THE PROPER FORCING AXIOM, PRIKRY FORCING, AND THE SINGULAR CARDINALS HYPOTHESIS

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Abstract. The purpose of this paper is to present some results which suggest that the Singular Cardinals Hypothesis follows from the Proper Forcing Axiom. What will be proved is that a form of simultaneous reflection follows from the Set Mapping Reflection Principle, a consequence of PFA. While the results fall short of showing that MRP implies SCH, it will be shown that MRP implies that if SCH fails first at κ then every stationary subset of $S^\omega_{\kappa^+} = \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \omega\}$ reflects. It will also be demonstrated that MRP always fails in a generic extension by Prikry forcing.

1. Introduction

A stationary subset S of a regular cardinal θ is said to *reflect* if there is a $\delta < \theta$ of uncountable cofinality such that $S \cap \delta$ is stationary in δ. Similarly, a family $\mathscr{F}$ of stationary sets is said to *simultaneously reflect* if there is a $\delta < \theta$ of uncountable cofinality such that $S \cap \delta$ is stationary in δ for every S in $\mathscr{F}$. Notice the cofinality of δ acts as an upper bound for the number of disjoint subsets of θ which can simultaneously reflect at δ.

Reflection and simultaneous reflection have been widely studied in set theory with a number of applications to areas such as cardinal arithmetic, descriptive set theory, and infinitary combinatorics. Our starting point will be the following theorem of Foreman, Magidor, and Shelah.

Theorem 1.1. [3] *Martin's Maximum implies that for every uncountable regular cardinal $\theta > \omega_1$ and every collection $\mathscr{F}$ of ω_1 many stationary subsets of $S^\omega_\theta = \{\alpha < \theta : \mathrm{cf}(\alpha) = \omega\}$ there is a $\delta < \theta$ of cofinality ω_1 which simultaneously reflects every element of $\mathscr{F}$. Moreover, it can be arranged that the union of $\mathscr{F}$ contains a club in δ.*

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Since for every regular uncountable θ there is a partition of S^ω_θ into disjoint stationary sets, they conclude that MM implies that $\theta^{\omega_1} = \theta$ for all regular $\theta \geq \omega_2$. By Silver's theorem [11] this in turn implies the Singular Cardinals Hypothesis — that $2^\kappa = \kappa^+$ for every singular strong limit κ.

In this paper, we will introduce and explore a new notion of reflection called *trace reflection* and prove a result analogous to Theorem 1.1.

Theorem 1.2. *(*MRP*) Suppose that $\Omega \subseteq S^\omega_\theta$ is a non-reflecting stationary set and that $\vec{C}$ avoids Ω. If $\mathscr{F}$ is a collection of stationary subsets of Ω and $\mathscr{F}$ has size ω_1 then there is a $\delta < \theta$ of cofinality ω_1 such that every element of $\mathscr{F}$ simultaneously trace reflects at δ.*

It will follow that MRP implies any failure of SCH must occur first at a singular cardinal κ such that every stationary subset of $S^\omega_{\kappa^+}$ reflects. The above theorem also has the following corollary.

Corollary 1.3. *Suppose that $M \subseteq V$ is an inner model with the same cardinals such that for some cardinal κ*

(1) $\mathrm{cf}(\kappa)^V = \omega < \mathrm{cf}(\kappa)^M = \kappa$ and

(2) every stationary subset of κ^+ in M is stationary in V.

Then MRP *fails in V. In particular,* MRP *fails in any generic extension by Prikry forcing.*[1]

This paper is intended to be self contained. Section 2 contains the definition of trace reflection and all of the necessary background on Todorčević's trace function. Section 3 provides the necessary background on the Set Mapping Reflection Principle which will figure prominently in the analysis. The main results then follow in Section 4.

The notation in the paper is mostly standard. All ordinals are von Neumann ordinals — the set of their predecessors. $H(\theta)$ is the collection of all sets of hereditary cardinality less than θ. If X is an uncountable set, $[X]^\omega$ is used to denote all countable subsets of X. See [5] or [6] for more background; see [4] for some information on the combinatorics of $[X]^\omega$, the club filter, and stationary subsets of $[X]^\omega$.

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[1]When I submitted this paper I was under the impression that it was unknown whether PFA always failed in a Prikry extension. Since the acceptance of this paper I have been made aware that this was not the case. Magidor has shown in an unpublished note that, by a slight modification of an argument of Todorčević, PFA implies $\square_{\kappa,\omega_1}$ fails for all $\kappa > \omega_1$. On the other hand, Cummings and Schimmerling have shown in [2] that after Prikry forcing at κ, $\square_{\kappa,\omega}$ and hence $\square_{\kappa,\omega_1}$ always holds.

2. Trace Reflection

In this section I will define trace reflection. First recall Todorčević's notion of a walk on a given cardinal θ (see [12] or [13]). A *C-sequence* is a sequence $\vec{C} = \langle C_\alpha : \alpha < \theta \rangle$ where θ is an ordinal, C_α is closed and cofinal in α for limit ordinals α, and $C_{\alpha+1} = \{\alpha\}$. A C-sequence $\vec{C}$ is said to *avoid* a subset $\Omega \subseteq \theta$ if C_α is disjoint from Ω for every limit $\alpha < \theta$. Notice that if $\Omega \subseteq \theta$ is a non-reflecting stationary set then there is a C-sequence on θ which avoids Ω. Conversely, any $\Omega \subseteq \theta$ which is avoided by a C-sequence cannot reflect.

For a given C-sequence, the *trace function* is defined recursively by

$$\mathrm{tr}(\alpha, \alpha) = \{\}$$

$$\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\beta\}.$$

Hence $\mathrm{tr}(\alpha, \beta)$ contains all ordinals "visited" in the walk from β down to α along the C-sequence except for the destination α.[2] The following property of the trace function captures some of its most important properties.

Fact 2.1. *If $\alpha < \beta$ and α is a limit then there is an $\alpha_0 < \alpha$ such that*

$$\mathrm{tr}(\alpha, \beta) \subseteq \mathrm{tr}(\gamma, \beta)$$

whenever $\alpha_0 < \gamma < \alpha$. If $\vec{C}$ avoids $\{\alpha\}$ then it can further be arranged that

$$\mathrm{tr}(\alpha, \beta) \cup \{\alpha\} \subseteq \mathrm{tr}(\gamma, \beta).$$

Proof. First, observe that if ξ is in $\mathrm{tr}(\alpha, \beta)$ then either $C_\xi \cap \alpha$ bounded or else α is in C_ξ. Furthermore, the latter can only occur if ξ is the least element of $\mathrm{tr}(\alpha, \beta)$. If $\alpha_0 < \alpha$ is an upper bound for every set $C_\xi \cap \alpha$ such that ξ is in $\mathrm{tr}(\alpha, \beta)$ and $\alpha \notin C_\xi$, then it is easily checked that α_0 has the desired properties (use induction on β). Such a bound exists since $\mathrm{tr}(\alpha, \beta)$ is finite. Finally, if $\vec{C}$ avoids $\{\alpha\}$ then α is not in C_ξ for any ξ in $\mathrm{tr}(\alpha, \beta)$. It is therefore possible to prove the stronger conclusion in this case. □

Let θ be an ordinal of uncountable cofinality. For a given C-sequence $\vec{C}$ of length θ, define $\mathscr{H}(\vec{C})$ to be the collection of all $X \subseteq \theta$ such that whenever $E \subseteq \theta$ is closed and unbounded, there are $\alpha < \beta$ in E with $\mathrm{tr}(\alpha, \beta) \cap X \neq \emptyset$. Clearly the complement of $\mathscr{H}(\vec{C})$ is a σ-ideal.

We say an element X of $\mathscr{H}(\vec{C})$ *trace reflects with respect to $\vec{C}$* if there is a $\delta < \theta$ of uncountable cofinality such that $X \cap \delta$ is in $\mathscr{H}(\vec{C} \upharpoonright \delta)$.

[2]The omission of the destination is not standard, but it simplifies the presentation at some points. For instance, Fact 2.1 requires this omission.

Simultaneous trace reflection is defined in a similar manner. If $\vec{C}$ is clear from the context, I will omit the phrase "with respect to $\vec{C}$." As with ordinary simultaneous reflection, if $\mathscr{F}$ is a disjoint family of elements of $\mathscr{H}(\vec{C})$ which simultaneously trace reflect at δ, then the cardinality of $\mathscr{F}$ is at most the cofinality of δ.

3. Set Mapping Reflection

Now I will recall some definitions associated with the Set Mapping Reflection Principle. For the moment let X be a fixed uncountable set and let θ be a regular cardinal such that $H(\theta)$ contains $[X]^\omega$. The set $[X]^\omega$ is equipped with a natural topology — the Ellentuck Topology — defined by declaring intervals of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

to be open where x is a finite subset of N. If M is a countable elementary submodel of $H(\theta)$ and Σ is a subset of $[X]^\omega$ then we say that Σ is *M-stationary* if $E \cap \Sigma \cap M$ is non-empty whenever $E \subseteq [X]^\omega$ is a closed unbounded set in M. If Σ is set mapping defined on a collection of countable elementary submodels of $H(\theta)$ then we say that Σ is *open stationary* if $\Sigma(M) \subseteq [X]^\omega$ is open and M-stationary for all relevant M.

A set mapping Σ as above *reflects* if there is a continuous $\in$-chain $\langle N_\nu : \nu < \omega_1\rangle$ in the domain of Σ such that for every limit $\nu > 0$, $N_\xi \cap X$ is in $\Sigma(N_\nu)$ for coboundedly many ξ in ν. If this happens then $\langle N_\nu : \nu < \omega_1\rangle$ is called a *reflecting sequence* for Σ. The axiom MRP asserts that every open stationary set mapping defined on a club reflects. In [8] it is shown that MRP is a consequence of PFA. It is also shown there that it implies $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and that $\square(\kappa)$ fails for all regular $\kappa > \omega_1$.

4. The main results

We now proceed to the proof of the main theorem.

Theorem. (MRP) *Suppose that $\Omega \subseteq S^\omega_\theta$ is a non-reflecting stationary set and that $\vec{C}$ avoids Ω. If $\mathscr{F}$ is a collection of stationary subsets of Ω and $\mathscr{F}$ has size ω_1 then there is a $\delta < \theta$ of cofinality ω_1 such that every element of $\mathscr{F}$ simultaneously trace reflects at δ.*

Proof. Let $\mathscr{F} = \{\Omega_\xi : \xi < \omega_1\}$ be given and let $\{S_\xi : \xi < \omega_1\}$ be a sequence of disjoint stationary sets such that $\xi < \min(S_\xi)$ and $\bigcup_{\xi<\omega_1} S_\xi$ contains a club. For M a countable elementary submodel of $H(2^{\theta^+})$ which contains $\mathscr{F}$, define $\Sigma_{\mathscr{F}}(M)$ to be the collection of all countable

$N \subseteq \theta$ such that either $N \cap \theta$ has a last element or else $\sup N < \sup(M \cap \theta)$ and

$$\operatorname{tr}(\sup N, \sup(M \cap \theta)) \cap \Omega_\delta \neq \emptyset$$

where δ is such that $M \cap \omega_1$ is in S_δ. That $\Sigma_{\mathscr{F}}(M)$ is open is a consequence of Fact 2.1.

Claim 4.1. *$\Sigma_{\mathscr{F}}(M)$ is M-stationary.*

Proof. Let $E \in M$ be a club of countable subsets of θ and let δ be such that $M \cap \omega_1$ is in S_δ. By elementarity and assumption that Ω_δ is stationary, there is an α in $\Omega_\delta \cap M$ such that for every $\alpha_0 < \alpha$, there is an N in $E \cap M$ such that $\alpha_0 < \sup(N) < \alpha$. By Fact 2.1 it is possible to find an N in $E \cap M$ such that α is in $\operatorname{tr}(\sup(N), \sup(M \cap \theta))$. □

Now, let $\langle N_\xi : \xi < \omega_1 \rangle$ be a reflecting sequence for $\Sigma_{\mathscr{F}}$ and put

$$E = \{\sup(N_\xi \cap \theta) : \xi < \omega_1\}$$
$$\delta = \sup E.$$

It suffices to show that for every $\xi < \omega_1$ and closed unbounded $E' \subseteq \delta$ that there are $\alpha < \beta$ in E' such that $\operatorname{tr}(\alpha, \beta) \cap \Omega_\xi$ is non-empty. Let $\beta < \delta$ be a limit point of $E \cap E'$ such that $N_\nu \cap \omega_1$ is in S_ξ where $\nu < \omega_1$ is such that $\beta = \sup(N_\nu \cap \theta)$. By virtue of $\langle N_\xi : \xi < \omega_1 \rangle$ reflecting $\Sigma_{\mathscr{F}}$ and the definition of E, there is a $\beta_0 < \beta$ such that if α is in E with $\beta_0 < \alpha < \beta$ then $\operatorname{tr}(\alpha, \beta) \cap \Omega_\xi$ is non-empty. Selecting α in $E \cap E'$ with $\beta_0 < \alpha < \beta$, we now have $\alpha < \beta$ both in E' with $\operatorname{tr}(\alpha, \beta) \cap \Omega_\xi$ non-empty as desired. □

We finish the section with proof of the corollary.

Corollary. *Suppose that $M \subseteq V$ is an inner model with the same cardinals such that for some cardinal κ*

(1) $\operatorname{cf}(\kappa)^V = \omega < \operatorname{cf}(\kappa)^M = \kappa$ and

(2) every stationary subset of κ^+ in M is stationary in V.

Then MRP *fails in V. In particular,* MRP *fails in any generic extension by Prikry forcing.*

Proof. Let $\vec{C}$ be a C-sequence in M of length $\theta = \kappa^+$ such that for every $\alpha < \theta$, C_α has ordertype at most κ and $\vec{C}$ avoids

$$\Omega = \{\alpha < \theta : \operatorname{cf}(\alpha)^M = \kappa\}.$$

Let $\{\Omega_\xi : \xi < \kappa\}$ be a partition in M of Ω into disjoint stationary sets. Pick an $X \subseteq \kappa$ in V which is countable and cofinal in κ. Now suppose towards a contradiction that MRP holds in V. By the main theorem there would be a $\delta < \theta$ of cofinality ω_1 such that Ω_ξ trace reflects at

δ for every ξ in X. Now observe that the cofinality of δ must be less than κ in M since otherwise it would have countable cofinality in V. Let $E \subseteq \delta$ be closed and unbounded with $|E| < \kappa$ and E in M. Put

$$X^* = \{\xi < \kappa : \exists \alpha, \beta \in E(\operatorname{tr}(\alpha, \beta) \cap \Omega_\xi \neq \emptyset)\}.$$

Certainly X^* is in M, has size less than κ (since the $\{\Omega_\xi : \xi < \kappa\}$ are all pairwise disjoint), and is cofinal (since it contains X). But this is a contradiction since κ is regular in M. $\square$

5. Concluding remarks

Assaf Sharon has announced that SCH can fail at κ (even for $\kappa = \aleph_\omega$) and yet every stationary subset of κ^+ reflects. Hence it is not possible to prove the conjecture by establishing a ZFC connection between the existence of a non-reflecting stationary subset of κ^+ and the failure of SCH at κ. Since the acceptance of this paper, Viale has shown that MRP also fails in extensions by forcings which have the weak Prikry property [14]. This class of forcings includes the forcing used by Sharon.

A different approach is to try to replace the assumption of a non-reflecting stationary subset of $S^\omega_{\kappa^+}$ with the existence of a good scale for κ. The motivating factor is that a good scale for κ always exists if SCH fails first at κ (see Main Claim 1.3, p 46 in [9]). One could attempt to refute the existence of good scales using MRP and thus prove MRP implies SCH (Magidor has shown that MM implies good scales do not exist). This is not possible,[3] however, since Magidor has shown from appropriate large cardinal assumptions that PFA is consistent with $\square^*(\kappa)$ holding at all cardinals κ above ω_2 [7]. Since $\square^*(\kappa)$ implies that there is a good scale for κ [1], this approach will not work.

Finally, there are some results which link reflection in $[\lambda]^\omega$ to SCH. In [15] Veličković showed that if $\theta > \omega_1$ is regular and stationary subsets of $[\theta]^\omega$ reflect to an internally closed unbounded set (strongly reflect in the language of [15]) then $\theta^\omega = \theta$. An immediate consequence is that PFA^+ implies SCH. Recently Shelah improved this result by showing that reflection of stationary subsets of $[\theta]^\omega$ to sets of size ω_1 is already sufficient to deduce $\theta^\omega = \theta$ [10].

References

[1] James Cummings, Matthew Foreman, and Menachem Magidor. Squares, scales and stationary reflection. *J. Math. Logic*, 1(1):35–98, 2001.

[2] James Cummings and Ernest Schimmerling. Indexed squares. *Israel J. Math.*, 131:61–99, 2002.

[3] I would like to thank John Krueger for pointing this out shortly before publication.

[3] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's Maximum, saturated ideals, and nonregular ultrafilters. I. *Ann. of Math. (2)*, 127(1):1–47, 1988.

[4] T. Jech. *Multiple forcing*, volume 88 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.

[5] Thomas Jech. *Set theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1997.

[6] Kenneth Kunen. *An introduction to independence proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. North-Holland, 1983.

[7] Menachem Magidor. Lectures on weak square principles and forcing axioms. Jerusalem logic seminar, 1995.

[8] Justin Tatch Moore. Set mapping reflection. *J. Math. Logic*, 5(1):87–98, 2005.

[9] Saharon Shelah. *Cardinal arithmetic*, volume 29 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.

[10] Saharon Shelah. Reflection implies SCH. Preprint, July 2004.

[11] Jack Silver. On the singular cardinals problem. In *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pages 265–268. Canad. Math. Congress, Montreal, Que., 1975.

[12] Stevo Todorčević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3–4):261–294, 1987.

[13] Stevo Todorčević. Coherent sequences. In *Handbook of Set Theory*. North-Holland, (in preparation).

[14] Matteo Viale. Prikry type forcings, the Singular Cardinals Hypothesis and the Proper Forcing Axiom. preprint, Aug. 2005.

[15] Boban Veličković. Forcing axioms and stationary sets. *Adv. Math.*, 94(2):256–284, 1992.