mention here.

First, Zenor showed that there was a relationship between the existence of certain S and L spaces.

Theorem 7.1. [27] *There is a strong S space iff there is a strong L space.*[9]

Hence the difference in the existence of S and L spaces lies in the properties of their finite powers. This gives some explanation as to why the existence of S and L spaces seem to be such similar hypotheses at first.

There are a number of results under $\mathrm{MA}_{\aleph_1}$ which limit the existence of S and L spaces.

[9] Here a space is a strong S (L) space iff all of its finite powers are S (L) spaces.

Theorem 7.2. [9] *(*$\mathrm{MA}_{\aleph_1}$*) There are no strong S or L spaces.*

Theorem 7.3. [16] *(*$\mathrm{MA}_{\aleph_1}$*) There are no first countable L spaces.*

Theorem 7.4. [2] *(*$\mathrm{MA}_{\aleph_1}$*) If K is a compact space which contains an L space, then K maps continuously onto* $[0,1]^{\aleph_1}$*.*

Finally, Todorcevic proved that S spaces do not exist assuming PFA.

Theorem 7.5. [18] *(*PFA*) If a regular Hausdorff space is hereditarily separable, then it is hereditarily Lindelöf.*

He also demonstrated that the existence of S and L spaces are different considerations.

Theorem 7.6. [21] *It is consistent with* $\mathrm{MA}_{\aleph_1}$ *that there are no S spaces and yet that there is an L space.*

Until now, however, it remained unclear whether Todorcevic's methods could be used to prove that PFA implies there are no L spaces. I will now show that this is not the case — that there is an L space which can be constructed without appealing to additional axioms of set theory.

Definition 7.7. For each $\beta < \omega_1$, define $w_\beta : \omega_1 \to \mathbb{T}$ by $w_\beta(\xi) = o(\xi, \eta)$ if $\xi < \beta$ and $w_\beta(\xi) = 1$ otherwise. Let $\mathscr{L}$ be the collection $\{w_\beta : \beta < \omega_1\}$ viewed as a subspace of $\mathbb{T}^{\omega_1}$.

The "+1" in the exponent of $o(\xi, \beta)$ is present so that $w_\beta(\xi) = 1$ iff $\xi \geq \beta$. This will simplify the discussion at certain points. Notice that $\mathscr{L}$ is non-separable.

If $X \subseteq \omega_1$, then let $\mathscr{L}_X$ be $\{w_\beta \restriction X : \beta \in X\}$ viewed as a subspace of $\mathbb{T}^X$. Since the map $w_\beta \mapsto w_\beta \restriction X$ defined on $\{w_\beta : \beta \in X\}$ is a bijection, we will write w_β for $w_\beta \restriction X$ when referring to elements of $\mathscr{L}_X$. It should be emphasized, however, that while the above bijection is continuous, its inverse may not be. Hence, even with this identification, $\mathscr{L}_X$ should not be viewed as a subspace of $\mathscr{L}$.

We will now see that $\mathscr{L}_X$ is Fréchet. Let $\mathscr{U}$ be the collection of open arcs in $\mathbb{T}$ whose endpoints are roots of 1 and which do not contain 1 as an element. Let $\mathscr{F}$ be the collection of all sets of the form $\{w \in \mathscr{L} : w(\alpha) \in U\}$ where U is an element of $\mathscr{U}$ and α is in X. Notice that $\mathscr{F}$ is point countable and that the topology on $\mathscr{L}_X$ is the smallest in which every element of $\mathscr{F}$ is clopen. The following theorem shows that $\mathscr{L}_X$ is always Fréchet.

Theorem 7.8. *If X is a set and $\mathscr{F}$ is a point countable point, separating family of subsets of X, then the topology on X defined by declaring elements of $\mathscr{F}$ to be clopen is countably tight and has every countable subspace metrizable. In particular the topology is Fréchet.*

Remark 7.9. I attribute this result to Zoltan Balogh. He once told me that "If there is an L space, then there is a countably tight one." While I never saw his proof, I think it is reasonable to assume this he may have proceeded along these lines.

Proof. That countable subspaces are metrizable follows from the assumptions and the well known fact that regular Hausdorff second countable spaces are metrizable. To see that the space is countably tight, let $A \subseteq X$ have an accumulation point x in X. Let M be a countable elementary submodel of $H(\theta)$ for θ regular and large enough so that X, $\mathscr{F}$, x, and A are all in M. It suffices to show that x is an accumulation point of $A \cap M$.

Suppose this is not the case and let U_i $(i < k)$ and V_i $(i < l)$ be elements of $\mathscr{F}$ such that

$$W = \bigcap_{i<k} U_i \setminus \bigcup_{j<l} V_j$$

contains x and is disjoint from $A \cap M$. Without loss of generality, we may assume that the empty set is not in $\mathscr{F}$. The important observation is this: if V is in $\mathscr{F}$, then $V \cap M$ is non-empty iff V is in M. This is because $\{V \in \mathscr{F} : y \in V\}$ is countable for all y in X and therefore a subset of M whenever y is in M (by Facts 5 and 7). Hence each U_i must be in M since x is in U_i and x is in M. Moreover, since we are only interested in having W be disjoint from $A \cap M$, we may assume without loss of generality that each V_j is in M. But then W must be in M and, by elementarity of M, there must be an element of $W \cap A$ which is in M, a contradiction. □

As mentioned above, if X is uncountable, then $\mathscr{L}_X$ can be shown to be an L space using Theorem 5.6 and the methods of [5]. I will prove the following theorem of independent interest and then derive this as a consequence.

Theorem 7.10. *If $X, Y \subseteq \omega_1$ have countable intersection, then there is no continuous injection from any uncountable subspace of $\mathscr{L}_X$ into $\mathscr{L}_Y$.*

Proof. Suppose for contradiction that such an injection g does exist. Then g is of the form $w_\beta \mapsto w_{f(\beta)}$ where $f : X_0 \to Y$ is an injection for some uncountable $X_0 \subseteq X$ which is, without loss of generality, disjoint from Y. For each $\xi < \omega_1$, let β_ξ and ζ_ξ be elements of X_0 and Y respectively such that $f(\beta_\xi) > \zeta_\xi$ and if $\xi < \xi'$ then $\beta_\xi < \zeta_{\xi'}$. Let $\Xi \subseteq \omega_1$ be uncountable such that for some open neighborhood V in $\mathbb{T}$, $g(w_{\beta_\xi})(\zeta_\xi)$ is not in the closure of V whenever ξ is in Ξ.

Applying continuity of g at w_{β_ξ} to

$$W_\xi = \{w \in \mathscr{L}_Y : w(\zeta_\xi) \notin \bar{V}\},$$

there is a basic open neighborhood U_ξ of w_{β_ξ} such that $U_\xi \subseteq g^{-1}W_\xi$. Applying the Δ-system lemma and the second countability of $\mathbb{T}$, there is an uncountable $\Xi' \subseteq \Xi$, open neighborhoods U_i $(i < k)$ in $\mathbb{T}$, and a_ξ in $[X]^k$ such that for all ξ in Ξ':

(1) $\{a_\xi : \xi \in \Xi'\}$ is a Δ-system with root a.
(2) The set
$$\{w \in \mathscr{L}_X : \forall i < k\big(w(a_\xi(i)) \in U_i\big)\}$$
has w_{β_ξ} as an element and is a subset of U_ξ.
(3) The inequality $\beta_\xi < f(\beta_\xi)$ does not depend on ξ.
(4) $|\zeta_\xi \cap a_\xi|$ does not depend on ξ.

Let $\mathscr{A}$ be the collection of all $a_\xi \cup \{\zeta_\xi\} \setminus a$ and $\mathscr{B}$ be the collection of all $\{\beta_\xi, f(\beta_\xi)\}$. Applying Theorem 5.6, it is possible to find $\xi < \xi'$ in Ξ' such that for all $i < k$

$$a_\xi \cup \{\zeta_\xi\} < \min(\beta_{\xi'}, f(\beta_{\xi'})),$$
$$w_{\beta_{\xi'}}(a_\xi(i)) = o(a_\xi(i), \beta_{\xi'}) \in U_i,$$
$$g(w_{\beta_{\xi'}}) = w_{f(\beta_{\xi'})}(\zeta_\xi) = o(\zeta_\xi, f(\beta_{\xi'})) \in V.$$

But now $w_{\beta_{\xi'}}$ is an element of U_ξ even though $g(w_{\beta_{\xi'}})$ is not in W_ξ, contradicting our choice of U_ξ. This finishes the proof. □

Corollary 7.11. *For every X, $\mathscr{L}_X$ is hereditarily Lindelöf.*

Proof. If not, then $\mathscr{L}_X$ would contain an uncountable discrete subspace. It is then possible to find disjoint $Y, Z \subseteq X$ such that $\mathscr{L}_Y$ and $\mathscr{L}_Z$ each contain uncountable discrete subspaces. But any function from a discrete space to another discrete space is continuous and we can then easily contradict Theorem 7.10. $\square$

Kunen has shown, however, that under $\mathrm{MA}_{\aleph_1}$ every L space contains an uncountable discrete subspace in one of its finite powers [9]. As might be expected, this happens at the first possible instance in our example.

Theorem 7.12. *For every uncountable $X \subseteq \omega_1$, ${\mathscr{L}_X}^2$ contains an uncountable discrete subspace.*

Proof. This essentially follows from the following observation which is of independent interest.

Proposition 7.13. *The tree*

$$T(o) = \{o(\cdot, \beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

is Aronszajn.

Proof. Theorem 5.6 implies that $T(o)$ does not contain an uncountable branch. It suffices to prove that all levels of $T(o)$ are countable. For this, it is sufficient to show that $T(\varrho_1)$ and $T(L)$ have countable levels. In the case of $T(\varrho_1)$, this is the content of Remark 2.8. In order to see that $T(L)$ has countable levels, let $\alpha < \omega_1$ be given. Let $\beta < \omega_1$ be greater than α. Using the compactness of $\alpha + 1$ and Fact 2, select a finite set $F_\beta \subseteq \alpha + 1$ containing 0 and α so that if $\gamma_0 < \gamma$ are consecutive elements of F_β, then $L(\gamma, \beta) < L(\xi, \gamma)$ whenever $\gamma_0 < \xi < \gamma$. It suffices to show that if

$$L(\cdot, \beta) \restriction F_\beta = L(\cdot, \beta') \restriction F_{\beta'},$$

then

$$L(\cdot, \beta) \restriction \alpha = L(\cdot, \beta') \restriction \alpha.$$

To see this, let ξ be an arbitrary element of $\alpha \setminus F_\beta$ and pick $\gamma_0 < \gamma$ in F_β so that $\gamma_0 < \xi < \gamma$. Applying Fact 1 we have

$$L(\xi, \beta) = L(\xi, \gamma) \cup L(\gamma, \beta) = L(\xi, \gamma) \cup L(\gamma, \beta') = L(\xi, \beta').$$

$\square$

To finish the proof of Theorem 7.12, select a sequence $(\beta^0_\xi, \beta^1_\xi)$ indexed by an uncountable set $\Xi \subseteq \omega_1$ such that for each ξ in Ξ the following conditions are satisfied:

(22) $\beta^0_\xi < \beta^1_\xi$ are both elements of X.
(23) There is a fixed $\varepsilon > 0$ such that

$$|w_{\beta^0_\xi}(\beta^0_\xi) - w_{\beta^1_\xi}(\beta^0_\xi)| = |1 - w_{\beta^1_\xi}(\beta^0_\xi)| \geq \varepsilon.$$

(24) If $\eta < \xi$ is in Ξ, then $\beta^0_\eta < \xi$.
(25) $w_{\beta^0_\xi} \restriction \xi = w_{\beta^1_\xi} \restriction \xi$.

This is possible since $T(o)$ has countable levels. Now consider the open neighborhoods

$$U_\xi = \{(u, v) \in \mathscr{L}_X : |w_{\beta^0_\xi}(\beta^0_\xi) - u(\beta^0_\xi)| + |v(\beta^0_\xi) - w_{\beta^1_\xi}(\beta^0_\xi)| < \varepsilon\}$$

of $(w_{\beta^0_\xi}, w_{\beta^1_\xi})$ in $\mathscr{L}_X{}^2$ for ξ in Ξ. If $\eta < \xi$ are in Ξ, then by (23) $(w_{\beta^0_\eta}, w_{\beta^1_\eta})$ is not in U_ξ since $w_{\beta^1_\eta}(\beta^0_\xi) = 1$. If $\xi < \eta$ are in Ξ, then by (24) $\beta^0_\xi < \eta$ and by (25)

$$w_{\beta^0_\eta}(\beta^0_\xi) = w_{\beta^1_\eta}(\beta^0_\xi).$$

Consequently $(w_{\beta^0_\eta}, w_{\beta^1_\eta})$ being in U_ξ would contradict (23) via the triangle inequality. Hence $(w_{\beta^0_\eta}, w_{\beta^1_\eta})$ is in U_ξ iff $\eta = \xi$ and therefore $\{(w_{\beta^0_\xi}, w_{\beta^1_\xi}) : \xi \in \Xi\}$ is an uncountable discrete subspace of $\mathscr{L}_X{}^2$ as desired. □

At this point, I will make the following conjecture.

Conjecture 4. *(*PFA*) If X is a non-separable, regular Hausdorff space, then X^2 contains an uncountable discrete subspace.*

While $\mathscr{L}$ is 0-dimensional, its closure in $\mathbb{T}^{\omega_1}$ is very far from being 0-dimensional.

Theorem 7.14. *If $E \subseteq \mathscr{L}$ is uncountable, then the closure of E in $\mathbb{T}^{\omega_1}$ contains a set of the form $\{f \in \mathbb{T}^{\omega_1} : f \restriction \alpha = \tau\}$ for some $\alpha < \omega_1$ and τ in $\mathbb{T}^\alpha$. In particular,* $\mathrm{cl}(E)$ *is not 0-dimensional.*

Remark 7.15. This example may be informative in studying the following problem of M. E. Rudin: Is it consistent that every locally connected perfect compact space is metrizable? See [4].

Proof. Let $E \subseteq \mathscr{L}$ be uncountable and put $T(E)$ be the set of all τ in $T(o)$ such that for uncountably many β in E, τ is a restriction of w_β. If τ is in $T(E)$, put E_τ to be the set of all w_β in E such that τ is a restriction of w_β.

Claim 7.16. *For each τ in $T(E)$ there is an $\alpha_\tau < \omega_1$ such that the projection map $\pi : \mathbb{T}^{\omega_1} \to \mathbb{T}^{\omega_1 \setminus \alpha_\tau}$ sends the closure of E_τ onto $\mathbb{T}^{\omega_1 \setminus \alpha_\tau}$.*

Proof. Suppose for contradiction that this is not the case. For each $\xi < \omega_1$, select a basic open neighborhood $W_\xi \subseteq \mathbb{T}^{\omega_1}$ such that $W_\xi \cap E_\tau$ is empty and membership to W_ξ depends only on coordinates greater than ξ. By refining the sequence W_ξ ($\xi < \omega_1$) it is possible to find open neighborhoods U_i ($i < k$) in $\mathbb{T}$ and an uncountable pairwise disjoint $\mathscr{A} \subseteq [\omega_1]^k$ such that if a is in $\mathscr{A}$, then there is no w_β in E_τ with

$$w_\beta(a(i)) \in U_i$$

for all $i < k$. Pairing $\mathscr{A}$ with $\mathscr{B} = \{\{\beta\} : w_\beta \in E_\tau\}$, we have a contradiction to Theorem 5.6. □

Let $\delta < \omega_1$ be such that if τ is an element of $T(E)$ of height less than δ, then $\alpha_\tau < \delta$. Since $T(o)$ has countable levels, such a δ exists. Since $T(E)$ is uncountable and downwards closed, there is a τ in $T(E)$ of height δ. Put

$$F = \bigcap_{\xi < \delta} \mathrm{cl}(E_{\tau \restriction \xi}).$$

Notice that the projection of each $\mathrm{cl}(E_{\tau\restriction\xi})$ onto the set of coordinates $\omega_1 \setminus \delta$ is onto $\mathbb{T}^{\omega_1 \setminus \delta}$. Since the fibers of this map are compact and since the intersection is decreasing, F must project onto $\mathbb{T}^{\omega_1 \setminus \delta}$. Since every element of F is an extension of τ,

$$F = \{f \in \mathbb{T}^{\omega_1} : f \text{ extends } \tau\}$$

is the desired subset of $\mathrm{cl}(E)$. □

The proof of Theorem 7.3 can be used to prove the following.

Theorem 7.17. *For every $X \subseteq \omega_1$, no uncountable subspace of $\mathscr{L}_X$ is first countable.*

Proof. The proof of Theorem 7.3 actually shows that if an L space is first countable, then there is a c.c.c. forcing which destroys it. It is easily checked that every uncountable subspace of $\mathscr{L}_X$ is an L space in every c.c.c. forcing extension. □

We also have the following consequence of Proposition 7.13.

Theorem 7.18. *If f is a continuous function from a subspace of $\mathscr{L}$ into a metric space, then the range of f is countable.*

Proof. Since $\mathscr{L} \subseteq \mathbb{T}^{\omega_1}$ is hereditarily Lindelöf and f is continuous, f depends only on a countable set of coordinates. The conclusion of the theorem now follows from Proposition 7.13. □

Theorem 7.19. *Any basis for the uncountable regular Hausdorff spaces must have cardinality greater than $\aleph_1$.*

Proof. The first conclusion follows from Theorem 7.10 and the observation that there is an almost disjoint family of uncountable subsets of ω_1 of cardinality $\aleph_2$. □

I will close this paper by referring the reader to [4] where a number of open problems related to this line of research are collected and discussed.

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Abstract. In this paper I will construct a non-separable hereditarily Lindelöf space (L space) without any additional axiomatic assumptions. The constructed space $\mathscr{L}$ is a subspace of $\mathbb{T}^{\omega_1}$ where $\mathbb{T}$ is the unit circle. It is shown to have a number of properties which may be of additional interest. For instance it is shown that the closure in $\mathbb{T}^{\omega_1}$ of any uncountable subset of $\mathscr{L}$ contains a canonical copy of $\mathbb{T}^{\omega_1}$.

I will also show that there is a function $f : [\omega_1]^2 \to \omega_1$ such that if $A, B \subseteq \omega_1$ are uncountable and $\xi < \omega_1$, then there are $\alpha < \beta$ in A and B respectively with $f(\alpha, \beta) = \xi$. Previously it was unknown whether such a function existed even if ω_1 was replaced by 2. Finally, I will prove that there is no basis for the uncountable regular Hausdorff spaces of cardinality $\aleph_1$.

The results all stem from the analysis of oscillations of coherent sequences $\langle e_\beta : \beta < \omega_1 \rangle$ of finite-to-one functions. I expect that the methods presented will have other applications as well.

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