A FIVE ELEMENT BASIS FOR THE UNCOUNTABLE LINEAR ORDERS

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Abstract. In this paper I will show that it is relatively consistent with the usual axioms of mathematics (ZFC) together with a strong form of the axiom of infinity (the existence of a supercompact cardinal) that the class of uncountable linear orders has a five element basis. In fact such a basis follows from the Proper Forcing Axiom, a strong form of the Baire Category Theorem. The elements are X, ω_1, ω_1^*, C, C^* where X is any suborder of the reals of cardinality $\aleph_1$ and C is any Countryman line. This confirms a longstanding conjecture of Shelah.

1. Introduction

Our focus in this paper will be to show that the Proper Forcing Axiom (PFA) implies that any uncountable linear order must contain an isomorphic copy of one of the following five orders: X, ω_1, ω_1^*, C, and C^*. Here X is any fixed set of reals of cardinality $\aleph_1$ and C is any fixed Countryman line. Such a list is called a *basis*.

The simplest example of an uncountable linear order is $\mathbb{R}$, the real line. This object serves as the prototype for the class of linear orders and as the canonical example of an uncountable set. Early on in modern set theory, Baumgartner proved the following deep result which suggested that it might be possible to prove more general classification results for uncountable linear orders.

Theorem 1.1. [3] *(*PFA*) If two sets of reals are $\aleph_1$-dense,*[1] *then they are isomorphic. In particular if X is a set of reals of cardinality $\aleph_1$,*

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[1] I.e. every interval meets them at a set of cardinality $\aleph_1$.

then X serves as a single element basis for the class of uncountable separable linear orders.

PFA is a strengthening of the Baire Category Theorem and is independent of the usual axioms of set theory. Frequently — as in Baumgartner's result above — this axiom can be used to find morphisms between certain structures or to make other combinatorial reductions (see [1], [3], [7], [24], [25], [27]). Some additional assumption is necessary in Baumgartner's result because of the following classical construction of Sierpiński.

Theorem 1.2. [19] *There is a set of reals X of cardinality continuum such that if $f \subseteq X^2$ is a continuous injective function, then f differs from the identity function on a set of cardinality less than continuum.*

From this it is routine to prove that under the Continuum Hypothesis there is no basis for the uncountable separable linear orders of cardinality less than $|\mathscr{P}(\mathbb{R})|$. This gives a complete contrast to the conclusion of Baumgartner's result.

The simplest example of a linear order which is separable only in the trivial instances is a well order. The uncountable well orders have a canonical minimal representative, the ordinal ω_1.[2] Similarly, the converse ω_1^* of ω_1 obtained by reversing the order relation forms a single element basis for all of the uncountable converse well orders.

Those uncountable linear orders which do not contain uncountable separable suborders or copies of ω_1 or ω_1^* are called *Aronszajn lines.*[3] They are classical objects considered long ago by Aronszajn and Kurepa who first proved their existence. Some time later Countryman made a brief but important contribution to the subject by asking whether there is an uncountable linear order C whose square is the union of countably many chains.[4] Such an order is necessarily Aronszajn. Furthermore, it is easily seen that no uncountable linear order can embed into both a Countryman line and its converse. Shelah proved that such orders exist in ZFC [16] and made the following conjecture:

Shelah's Conjecture. [16] *(*PFA*) Every Aronszajn line contains a Countryman suborder.*

This soon developed into the following equivalent basis conjecture — see [22].

[2] The canonical representation of well orders mentioned here is due to von Neumann.

[3] Or *Specker types.*

[4] Here *chain* refers to the coordinate-wise partial order on C^2.

Conjecture. [4] *(*PFA*) The orders X, ω_1, ω_1^*, C and C^* form a five element basis for the uncountable linear orders any time X is a set of reals of cardinality $\aleph_1$ and C is a Countryman line.*

Notice that by our observations such a basis is necessarily minimal.

This problem was exposited, along with some other basis problems for uncountable structures, in Todorčević's [21]. It also appears as Question 5.1 in Shelah's problem list [18]. A related and inspirational analysis of Aronszajn trees was also carried out in [22].

In this paper I will prove Shelah's conjecture. In doing so, I will introduce some new methods for applying PFA which may be relevant to solving other problems. I would like to thank Ilijas Farah, Jean Larson, Paul Larson, Bill Mitchell, and Boban Veličković for carefully reading the paper and offering their suggestions and comments. I would also like to thank Jörg Brendle for supporting my visit to Japan where I presented the results of this paper in a series of lectures at Kobe University in December 2003.

This work is dedicated to Fennel Moore.

2. Background

This paper should be readily accessible to anyone who is well versed in set theory and the major developments in the field in the 70s and 80s. The reader is assumed to have proficiency in the areas of Aronszajn tree combinatorics, forcing axioms, the combinatorics of $[X]^{\aleph_0}$, and Skolem hull arguments. Jech's [12] and Kunen's [14] serve as good references on general set theory. They both contain some basic information on Aronszajn trees; further reading on Aronszajn trees can be found in [5], [20], and [26]. The reader is referred to [6], [16], [22], [23], or [26] for information on Countryman lines. It should be noted, however, that knowledge of the method of minimal walks will not be required.

The set theoretic assumption we will be working with is the Proper Forcing Axiom. We will be heavily utilizing Todorcevic's method of building proper forcings using models as side conditions. Both [25] and the section on PFA in [24] serve as good concise references on the subject. See [15] for information on the Mapping Reflection Principle. For basic forcing technology, the reader is referred to [11] and [14]. Part III of Jech's [11] gives a good exposition on the combinatorics of $[X]^{\aleph_0}$, the corresponding closed unbounded (or *club*) filter, and related topics.

The notation in this paper is mostly standard. If X is an uncountable set, then $[X]^{\aleph_0}$ will be used to denote the collection of all countable subsets of X. All ordinals are von Neumann ordinals — they are the set of their predecessors under the $\in$ relation. The collections $H(\theta)$ for regular cardinals θ consist of those sets of hereditary cardinality less than θ. Hence $H(2^{\theta^+})$ contains $H(\theta^+)$ as an element and $\mathscr{P}(H(\theta^+))$ as a subset. Often when I refer to $H(\theta)$ in this paper I will really be referring to the structure $(H(\theta), \in, \triangleleft)$ where $\triangleleft$ is some fixed well ordering of $H(\theta)$ which can be used to generate the Skolem functions.

3. The axioms

The working assumption in this paper will be the Proper Forcing Axiom introduced by Shelah and proved relatively consistent from a supercompact cardinal. We will often appeal to the bounded form of this axiom isolated by Goldstern and Shelah [9]. We will use an equivalent formulation due to Bagaria [2]:

BPFA: If ϕ is a formula in language of $H(\aleph_1^+)$ with only bounded quantifiers and there is a proper partial order which forces $\exists X \phi(X)$, then $H(\aleph_1^+)$ already satisfies $\exists X \phi(X)$.

At a crucial point in the proof we will also employ the Mapping Reflection Principle introduced recently in [15]. In order to state it we will need the following definitions.

Definition 3.1. If X is an uncountable set, then there is a natural topology — the *Ellentuck topology* — on $[X]^{\aleph_0}$ defined by declaring

$$[x, N] = \{Y \in [X]^{\aleph_0} : x \subseteq Y \subseteq N\}$$

to be open whenever N is in $[X]^{\aleph_0}$ and x is a finite subset of N.

This topology is regular and 0-dimensional. Moreover, the closed and cofinal sets generate the club filter on $[X]^{\aleph_0}$.

Definition 3.2. If M is an elementary submodel of some $H(\theta)$ and X is in M, then we say a subset $\Sigma \subseteq [X]^{\aleph_0}$ is *M-stationary* if whenever $E \subseteq [X]^{\aleph_0}$ is a club in M, the intersection $\Sigma \cap E \cap M$ is non-empty.

Definition 3.3. If Σ is a set mapping defined on a set of countable elementary submodels of some $H(\theta)$ and there is an X such that $\Sigma(M) \subseteq [X]^{\aleph_0}$ is open and M-stationary for all M, then we say Σ is an *open stationary set mapping.*

The Mapping Reflection Principle is the following statement:

MRP: If Σ is an open stationary set mapping defined on a club of models, then there is a continuous $\in$-chain $\langle N_\xi : \xi < \omega_1 \rangle$ in the domain of Σ such that for every $\nu > 0$ there is a $\nu_0 < \nu$ such that $N_\xi \cap X$ is in $\Sigma(N_\nu)$ whenever $\nu_0 < \xi < \nu$.

The sequence $\langle N_\xi : \xi < \omega_1 \rangle$ postulated by this axiom will be called a *reflecting sequence* for the set mapping Σ.

4. A combinatorial reduction

Rather than prove Shelah's basis conjecture directly, I will appeal to the following reduction.

Theorem 4.1. *(*BPFA*) The following are equivalent:*

1. *The uncountable linear orders have a five element basis.*
2. *There is an Aronszajn tree T such that for every $K \subseteq T$ there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$*[5] *is either contained in or disjoint from K.*

Remark. This result seems to essentially be folklore; reader interested in the historical aspects of this are referred to p 79 of [1], [4], [16]. A detailed proof of this theorem can be found in the last section of [22]. I will sketch the proof for completeness.

The implication *(1) implies (2)* does not require BPFA and in fact (1) implies that the conclusion of (2) holds for an arbitrary Aronszajn tree T. To see why it is true, suppose that $(T, \leq)$ is an Aronszajn tree equipped with a lexicographical order and suppose that $K \subseteq T$ witnesses a failure of (2). If $(T, \leq)$ doesn't contains a Countryman suborder, then (1) must fail. So without loss of generality, we may assume that $(T, \leq)$ is Countryman.

Define $s \leq' t$ iff $s \wedge t$ is in K and $s \leq t$ or $s \wedge t$ is not in K and $t \leq s$. It is sufficient to check that neither $(T, \leq)$ nor its converse $(T, \geq)$ embeds an uncountable suborder of $(T, \leq')$. This is accomplished with two observations. First, since $(T, \leq)$ and its converse are Countryman, any such embedding can be assumed to be the identity map. Second, if $\leq$ and $\leq'$ agree on $X \subseteq T$, then $\wedge(X) \subseteq K$; disagreement on X results in $\wedge(X) \cap K = \emptyset$.

For the implication *(2) implies (1)* we first observe that, by Baumgartner's result mentioned above, it suffices to show that the Aronszajn lines have a two element basis. Fix a Countryman line C which is a lexicographical order $\leq$ on an Aronszajn tree T. The club isomorphism of Aronszajn trees under BPFA [1] together with some further appeal to $\mathrm{MA}_{\aleph_1}$ implies that any Aronszajn line contains a suborder isomorphic to some $(X, \leq')$ where $X \subseteq T$ is uncountable and binary and $\leq'$ is a — possibly different — lexicographical order on T. Statement (2) is used to compare $\leq$ and $\leq'$ and find an uncountable $Y \subseteq X$ on which these orders always agree or always disagree. Applying $\mathrm{MA}_{\aleph_1}$, C embeds into all its uncountable suborders, thus finishing the proof.

[5]This will be defined momentarily.

5. The proof of the main result

In this section we will prove the basis conjecture of Shelah by proving the following result and appealing to Theorem 4.1.

Theorem 5.1. (PFA) *There is an Aronszajn tree T such that if $K \subseteq T$, then there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ is either contained in or disjoint from K.*

The proof will be given as a series of lemmas. In each case, I will state any set theoretic hypothesis needed to prove a lemma. This is not so much to split hairs but because I feel that it will help the reader better understand the proof.

For the duration of the proof, we will let T be a fixed Aronszajn tree which is contained in the complete binary tree, coherent, closed under finite changes, and special.[6] It will be convenient to first make some definitions and fix some notation.

Definition 5.2. If s and t are two elements of T, then $\mathrm{diff}(s,t)$ is the set of all ξ such that $s(\xi)$ and $t(\xi)$ are defined and not equal. If $F \subseteq T$, then $\mathrm{diff}(F)$ is the union of all $\mathrm{diff}(s,t)$ such that s and t are in F. The *coherence* of T is the assertion that $\mathrm{diff}(s,t)$ is a finite set for all s, t in T.

Definition 5.3. If X is a subset of T and $\delta < \omega_1$, then $X \restriction \delta$ is the set of all $t \restriction \delta$ such that t is in X. Here $t \restriction \delta$ is just functional restriction.

Definition 5.4. If s and t are in T, then $\Delta(s,t)$ is the least element of $\mathrm{diff}(s,t)$. If s and t are comparable, we leave $\Delta(s,t)$ undefined.[7] If $Z \subseteq T$ and t is in T, then $\Delta(Z,t) = \{\Delta(s,t) : s \in Z\}$.

Definition 5.5. If X is a finite subset of T, then $X(j)$ will denote the j^{th} least element of X in the lexicographical order inherited from T.

Definition 5.6. If s, t are incomparable in T, then the meet of s and t — denoted $s \wedge t$ — is the restriction $s \restriction \Delta(s,t) = t \restriction \Delta(s,t)$. If X is a subset of T, then $\wedge(X) = \{s \wedge t : s, t \in X\}$.[8]

The following definition provides a useful means of measuring subsets of an elementary submodel's intersection with ω_1.

[6] The tree $T(\rho_3)$ of [23] is such an example. Coherence is defined momentarily.

[7] This is somewhat non-standard but it will simplify the notation at some points. For example, in the definition of $\Delta(Z,t)$ we only collect those values where Δ is defined.

[8] The domain of $\wedge$ is the same as the domain of Δ; the set of all incomparable pairs of elements of T.

Definition 5.7. If P is a countable elementary submodel of $H(\aleph_1^+)$ containing T as an element, define $\mathscr{I}_P(T)$ to be the collection of all $I \subseteq \omega_1$ such that for some uncountable $Z \subseteq T$ in P and some t of height $P \cap \omega_1$ which is in the downward closure of Z, the set $\Delta(Z, t)$ is disjoint from I.

The following propositions are routine to verify using the coherence of T and its closure under finite changes (compare to the proof that $\mathscr{U}(T)$ is a filter in [22] or [26]).

Proposition 5.8. *If I is in $\mathscr{I}_P(T)$ and t is in T with height $P \cap \omega_1$, then there is a $Z \subseteq T$ in P such that t is in the downward closure[9] of Z and $\Delta(Z, t)$ is disjoint from I.*

Proposition 5.9. *If I is in $\mathscr{I}_P(T)$, Z_0 is a subset of T in P and t is an element of the downward closure of Z_0 of height $P \cap \omega_1$, then there is a $Z \subseteq Z_0$ in P which also contains t in its downward closure and satisfies $\Delta(Z, t) \cap I$ is empty.*

Proposition 5.10. *$\mathscr{I}_P(T)$ is a proper ideal on ω_1 which contains $I \subseteq \omega_1$ whenever $I \cap P$ is bounded in $\omega_1 \cap P$.*

Proposition 5.11. *Suppose P is a countable elementary submodel of $H(\aleph_1^+)$ such that $Z \subseteq T$ is an element of P, and there is a $t \in T$ of height $P \cap \omega_1$ in the downward closure of Z. Then Z is uncountable.*

Let $K \subseteq T$ be given. The following definitions will be central to the proof. The first is the naïve approach to forcing an uncountable X such that $\wedge(X)$ is contained in K.

Definition 5.12. $\mathscr{H}(K)$ is the collection of all finite $X \subseteq T$ such that $\wedge(X)$ is contained in K.[10]

It is worth noting that $\mathscr{H}(K)$ is the correct forcing to work with if K is union of levels of T; this is demonstrated in [22]. This and other successes in [22] emboldened me to attempt the more general case in which K is an arbitrary subset of T.

The second is the notion of rejection which will be central in the analysis of $\mathscr{H}(K)$. For convenience we will let $\mathscr{E}$ denote the collection of all clubs $E \subseteq [H(\aleph_1^+)]^{\aleph_0}$ which consist of elementary submodels which contain T and K as elements. Let E_0 denote the element of $\mathscr{E}$ which consists of all such submodels.

[9]The *downward closure* of Z is the collection of all s such that $s \leq s^*$ for some s^* in Z

[10]A collection of finite sets such as this becomes a forcing notion when given the order of reverse inclusion ($q \leq p$ means that q is stronger than p). A collection of ordered pairs of finite sets becomes a forcing by coordinate-wise reverse inclusion.

Definition 5.13. If X is a finite subset of T, then let $K(X)$ denote the set of all $\gamma < \omega_1$ such that for all t in X, if γ is less than the height of t, then $t \restriction \gamma$ is in K.

Definition 5.14. If P is in E_0 and X is a finite subset of T, then we say that P *rejects* X if $K(X)$ is in $\mathscr{I}_P(T)$.

The following trivial observations about P in E_0 and finite $X \subseteq T$ are useful and will be used tacitly at times in the proofs which follow.

Proposition 5.15. *If P does not reject X, then it does not reject any of its restrictions $X \restriction \gamma$.*

Proposition 5.16. *P rejects X iff it rejects $X \restriction (P \cap \omega_1)$ iff it rejects $X \setminus P$.*

Proposition 5.17. *If X is in P, then P does not reject X.*

The forcing notion $\partial(K)$ which we are about to define seeks to add a subset of T in which rejection is rarely encountered.[11]

Definition 5.18. $\partial(K)$ consists of all pairs $p = (X_p, \mathscr{N}_p)$ such that:

(1) $\mathscr{N}_p$ is a finite $\in$-chain such that if N is in $\mathscr{N}$, then T and K are in N and N is the intersection of a countable elementary submodel of $H(2^{2^{\aleph_1}+})$ with $H(2^{\aleph_1 +})$.

(2) $X_p \subseteq T$ is a finite set and if N is in $\mathscr{N}_p$, then there is an E in $\mathscr{E} \cap N$ such that X_p is not rejected by any element of $E \cap N$.

We will also be interested in the suborder

$$\partial\mathscr{H}(K) = \{p \in \partial(K) : X_p \in \mathscr{H}(K)\}$$

which seems to be the correct modification of $\mathscr{H}(K)$ from the point of view of forcing the conclusion of the main theorem.

In order to aid in the presentation of the lemmas, I will make the following definition.

Definition 5.19. $\partial(K)$ is *canonically proper* if whenever M is a countable elementary submodel of $H\big(|2^{\partial(K)}|^+\big)$ and $\partial(K)$ is in M, any condition p which satisfies $M \cap H(2^{\aleph_1 +})$ is in $\mathscr{N}_p$ is $(M, \partial(K))$-generic. An analogous definition is made for $\partial\mathscr{H}(K)$.

We will eventually prove that, assuming the Proper Forcing Axiom, $\partial\mathscr{H}(K)$ is canonically proper. The following lemma shows that this is sufficient to finish the argument.

[11]The symbol ∂ is being used here because there is a connection to the notion of a Cantor-Bendixon derivative. In a certain sense we are removing the parts of the partial order $\mathscr{H}(K)$ which are causing it to be improper.

Lemma 5.20. *(*BPFA*) If $\partial\mathscr{H}(K)$ is canonically proper, then there is an uncountable $X \subseteq T$ such that $\wedge(X)$ is either contained in K or disjoint from K.*

Remark. This conclusion is sufficient since the properties of T imply that X contains an uncountable antichain.

Proof. Let M be a countable elementary submodel of $H\big(|2^{\partial\mathscr{H}(K)}|^+\big)$ containing $\partial\mathscr{H}(K)$ as an element. Let t be an element of T of height $M \cap \omega_1$. If

$$p = \big(\{t\}, \{M \cap H(2^{\aleph_1}{}^+)\}\big)$$

is a condition in $\partial\mathscr{H}(K)$, then it is $(M, \partial\mathscr{H}(K))$-generic by assumption. Consequently p forces that the interpretation of

$$\dot{X} = \{s \in T : \exists q \in \dot{G}(s \in X_q)\}$$

is uncountable. Since $\dot{X}$ will then be forced to have the property that $\wedge(\dot{X}) \subseteq \check{K}$, we can apply BPFA to find such an X in V.

Now suppose that p is not a condition. It follows that there is a countable elementary submodel P of $H(\aleph_1^+)$ in M such that T is in P and $K(\{t\})$ is in $\mathscr{I}_P(T)$. Therefore there is a $Z \subseteq T$ in P such that $t \restriction (P \cap \omega_1)$ is in the downward closure of Z and for all s in Z, $s \wedge t$ is not in K. Let Y consist of all those w in $\wedge(Z)$ such that if u, v are incomparable elements of Z and $u \wedge v \leq w$, then $u \wedge v$ is not in K. Notice that Y is an element of P. Y is uncountable since it contains $s \wedge t$ for every s in $P \cap Z$ which is incomparable with t and the heights of elements of this set are easily seen to be unbounded in $P \cap \omega_1$. We are therefore finished once we see that $\wedge(Y)$ is disjoint from K. To this end, suppose that w_0 and w_1 are incomparable elements of Y. Let u_0, u_1, v_0, v_1 be elements of Z such that u_i and v_i are incomparable and $w_i = u_i \wedge v_i$. Since w_0 and w_1 are incomparable,

$$u_0\big(\Delta(w_0, w_1)\big) = w_0\big(\Delta(w_0, w_1)\big) \neq w_1\big(\Delta(w_0, w_1)\big) = v_1\big(\Delta(w_0, w_1)\big).$$

It follows that $u_0 \wedge v_1 = w_0 \wedge w_1$. Since w_0 extends $u_0 \wedge v_1$ and is in Y, it must be that $u_0 \wedge v_1$ is not in K. Hence $w_0 \wedge w_1$ is not in K. This completes the proof that $\wedge(Y)$ is disjoint from K. □

The following lemma is the reason for our definition of rejection. It will be used at crucial points in the argument.

Lemma 5.21. *Suppose that E is in $\mathscr{E}$ and $\langle X_\xi : \xi < \omega_1 \rangle$ is a sequence of disjoint n-element subsets of T so that no element of E rejects any X_ξ for $\xi < \omega_1$. Then there are $\xi \neq \eta < \omega_1$ such that $X_\xi(j) \wedge X_\eta(j)$ is in K for all $j < n$.*

Proof. By the pressing down lemma we can find a $\zeta < \omega_1$ and a stationary set $\Xi \subseteq \omega_1$ such that:

(1) For all ξ in Ξ, X_ξ contains only elements of height at least ξ.
(2) $X_\xi(j) \restriction \zeta = X_\eta(j) \restriction \zeta$ for all $j < n$ and $\xi, \eta \in \Xi$.
(3) For all ξ in Ξ the set $\operatorname{diff}(X_\xi \restriction \xi)$ is contained in ζ.

Now let P be an element of E which contains $\langle X_\xi : \xi \in \Xi \rangle$. Let η be an element of Ξ outside of P and pick a ξ in $\Xi \cap P$ such that $X_\xi(0) \restriction \xi$ and $X_\eta(0) \restriction \eta$ are incomparable and for all $j < n$

$$X_\eta(j) \restriction \Delta\big(X_\eta(0), X_\xi(0)\big)$$

is in K. This is possible since otherwise $Z = \{X_\xi(0) \restriction \xi : \xi \in \Xi\}$ and $t = X_\eta(0) \restriction (P \cap \omega_1)$ would witness $K(X_\eta)$ is in $\mathscr{I}_P(T)$ and therefore that P rejects X_η.

Notice that if $j < n$, then

$$\Delta\big(X_\eta(j), X_\xi(j)\big) = \Delta\big(X_\eta(0), X_\xi(0)\big)$$

since

$$\operatorname{diff}(X_\xi \restriction \xi) \cup \operatorname{diff}(X_\eta \restriction \eta) \subseteq \zeta,$$
$$X_\eta(j) \restriction \zeta = X_\xi(j) \restriction \zeta.$$

Hence the meets

$$X_\xi(j) \wedge X_\eta(j) = X_\eta(j) \restriction \Delta\big(X_\xi(0), X_\eta(0)\big)$$

are in K for all $j < n$. $\square$

The next lemma draws the connection between $\partial\mathscr{H}(K)$ and the forcing $\partial(K)$. We will then spend the remainder of the paper analyzing $\partial(K)$.

Lemma 5.22. *(*BPFA*) If $\partial(K)$ is canonically proper, so is $\partial\mathscr{H}(K)$.*

Proof. We will show that otherwise the forcing $\partial(K)$ introduces a counterexample to Lemma 5.21 which would then exist in V by an application of BPFA. Let M be a countable elementary submodel of $H\big(|2^{\partial(K)}|^+\big)$ which contains K as an element and let $r \in \partial\mathscr{H}(K)$ be such that $M \cap H(2^{\aleph_1{}^+})$ is in $\mathscr{N}_r$ and yet r is not $(M, \partial\mathscr{H}(K))$-generic. By extending r if necessary, we may assume that there is a dense open set $\mathscr{D} \subseteq \partial\mathscr{H}(K)$ in M which contains r such that if q is in $\mathscr{D} \cap M$, then q is $\partial\mathscr{H}(K)$-incompatible with r.

Let $E \in \mathscr{E} \cap M$ be such that no element of $E \cap M$ rejects X_r and let E' be the elements of E which are the union of their intersection with E. Put $Y_r = (X_r \setminus M) \restriction (M \cap \omega_1)$.

Claim 5.23. *No element of E' rejects Y_r.*

Proof. First observe that no element of $E' \cap M$ rejects Y_r; the point is to generalize this to arbitrary elements of E'. Let P be an element of E'. We need to verify that $K(Y_r)$ is not in $\mathscr{I}_P(T)$. If $P \cap \omega_1$ is greater than $M \cap \omega_1$, then $Y_r \subseteq P$ and this is trivial. Now suppose that $Z \subseteq T$ is in P and t is an element of T of height $P \cap \omega_1$ which is in the downward closure of Z. Let P_0 be an element of $E \cap P$ which contains Z as a member. Such a P_0 will satisfy

$$P_0 \cap \omega_1 < P \cap \omega_1 \leq M \cap \omega_1.$$

Let $\nu = P_0 \cap \omega_1$. If $\Delta\big(Z, t \restriction (P \cap \omega_1)\big)$ is disjoint from $K(Y_r)$, then it witnesses $K(Y_r \restriction \nu)$ is in $\mathscr{I}_{P_0}(T)$. But then we could use the elementarity of M to find such a P_0 in $M \cap E$, which is contrary to our choice of E. Hence no element of E' rejects Y_r. $\square$

Let $\zeta \in M \cap \omega_1$ be an upper bound for $\operatorname{diff}(Y_r)$ and let $n = |Y_r|$. If $j < n$, let $A_j \subseteq T$ be an antichain in M which contains $Y_r(j)$.[12] Put $\mathscr{D}_*$ to be the collection of all q in $\mathscr{D}$ such that:

(4) $X_r \cap M = X_q \cap N(q)$ where $N(q)$ is the least element of $\mathscr{N}_q$ which is not in $\mathscr{N}_r \cap M$.
(5) $Y_q \restriction \zeta = Y_r \restriction \zeta$ where $Y_q = \big(X_q \setminus N(q)\big) \restriction \big(N(q) \cap \omega_1\big)$.
(6) No element of E' rejects Y_q.
(7) $Y_q(j)$ is in A_j whenever $j < n$.

Note that $\mathscr{D}_*$ is in M.

Let G be a $\partial(K)$-generic filter which contains r. Notice that r is $(M, \partial(K))$-generic. Working in $V[G]$, let $\mathscr{F}$ be the collection of all Y_q where q is in $\mathscr{D}_* \cap G$. Now $M[G \cap M]$ is an elementary submodel of $H\big(|2^{\partial(K)}|^+\big)[G]$[13] which contains $\mathscr{F}$ as an element but not as a subset (since Y_r is in $\mathscr{F}$). Therefore $\mathscr{F}$ is uncountable. Notice that every element of $\mathscr{F}$ has the property that it is in $\mathscr{H}(K)$ but that for every countable $\mathscr{F}_0 \subseteq \mathscr{F}$ there is a Y_q in $\mathscr{F} \setminus \mathscr{F}_0$ such that $Y_q \cup Y_{q_0}$ is not in $\mathscr{H}(K)$ for any Y_{q_0} in $\mathscr{F}_0$. This follows from the elementarity of $M[G \cap M]$ and from the fact that $Y_r \cup Y_q$ is not in $\mathscr{H}(K)$ for any Y_q in $\mathscr{F} \cap M[G \cap M]$. Now it is possible to build an uncountable sequence $\langle X_\xi : \xi \in \Xi \rangle$ of elements of $\mathscr{F}$ such that:

(8) X_ξ has size n for all $\xi \in \Xi$ and is a subset of the ξ^{th} level of T.
(9) $X_\xi \cup X_\eta$ is not in $\mathscr{H}(K)$ whenever $\xi \neq \eta$ are in Ξ.
(10) There is a $\zeta < \omega_1$ such that $X_\xi \restriction \zeta = X_\eta \restriction \zeta$ has size n for all $\xi, \eta < \omega_1$.

[12] Here we are using that T is special.
[13] By Theorem 2.11 of [17].

It follows from item 9 that if $\xi < \eta < \omega_1$, then there are $j, j' < n$ such that $X_\xi(j) \wedge X_\eta(j')$ is not in K. By item 10, it must be the case that $j = j'$ since this condition ensures that

$$X_\xi(j) \wedge X_\eta(j') = X_\xi(j) \wedge X_\xi(j')$$

whenever $j \neq j' < n$ and hence this meet would be in K by virtue of X_ξ being in $\mathscr{H}(K)$. Applying BPFA we get a sequence of sets satisfying 1–3 in V and therefore a contradiction to Lemma 5.21 since no elements of $\mathscr{F}$ are rejected by any member of E'. Hence $\partial\mathscr{H}(K)$ must also be canonically proper. □

Next we have a typical "models as side conditions" lemma.

Lemma 5.24. *If $\partial(K)$ is not canonically proper, then there are disjoint sets $\mathscr{A}$, $\mathscr{B}$ and a function $Y : \mathscr{A} \cup \mathscr{B} \to [T]^{<\aleph_0}$ such that:*

1. *$\mathscr{A} \subseteq [H({2^{\aleph_1}}^+)]^{\aleph_0}$ and $\{N \cap H(\aleph_1^+) : N \in \mathscr{A}\}$ is stationary.*
2. *$\mathscr{B} \subseteq [H({2^{2^{\aleph_1}}}^+)]^{\aleph_0}$ is stationary.*
3. *If M is in $\mathscr{A} \cup \mathscr{B}$, then $\big(Y(M), \{M \cap H({2^{\aleph_1}}^+)\}\big)$ is in $\partial(K)$.*
4. *For every M in $\mathscr{B}$ and N in $\mathscr{A} \cap M$, $\big(Y(N) \cup Y(M), \{N\}\big)$ is not a condition in $\partial(K)$.*

Proof. Let M be a countable elementary submodel of $H\big(|2^{\partial(K)}|^+\big)$ and r in $\partial(K)$ be a condition which is not $(M, \partial(K))$-generic such that $M \cap H({2^{\aleph_1}}^+)$ is in $\mathscr{N}_r$. By extending r if necessary, we can find dense open $\mathscr{D} \subseteq \partial(K)$ in M which contains r such that no element of $\mathscr{D} \cap M$ is compatible with r. Furthermore we may assume that if q is in $\mathscr{D}$, N is in $\mathscr{N}_q$, and t is in X_q, then $t \restriction (N \cap \omega_1)$ is also in X_q.

Define $r_0 = (X_r \cap M, \mathscr{N}_r \cap M)$. If q is in $\mathscr{D}$, let $N(q)$ be the $\in$-least element of $\mathscr{N}_q \setminus \mathscr{N}_{r_0}$. Let $k = |\mathscr{N}_r \setminus \mathscr{N}_{r_0}|$ and ζ be the maximum of all ordinals of the forms $\mathrm{ht}(s) + 1$ for $s \in X_{r_0}$ and $N \cap \omega_1$ for $N \in \mathscr{N}_{r_0}$. Let $\mathscr{T}_k$ be the set of all $q \leq r_0$ in $\mathscr{D}$ such that:

(11) $X_q \cap N(q) = X_{r_0}$ and $\mathscr{N}_q \cap N(q) = \mathscr{N}_{r_0}$.
(12) $X_q \restriction \zeta = X_r \restriction \zeta$.
(13) $|X_q| = |X_r| = m$ and $|\mathscr{N}_q \setminus \mathscr{N}_{r_0}| = k$.

Let $N_i(q)$ denote the i^{th} $\in$-least element of $\mathscr{N}_q \setminus N(q)$ and define $\mathscr{T}_i$ recursively for $i \leq k$. Given $\mathscr{T}_{i+1}$, define $\mathscr{T}_i$ to be the collection of all q such that

$$\{N_{i+1}(q^*) \cap H(\aleph_1^+) : q^* \in \mathscr{T}_{i+1} \text{ and } q = q^* \restriction N_{i+1}(q^*)\}$$

is stationary where

$$q^* \restriction N_{i+1}(q^*) = \big(X_{q^*} \cap N_{i+1}(q^*), \mathscr{N}_{q^*} \cap N_{i+1}(q^*)\big).$$

Let $\mathscr{T}$ be the collection of all q in $\bigcup_{i \leq k} \mathscr{T}_i$ such that if q is in $\mathscr{T}_i$, then $q \restriction N_{i'+1}(q)$ is in $\mathscr{T}_{i'}$ for all $i' < i$.

Claim 5.25. *r is in $\mathscr{T}$.*

Proof. If q is in $\partial(K)$, define

$$\tilde{q} = \big(X_q, \{N \cap H(\aleph_1^+) : N \in \mathscr{N}_q\}\big).$$

While elements of $\mathscr{N}_r \setminus M$ need not contain $\mathscr{T}_i$ as an element for a given $i \leq k$, they do contain $\tilde{\mathscr{T}}_i = \{\tilde{q} \in \mathscr{T}_i\}$ as an element for each $i \leq k$. Define $r_k = r$ and $r_i = r_{i+1} \restriction N_{i+1}(r)$. Suppose that r_{i+1} is in $\mathscr{T}_{i+1}$. Since $\tilde{\mathscr{T}}_{i+1}$ and r_i are in $N_{i+1}(r) = N_{i+1}(r_{i+1})$ and since $N_{i+1}(r) \cap H(\aleph_1^+)$ is in every club in $\mathscr{E} \cap N_{i+1}(r)$, it follows by elementarity of $N_{i+1}(r)$ that the set

$$\{N_{i+1}(q^*) \cap H(\aleph_1^+) : q^* \in \mathscr{T}_{i+1} \text{ and } r_i = q^* \restriction N_{i+1}(q^*)\} =$$
$$\{N_{i+1}(\tilde{q}^*) : \tilde{q}^* \in \tilde{\mathscr{T}}_{i+1} \text{ and } \tilde{r}_i = \tilde{q}^* \restriction N_{i+1}(\tilde{q}^*)\}$$

is stationary. Hence r_i is in $\mathscr{T}_i$. □

Notice that $\mathscr{T}$ is in M. $\mathscr{T}$ has a natural tree order associated with it induced by restriction. Since no element of $\mathscr{T}_k \cap M$ is compatible with r and since r_0 is in $\mathscr{T} \cap M$, there is a q in $\mathscr{T} \cap M$ which is maximal in the tree order such that q is compatible with r but such that none of q's immediate successors in $\mathscr{T} \cap M$ are compatible with r. Let l denote the height of q in $\mathscr{T}$ and put $\mathscr{A}$ to be equal to the set of all $N_{l+1}(q^*)$ such that q^* is an immediate successor of q in $\mathscr{T}$. Notice that if q^* is in $\mathscr{T}_{l+1}$ and q is a restriction of q^*, then q^* is in $\mathscr{T}$. Hence we have arranged that $\{N \cap H(\aleph_1^+) : N \in \mathscr{A}\}$ is stationary. For each N in $\mathscr{A}$, select a fixed q^* which is an immediate successor of q in $\mathscr{T}$ such that $N_{l+1}(q^*) = N$ and put

$$Y(N) = X_{q^*} \setminus X_q.$$

Claim 5.26. *For all N in $\mathscr{A} \cap M$ the pair $\big(X_r \cup Y(N), \{N\}\big)$ is not a condition in $\partial(K)$.*

Proof. Let N be in $\mathscr{A} \cap M$ and fix an immediate successor q^* of q in $\mathscr{T}$ such that $N_{l+1}(q^*) = N$ and $Y(N) = X_{q^*} \setminus X_q$. Observe that

$$(X_r \cup X_{q^*}, \mathscr{N}_{q^*} \cup \mathscr{N}_r)$$

is not a condition in $\partial(K)$ but that

$$(X_r \cup X_q, \mathscr{N}_q \cup \mathscr{N}_r)$$

is a condition. Furthermore, $(X_r \cup X_{q^*}, \mathscr{N}_{q^*} \cup \mathscr{N}_r)$ fails to be a condition only because it violates item 2 in the definition of $\partial(K)$. Observe that

$\mathscr{N}_{q^*} \setminus \mathscr{N}_q = \{N\}$. If N' is an element of $\mathscr{N}_r \cup \mathscr{N}_q$, then the sets of restrictions

$$\{t \restriction (N' \cap \omega_1) : t \in X_r \cup X_{q^*}\},$$
$$\{t \restriction (N' \cap \omega_1) : t \in X_r \cup X_q\}$$

are equal by definitions of $\mathscr{T}_l$ and q and by our initial assumptions about the closure of X_q for q in $\mathscr{D}$ under taking certain restrictions. Since $(X_r \cup X_q, \mathscr{N}_r \cup \mathscr{N}_q)$ is a condition, such an N' cannot witness the failure of 2. Therefore it must be the case that the reason $(X_r \cup X_{q^*}, \mathscr{N}_{q^*} \cup \mathscr{N}_r)$ is not in $\partial(K)$ is that N witnesses a failure of item 2. Now, the elements of X_{q^*} which have height at least $N \cap \omega_1$ are exactly those in $Y(N) = X_{q^*} \setminus X_q$. This finishes the claim. □

Notice that by elementarity of M, $Y \restriction \mathscr{A}$ can be chosen to be in M. Now M satisfies "There is a stationary set of countable elementary submodels M_* of $H(2^{2^{\aleph_1}+})$ such that for some $Y(M_*)$ with $\big(Y(M_*), \{M_* \cap H(2^{\aleph_1 +})\}\big)$ in $\partial(K)$ we have that for every N in $\mathscr{A} \cap M_*$ the pair $\big(Y(N) \cup Y(M_*), \{N\}\big)$ is not a condition in $\partial(K)$." By elementarity of M, we are finished. □

The following definition will be useful.

Definition 5.27. A function h is a *level map* if its domain is a subset of ω_1 and $h(\delta)$ is a finite subset of the δ^{th} level of T whenever it is defined.

The next proposition is useful and follows easily from the fact that all levels of T are countable.

Proposition 5.28. *If $N \cap H(\aleph_1^+)$ is in E_0, $\delta = N \cap \omega_1$, and X is a finite subset of the δ^{th} level of T, then there is a level map h in N such that $h(\delta) = X$.*

The next lemma will represent the only use of MRP in the proof.

Lemma 5.29. (MRP) *Suppose that M is a countable elementary submodel of $H(2^{2^{\aleph_1}+})$ which contains T and K as members. If X is a finite subset of T, then there is an E in $\mathscr{E} \cap M$ such that either every element of $E \cap M$ rejects X or no element of $E \cap M$ rejects X.*

Remark. Notice that the latter conclusion is just a reformulation of the statement that $\big(X, \{M \cap H(2^{\aleph_1 +})\}\big)$ is a condition in $\partial(K)$.

Proof. Let $\delta = M \cap \omega_1$. Without loss of generality, we may assume that $X = X \restriction \delta$. Applying Proposition 5.28, select a level map g in M such that $g(\delta) = X$. If N is a countable elementary submodel of $H(2^{\aleph_1 +})$

with T and K as members, define $\Sigma(N)$ as follows. If the set of all P in E_0 which reject $g(N \cap \omega_1)$ is N-stationary, then put $\Sigma(N)$ to be equal to this set unioned with the complement of E_0. If $\Sigma(N)$ is defined in this way, it will be said to be defined *non-trivially*. Otherwise put $\Sigma(N)$ to be the interval $[\emptyset, N \cap H(\aleph_1^+)]$.

Observe that Σ is an open stationary set mapping which is moreover an element of M. Applying MRP and the elementarity of M, it is possible to find a reflecting sequence $\langle N_\xi : \xi < \omega_1 \rangle$ for Σ which is an element of M. Let E be the collection of all $\overline{P}$ in E_0 which contain

(14) the sequence $\langle N_\xi \cap H(\aleph_1^+) : \xi < \omega_1 \rangle$ and
(15) some $\delta_0 < N \cap \omega_1$ such that $N_\xi \cap H(\aleph_1^+)$ is in $\Sigma(N_\delta)$ whenever ξ is in (δ_0, δ).

Notice that E is in $M \cap \mathscr{E}$.

To finish the proof, suppose that the set of all P in $M \cap E_0$ which reject X is M-stationary (i.e. the second conclusion does not hold).

Claim 5.30. *$\Sigma(N_\delta)$ is defined non-trivially.*

Proof. Suppose that $E' \subseteq E_0$ is a club in N_δ. Since the reflecting sequence is continuous, N_δ is a subset of M and therefore E' is also in M. By assumption, there is a P in $E' \cap M$ such that P rejects X. Let $\nu = P \cap \omega_1$ and note that $\nu < \delta$. Applying elementarity of N_δ, Proposition 5.16, and the fact that $X \restriction \nu$ is in N_δ, it is possible to find such a P in $E' \cap N_\delta$ which rejects $X \restriction \nu$ — and hence X. It follows that $\Sigma(N_\delta)$ is defined non-trivially. $\square$

Now suppose that $\overline{P}$ is in $E \cap M$. We are finished once we see that $\overline{P}$ rejects X. Let $\nu = \overline{P} \cap \omega_1$. Since $\delta_0 < \nu < \delta$, $P_\nu = N_\nu \cap H(\aleph_1^+)$ is in $\Sigma(N_\delta)$. So P_ν rejects X or — equivalently — $K(X)$ is in $\mathscr{I}_{P_\nu}(T)$. Observe that $P_\nu \cap \omega_1 = \overline{P} \cap \omega_1$ and $P_\nu \subseteq \overline{P}$ by continuity of the reflecting sequence. Hence $\mathscr{I}_P(T) \subseteq \mathscr{I}_{\overline{P}}(T)$. It follows that $\overline{P}$ rejects X. $\square$

The next lemma finishes the proof of the main theorem.

Lemma 5.31. *($\mathrm{MRP} + \mathrm{MA}_{\aleph_1}$) There are no $\mathscr{A}$, $\mathscr{B}$, and Y which satisfy the conclusion of Lemma 5.24. In particular, $\partial(K)$ is canonically proper.*

Proof. We will assume that there are such $\mathscr{A}$, $\mathscr{B}$, and Y and derive a contradiction by violating Lemma 5.21. Without loss of generality we may suppose that elements of $\mathscr{B}$ contain $\mathscr{A}$ as a member. By modifying Y we may assume that all elements of $Y(M)$ have height $M \cap \omega_1$ whenever M is in $\mathscr{A} \cup \mathscr{B}$. Further, we may assume that $Y(M)$

has the same fixed size n for all M in $\mathscr{B}$ and that there is a ζ_0 and $E_* \in \mathscr{E}$ such that:

(16) If M is in $\mathscr{B}$, then $\operatorname{diff}(Y(M)) \subseteq \zeta_0$.
(17) If M, M' are in $\mathscr{B}$, then $Y(M) \restriction \zeta_0 = Y(M') \restriction \zeta_0$.
(18) If M is in $\mathscr{B}$, then E_* is in M and no element of E_* rejects $Y(M)$.

This is achieved by the pressing down lemma and the proof of Claim 5.23.[14] Let $\mathscr{F}$ be the collection of all finite $X \subseteq T$ such that all elements of X have the same height $\gamma > \zeta_0$ and the set

$$\{M \in \mathscr{B} : Y(M) \restriction \gamma = X\}$$

is stationary. Notice that, for a fixed γ, we can define $\mathscr{B}$ to be a union over the finite subsets X of T_γ of the collection

$$\mathscr{B}[X] = \{M \in \mathscr{B} : Y(M) \restriction \gamma = X\}$$

and hence at least one such $\mathscr{B}[X]$ must be stationary. Consequently $\mathscr{F}$ must be uncountable. Also, no element of E_* rejects any element of $\mathscr{F}$. Now define $\mathscr{Q}$ to be the collection of all finite $F \subseteq \mathscr{F}$ such that if $X \neq X'$ are in F, then the heights of elements of X and X' are different and there is a $j < n$ such that $X(j) \wedge X'(j)$ is not in K.

Claim 5.32. *(*MRP*) $\mathscr{Q}$ satisfies the countable chain condition.*

Proof. Suppose that $\langle F_\xi : \xi < \omega_1 \rangle$ is a sequence of distinct elements of $\mathscr{Q}$. We will show that $\{F_\xi : \xi < \omega_1\}$ is not an antichain in $\mathscr{Q}$. By a Δ-system argument, we may assume that the sequence consists of disjoint sets of the same cardinality m. Let $F_\xi(i)$ denote the i^{th}-least element of F_ξ in the order induced by T's height function. If $j < n$, let $F_\xi(i,j)$ denote the j^{th} element of $F_\xi(i)$ in the lexicographical order on $F_\xi(i)$ (i.e. $F_\xi(i,j) = F_\xi(i)(j)$).

Let N be an element of $\mathscr{A}$ which contains ζ_0 and $\langle F_\xi : \xi < \omega_1 \rangle$ as members. Put $\delta = N \cap \omega_1$ and fix a β in $\omega_1 \setminus N$. Let E be a club in N such that $Y(N)$ is not rejected by any element of $E \cap N$.

For each $i < m$, pick an M_i in $\mathscr{B}$ such that $N \in M_i$ and $F_\beta(i)$ is a restriction of $Y(M_i)$. Let M be a countable elementary submodel of $H(2^{2^{\aleph_1}+})$ such that $N = M \cap H(2^{\aleph_1+})$. Applying Lemma 5.29 to M for each $i < m$ and intersecting clubs,[15] it is possible to find a P in

[14]To get the last item, find an E_1 in $\mathscr{E}$ such that no element of $E_1 \cap M$ rejects $Y(M)$ for stationary many M in $\mathscr{B}$, put E_* to be the elements P of E_1 which are equal to the union of their intersection with E_1.

[15]This is the only place where Lemma 5.29 and hence MRP is applied.

$E \cap N$ such that

$$I = \bigcup_{i<m} K\big(Y(N) \cup Y(M_i)\big) \in \mathscr{I}_P(T).$$

Put $\nu = P \cap \omega_1$. Pick a $\zeta < \nu$ such that $\operatorname{diff}(\cup F_\beta \restriction \nu)$ is contained in ζ and if u, v are distinct elements of $\cup F_\beta \restriction \nu$, then $u \restriction \zeta$ and $v \restriction \zeta$ are distinct.

Subclaim 5.33. *There is a sequence $\langle \alpha_\xi : \xi < \omega_1 \rangle$ in P such that for each $\xi < \omega_1$ we have the following conditions:*

1. $\xi \leq \alpha_\xi$,
2. $\cup F_{\alpha_\xi} \restriction \zeta = \cup F_\beta \restriction \zeta$,
3. $\operatorname{diff}(\cup F_{\alpha_\xi} \restriction \xi) = \operatorname{diff}(\cup F_\beta \restriction \nu)$, *and*
4. $\cup F_{\alpha_\nu} \restriction \nu = \cup F_\beta \restriction \nu$.

Proof. The only part which is non-trivial is to get the sequence to be a member of P and to satisfy item 4. By Proposition 5.28, there is a level map g in P such that $g(\nu) = \cup F_\beta \restriction \nu$. Now working in P, we can define α_ξ to be an ordinal such that $\cup F_{\alpha_\xi} \restriction \xi = g(\xi)$ if $g(\xi)$ is defined, is a restriction of this form, and satisfies $\operatorname{diff}(g(\xi)) = \operatorname{diff}(\cup F_\beta \restriction \nu)$ and $g(\xi) \restriction \zeta = \cup F_\beta \restriction \zeta$. If α_ξ is left undefined, then simply select a α_ξ with the necessary properties. Notice that α_ν is defined using g. □

Subclaim 5.34. *There is an uncountable $\Xi \subseteq \omega_1$ in P such that if*

$$Z = \{F_{\alpha_\xi}(0,0) \restriction \xi : \xi \in \Xi\}$$

and $t = F_\beta(0,0) \restriction \nu$, then t is in the downwards closure of Z and $\Delta(Z, t)$ is disjoint from

$$I = \bigcup_{i<m} K\big(Y(N) \cup Y(M_i)\big).$$

Proof. By Proposition 5.9 there is a $\Xi_0 \subseteq \omega_1$ such that for some t_0 in T of height ν in the downward closure of $Z_0 = \{F_{\alpha_\xi}(0,0) \restriction \xi : \xi \in \Xi_0\}$ the set $\Delta(Z_0, t_0)$ is disjoint from I. Let Z_1 be all elements s in T obtained from some $F_{\alpha_\xi}(0,0) \restriction \xi$ by changing its values on the finite set

$$\operatorname{diff}\big(F_{\alpha_\nu}(0,0) \restriction \nu, t_0\big) \cap \xi.$$

Let Ξ be the collection of all ξ such that $F_{\alpha_\xi}(0,0) \restriction \xi$ is an initial part of some element of Z_1. Notice that Ξ is in P and is uncountable since it contains ν. Furthermore, if $t = F_{\alpha_\nu}(0,0) \restriction \nu = F_\beta(0,0) \restriction \nu$, then $\Delta(Z, t)$ is contained in

$$\Delta(Z_1, t) = \Delta(Z_0, t_0)$$

and hence is disjoint from I. □

The key observation — and the reason why the main theorem goes through — is the following. Since P is in $E \cap N$, it does not reject $Y(N)$ and therefore it is the case that there is a ξ in $\Xi \cap P$ such that for all $j < |Y(N)|$ the restriction

$$Y(N)(j) \restriction \Delta\big(F_{\alpha_\xi}(0,0), t\big)$$

is in K where $t = F_\beta(0,0) \restriction \nu$. By the choice of Ξ this means that for all $i < m$ there is a $j < n$ such that

$$F_\beta(i,j) \restriction \Delta\big(F_{\alpha_\xi}(0,0), t\big) = F_\beta(i,j) \wedge F_{\alpha_\xi}(i,j)$$

is not in K. Let $\alpha = \alpha_\xi$.

Now we claim that $F_\alpha \cup F_\beta$ is in $\mathscr{Q}$. To see this, suppose that $i, i' < m$. If $F_\beta(i) \restriction \nu \neq F_\beta(i') \restriction \nu$, then pick a $j < n$ such that $F_\alpha(i,j) \wedge F_\alpha(i',j)$ is not in K. Since

$$\Delta\big(F_\alpha(i',j), F_\beta(i',j)\big) \geq \zeta > \Delta\big(F_\alpha(i,j), F_\alpha(i',j)\big)$$

it must be the case that

$$\Delta\big(F_\alpha(i,j), F_\beta(i',j)\big) = \Delta\big(F_\alpha(i,j), F_\alpha(i',j)\big)$$

and so

$$F_\alpha(i,j) \wedge F_\beta(i',j) = F_\alpha(i,j) \wedge F_\alpha(i',j)$$

is not in K.

If $F_\beta(i) \restriction \nu = F_\beta(i') \restriction \nu$, then we have that for all $j < n$ that

$$\Delta\big(F_\alpha(i,j), F_\beta(i',j)\big) = \Delta\big(F_\alpha(i,j), F_\beta(i,j)\big) = \Delta\big(F_\alpha(0,0), F_\beta(0,0)\big).$$

By arrangement there is a j such that

$$F_\alpha(i,j) \wedge F_\beta(i,j) = F_\beta(i,j) \restriction \Delta\big(F_\alpha(0,0), F_\beta(0,0)\big)$$

is not in K. Hence for all $i, i' < m$ there is a $j < n$ such that

$$F_\alpha(i,j) \wedge F_\beta(i',j)$$

is not in K and therefore we have that $F_\alpha \cup F_\beta$ is in $\mathscr{Q}$. $\square$

Applying $\mathrm{MA}_{\aleph_1}$ to the forcing $\mathscr{Q}$ it is possible to find an uncountable $\mathscr{F}_0 \subseteq \mathscr{F}$ such that whenever $X \neq X'$ are in $\mathscr{F}_0$, there is a $j < n$ such that $X(j) \wedge X'(j)$ is not in K. This contradicts Lemma 5.21 since no element of E_* rejects any element of $\mathscr{F}$. $\square$

6. Closing remarks

The use of MRP in the argument above is restricted to proving Lemma 5.29. Working from a stronger assumption, the following abstract form of the lemma can be deduced. The interested reader is encouraged to supply a proof and see why a stronger assumption is apparently needed for the abstract statement while MRP suffices in the proof of Lemma 5.29.

0-1 law for open set mappings. *(*SMRP[16]*) Suppose that Σ is an open set mapping defined on a club and that Σ has the following properties:*

1. *If N is in the domain of Σ, then $\Sigma(N)$ is closed under end extensions.*[17]
2. *If N and $\overline{N}$ are in the domain of Σ and $\overline{N}$ is an end extension of N, then $\Sigma(N) = \Sigma(\overline{N}) \cap N$.*

Then for a closed unbounded set of N in the domain of Σ, there is a club $E \subseteq [X_\Sigma]^{\aleph_0}$ in N such that $E \cap N$ is either contained in or disjoint from $\Sigma(N)$.

It seems quite possible that this *0-1 law* will be useful in analyzing related problems such as Fremlin's problem on perfectly normal compacta (see [10], [21]).

The conventional wisdom had been that if it were possible to prove the consistent existence of a five element basis for the uncountable linear orders, then such a basis would follow from BPFA. MRP has considerable consistency strength [15], while BPFA can be forced if there is a reflecting cardinal [9]. The following is left open.

Question 6.1. *Does* BPFA *imply Shelah's conjecture?*

Recently König, Larson, Veličković, and I have shown that a certain saturation property of Aronszajn trees taken together with BPFA implies Shelah's conjecture [13]. This saturation property can be forced if there is a Mahlo cardinal. This considerably reduces the upper bound on the consistency strength of Shelah's conjecture to that of a reflecting Mahlo cardinal. It is possible, however, that Shelah's conjecture cannot follow from BPFA simply on grounds of its consistency strength. It should be remarked though that Shelah's conjecture is not known to have any large cardinal strength.

[16]SMRP is the *Strong Mapping Reflection Principle* obtained by replacing "club" in the statement of MRP with "projective stationary" (see [8]). This axiom follows from Martin's Maximum via the same proof that MRP follows from PFA (see [15]).

[17]Here we define $\overline{N}$ *end extends* N as meaning that $N \cap \omega_1 = \overline{N} \cap \omega_1$ and $N \subseteq \overline{N}$.

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