Lecture 1

Descriptive set theory concerns the study of “regular” or “tame” subsets of separable complete metric spaces. We will begin by developing a better understanding of which topological spaces admit separable, complete metrics.

**Definition 1.** A topological space $X$ is *Polish* if it is separable and completely metrizable.

It is straightforward to characterize when a subspace of a complete metric space is complete in the inherited metric.

**Proposition 1.** If $(X,d)$ is a metric space and $E \subseteq X$, then $E$ is closed if and only if $(E,d)$ is complete.

In particular, $(0,\infty)$, $\mathbb{R}\setminus\mathbb{Q}$, and $\mathbb{Q}$ all fail to be complete when given the usual metric on $\mathbb{R}$. The question of when a subspace of a Polish space is Polish (as witnessed by a different metric) is more subtle. For instance any logarithm function is a homeomorphism between $(0,\infty)$ and $\mathbb{R}$. We will see momentarily that $\mathbb{R}\setminus\mathbb{Q}$ is also Polish while $\mathbb{Q}$ is not. Specifically, Alexandroff proved that the Polish subspaces of a Polish space are exactly the subsets which are countable intersections of open sets.

The next proposition will be useful in Alexandroff’s characterization and is of independent interest.

**Proposition 2.** If $(X_n \mid n < \infty)$ is a sequence of Polish spaces, then $\prod_{n=0}^{\infty} X_n$ is Polish.

**Proof.** For each $n$, let $d_n$ be a complete metric on $X_n$ which generates its topology. If $x, y \in \prod_{n=0}^{\infty} X_n$, define

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n-1} \min(d_n(x_n, y_n), 1).$$

It can be checked that the product topology is the same as the topology induced by $d$. 

\[ \square \]
Next we handle an important special case of the characterization.

**Lemma 1.** If $X$ is Polish and $U \subseteq X$ is open, then $U$ is Polish.

**Proof.** Set $E = X \setminus U$ and define
\[ \Gamma = \{(x, r) \in X \times \mathbb{R} \mid d(x, E) \cdot r = 1\}. \]
It is readily verified that $\Gamma$ is closed and that the projection onto the first coordinate is a homeomorphism between $\Gamma$ and $U$. Since $X$ and $\mathbb{R}$ are Polish, so is $X \times \mathbb{R}$. Hence $\Gamma$ is closed, it is Polish as well. Since being Polish is a topological property, it is perserved by homeomorphisms and thus $U$ is Polish. \( \square \)

We are now ready to prove the characterization. Recall that a subset $Y$ of a topological space is a $G_\delta$-set if it is a countable intersection of open sets.

**Theorem 1** (Alexandroff). If $X$ is a Polish space and $Y \subseteq X$, then $Y$ is Polish if and only if it is a $G_\delta$-set.

**Proof.** First suppose that $Y$ is a $G_\delta$-set and let $Y = \bigcup_{n=0}^{\infty} U_n$ where $U_n \subseteq X$ is open. By Lemma 1, each $U_n$ is Polish and by Lemma 1, $\prod_{n=0}^{\infty} U_n$ is Polish. Let $\Delta \subseteq \prod_{n=0}^{\infty} X$ consist of the constant sequences. The projection onto the first coordinate defines a homeomorphism between $\Delta$ and $X$. Furthermore, this homeomorphism restricts to a homeomorphism from $\Delta \cap \prod_{n=0}^{\infty} U_n$ to $Y$. Since $\Delta \cap \prod_{n=0}^{\infty} U_n$ is a closed subspace of $\prod_{n=0}^{\infty} U_n$ it is also Polish. Hence $Y$ is Polish.

Now suppose that $Y \subseteq X$ is Polish when given the subspace topology. Let $d'$ be the complete metric on $Y$ which witnesses this. Define $\mathcal{U}_n$ to be all open subsets $U$ of $X$ such that:
- the $d$-diameter of $U$ is less than $1/n$;
- $U \cap Y$ is nonempty and has $d'$-diameter less than $1/n$.

Set $W_n = \bigcup \mathcal{U}_n$. Since $\mathcal{U}_n$ is a cover of $Y$ for each $n$, $Y \subseteq \bigcap_{n=0}^{\infty} W_n$. It is therefore sufficient to show that $\bigcap_{n=0}^{\infty} W_n \subseteq Y$. Toward this end, let $y \in \bigcap_{n=0}^{\infty} W_n$. For each $n$, let $U_n \in \mathcal{U}_n$ be such that $y \in U_n$. Observe that $y$ is in the closure of $Y$ since every element of $W_n$ is of distance at most $1/n$ from $Y$. For each $n$, pick $y_n \in Y \cap \bigcap_{i \leq n} U_i$. Clearly $y_n \to y$. Furthermore, $(y_n)_n$ is Cauchy since if $n \in \mathbb{N}$, then $\{y_i \mid n \leq i \} \subseteq U_n$ which has diameter less than $1/n$. \( \square \)
Lecture 2

We begin by recalling the following classical metrization result from point-set topology. This theorem gives a very useful criteria for determining when a topological space is metrizable.

**Theorem 2.** A regular topological space is second countable (i.e. has a countable base of open sets) if and only if it is both separable and metrizable.

The focus of this lecture is to prove embedding theorems for Polish spaces. The first theorem shows that any separable metric space can be embedded into a compact metric space. Notice the separability assumption is necessary since every compact metric space is separable and every subspace of a separable metric space is separable.

**Theorem 3.** If $X$ is a separable metric space, then $X$ is embeddable into $[0,1]^\mathbb{N}$.

**Proof.** Let $d$ be a complete metric on $X$ which generates the topology and which is bounded by 1. Let $(z_n \mid n \in \mathbb{N})$ enumerate a countable dense subset of $X$ and define a function $\phi : X \to \mathbb{R}^\mathbb{N}$ by $\phi(x)(n) = d(x, z_n)$. We must show that $\phi$ is one-to-one, continuous, and that $\phi^{-1}$ is continuous.

To see that $\phi$ is one-to-one, suppose that $x \neq y$ and let $n$ be such that $d(x, z_n) < d(x, y)/2$. It follows that $d(x, z_n) < d(y, z_n)$ and hence that $\phi(x) \neq \phi(y)$.

That $\phi$ is continuous follows from the fact that $d$ is continuous. In order to see that $\phi^{-1}$ is continuous, suppose that $(x_m)_m$ is a sequence such that $\phi(x_m) \to \phi(x)$ for some $x \in X$. Let $\varepsilon > 0$ be given and let $n$ be such that $d(x, z_n) < \varepsilon/3$. Find an $m_0$ such that if $m > m_0$, then $|d(x_m, z_n) - d(x, z_n)| < \varepsilon/3$. If $m > m_0$, then

$$d(x_m, x) \leq d(x_m, z_n) + d(x, z_n)$$

$$= |d(x_m, x_n) - d(x_n, z_n) + d(x, z_n)| + d(x, z_n)$$

$$\leq |d(x_m, x_n) - d(x, z_n)| + 2d(x, z_n) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $x_m \to x$. □

**Corollary 1.** If $X$ is a compact metric space, then $X$ is homeomorphic to a closed subspace of $[0,1]^\mathbb{N}$.

**Proof.** By the previous theorem, $X$ is homeomorphic to a subspace of $[0,1]^\mathbb{N}$. Since $X$ is compact, so is this subspace; since compact subspaces are always closed, we are finished. □
If we replace $[0, 1]^N$ by the Polish space $\mathbb{R}^N$, then we can obtain an analogous embedding theorem for Polish spaces.

**Theorem 4.** If $X$ is a Polish space, then $X$ is homeomorphic to a closed subspace of $\mathbb{R}^N$.

**Proof.** By Theorem 3, we may assume that $X \subseteq [0, 1]^N \subseteq \mathbb{R}^N$. Since $X$ is Polish, it is a $G_\delta$-set inside $\mathbb{R}^N$. Let $(E_n)_n$ be a sequence of closed sets such that $\bigcup_{n=0}^{\infty} E_n = \mathbb{R}^N \setminus X$. Define

$$Y = \{(x, r) \in \mathbb{R}^N \times \mathbb{R}^N : \forall n(d(x, E_n) \cdot r_n = 1)\}.$$ 

Notice that $Y \subseteq \mathbb{R}^N \times \mathbb{R}^N \approx \mathbb{R}^N$ is closed. It therefore suffices to show that $X \approx Y$. Observe that if $(x, r) \in Y$, then $x \in X$ and if $x \in X$, then there is a unique $r \in \mathbb{R}^N$ such that $(x, r) \in Y$. Since the first coordinate projection map is both continuous and open, its restriction to the closed set $Y$ is a homeomorphism. □

Next we turn to the spaces $2^\mathbb{N} = \{0, 1\}^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$, which play a dual role in understanding Polish spaces. If $S$ is any set, define $\Delta : S^\mathbb{N} \to \mathbb{N} \cup \{\infty\}$ by

$$\Delta(x, y) = \min(\{n \in \mathbb{N} \mid x(n) \neq y(n)\} \cup \{\infty\}).$$

This gives rise to a metric: $d(x, y) = 2^{-\Delta(x, y)}$. This function satisfies a strong form of the triangle inequality:

$$d(x, z) \leq \max(d(x, y), d(y, z)).$$

Moreover $d$ is complete and the induced topology on $S^\mathbb{N}$ is separable if and only if $S$ is countable.

The following theorems characterize the spaces $2^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ in terms of their topological properties. Recall that a topological space is $0$-dimensional if it has a base of sets which are clopen (i.e. both closed and open).

**Theorem 5.** $2^\mathbb{N}$ is homeomorphic to any compact metric space which is 0-dimensional and has no isolated points.

**Theorem 6.** $\mathbb{N}^\mathbb{N}$ is homeomorphic to any Polish space which is 0-dimensional and has no compact neighborhoods.

The space $2^\mathbb{N}$ is commonly referred to as the Cantor set. It is homeomorphic to the standard “middle third” Cantor set contained in the unit interval. The space $\mathbb{N}^\mathbb{N}$ is commonly known as **Baire space**. By the above characterization, it is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. 
In the previous lecture we saw that every Polish space embeds as a closed subspace of $\mathbb{R}^\omega$ and that every compact metric space embeds as a closed subspace of $[0, 1]^\omega$. In this lecture we will show that every Polish space is a continuous image of $\omega^\omega$ and that every compact metric space is a continuous image of $2^\omega$. Moreover if the space is assumed to have no isolated points, then the continuous function can be taken to be one-to-one.

Before going further, it will be useful to pause and define some terminology and notation which will be used later. An ordinal is a set $\alpha$ which has the following properties:

- (transitive set) if $\gamma \in \beta \in \alpha$, then $\gamma \in \alpha$;
- (linearly ordered) if $\beta \neq \gamma$ are in $\alpha$, then $\beta \in \gamma$ or $\gamma \in \beta$.

It can be readily checked that:

- the emptyset is an ordinal, which is usually denoted 0 when viewed as such;
- if $\alpha$ is an ordinal, then so is $\alpha + 1 := \alpha \cup \{\alpha\}$ and if $A$ is a set of ordinals, then $\bigcup A$ is an ordinal.

Furthermore, we have the following key feature of ordinals.

**Theorem 7.** If $\alpha \neq \beta$ are ordinals, then $\alpha \in \beta$ or $\beta \in \alpha$. Moreover, every nonempty subset of the ordinals has an $\in$-least element.

Thus we regard the class of all ordinals as being well ordered by $\in$ (even though formally the class of all ordinals is a proper class and not a set).

The finite ordinals naturally correspond to the nonnegative integers: $n := \{0, \ldots, n - 1\}$. The set of finite ordinals is denoted $\omega$ and we will generally write $\omega$ instead of $\mathbb{N}$ from this point forward. Unless otherwise stated, counting and indexing will start at 0 and the variables $i, j, k, l, m, n$ will range over $\omega$.

A sequence is a function whose domain is an ordinal. The length of a sequence $s$ is defined to be its domain and is denotes $lh(s)$. In this course, we will generally only work with sequences of length at most $\omega$. If $S$ is a set, we will use $S^{<\omega}$ to denote the set of all finite length sequences whose range is contained in $S$. We will use $s \cdot t$ to denote the concatenation of $s$ and $t$ (the meaning should be clear enough when $s$ is finite and $t$ has length at most $\omega$ and we will not need any further generality than this; a general definition is possible, however). Either $\langle a_0, a_1, \ldots, a_{n-1} \rangle$ or $\langle a_i \mid i < n \rangle$ will be used to denote the sequence whose $i^{th}$ entry is $a_i$. The sequence of length 0 will be denoted $\varepsilon$. 
We now return to our regular discussion. Recall from last time that \(d(a, b) = 2^{-\Delta(a, b)}\) defines a complete metric on \(S^\omega\). This is compatible with the product topology if \(S\) is given the discrete topology. Furthermore, the open balls in \(S^\omega\) all have the form
\[
[s] := \{a \in S^\omega | a \upharpoonright lh(s) = s\}
\]
where \(s \in S^{<\omega}\). Notice that these sets are clopen.

**Theorem 8.** If \(X\) is a 0-dimensional Polish space with no compact neighborhoods, then \(X\) is homeomorphic to \(\omega^\omega\).

**Proof.** Let \(\{W_n | n \in \omega\}\) be a base for \(X\) consisting of clopen sets and fix a compatible complete metric \(d\) on \(X\). Recursively construct, for each \(s \in \omega^{<\omega}\), \(U_s \subseteq X\) such that:

- \(U_s = X\);
- for each \(s\), \(\langle U_{s^-\langle n \rangle} | n \in \omega\rangle\) is a partition of \(U_s\) into nonempty clopen sets;
- \(U_s\) is contained in or disjoint from \(W_{lh(s)}\);
- the \(d\)-diameter of \(U_s\) is at most \(2^{-lh(s)}\).

Given \(U_s\), construct \(\{U_{s^-\langle n \rangle} | n \in \omega\}\) as follows. By our assumption, we know that there is an open cover \(U_s\) which has no finite subcover. By refining the cover if necessary, we may assume that each of its elements are basic clopen sets of diameter less than \(2^{-lh(s)-1}\), that each are contained in \(U_s\), and that each are contained in or disjoint from \(W_{lh(s)+1}\). Since clopen sets are closed under finite boolean combinations and since there are only countably many basic open sets, we may further refine this cover to a partition. Let \(\langle U_{s^-\langle n \rangle} | n \in \omega\rangle\) enumerate this partition without repetition.

Observe that the conditions of the construction imply that each \(W_n\) is a union of elements of \(\{U_s | s \in \omega^{<\omega}\}\) and hence \(\{U_s | s \in \omega^{<\omega}\}\) forms a base for the topology on \(X\).

**Claim 1.** For each \(a \in \omega^\omega\), \(\bigcap_{n=0}^\infty U_{a|n}\) is a singleton.

**Proof.** Observe that if \(x_m \in U_{a|m}\) for each \(m \in \omega\), then since the diameters of the \(U_{a|n}\)'s tend to 0, the sequence \(\langle x_m | m \in \omega\rangle\) is Cauchy. This implies in particular that the intersection \(\bigcap_{n=0}^\infty U_{a|n}\) contains at most one element. Since \(d\) is complete, \(x_m \to x\) for some \(x\). Since \(U_{a|n}\) is closed for all \(n\), \(x \in \bigcap_{n=0}^\infty U_{a|n}\). \(\square\)

Define \(\phi(a)\) to be the unique element of \(\bigcap_{n=0}^\infty U_{a|n}\). Since, \(U_\varepsilon = X\) and \(\langle U_{s^-\langle n \rangle} | n \in \omega\rangle\) is a partition of \(U_s\) for each \(s \in \omega^{<\omega}\), it follows that \(\phi\) is one-to-one and onto. Furthermore, \(\phi^{-1}(U_s) = [s]\). Since \(\{U_s | s \in \omega^{<\omega}\}\) and \(\{[s] | s \in \omega^{<\omega}\}\) are bases for the topologies of \(X\) and \(\omega^\omega\), respectively, it follows that both \(\phi\) and \(\phi^{-1}\) are continuous. \(\square\)
A similar proof can be used to show that every Polish space is a continuous image of $\omega^\omega$; we will state and prove a stronger result in the next lecture. We now turn to the role that $2^\omega$ plays in the class of compact metric spaces.

**Theorem 9.** If $E$ is a compact metric space, then there is a continuous surjection $\phi : 2^\omega \to E$. Moreover if $E$ has no isolated points and is 0-dimensional, then $\phi$ can be taken to be a homeomorphism.

**Proof.** Let $\{W_n \mid n \in \omega\}$ be a basis for the topology of $E$; if $E$ is 0-dimensional, then arrange that each $W_n$ is clopen. Recursively construct, for each finite binary sequence $s$, a nonempty open set $U_s \subseteq E$ such that:

1. $U_s = E$, where $\varepsilon$ is the null sequence and if $E$ is 0-dimensional then each $U_s$ is clopen;
2. $U_{s^{-}(0)} \cup U_{s^{-}(1)} \subseteq U_s \subseteq \overline{U_{s^{-}(0)}} \cup \overline{U_{s^{-}(1)}}$;
3. $U_s$ is contained $W_n$ or is or disjoint from $W_n$ where $n$ is the length $|s|$ of $s$;
4. if $U_s$ is not a singleton, then $U_{s^{-}(0)} \cap U_{s^{-}(1)} = \emptyset$.

Observe that if $x \neq y$ are in $E$, then there is an $n$ such that $x \in W_n$ and $y$ is not in $W_n$. In particular, if $s$ has length at least $n$, then $x$ and $y$ are not both in $\overline{U_s}$. On the other hand, if $a \in 2^\omega$, then $\{\overline{U_{a\upharpoonright n}} \mid n \in \omega\}$ is a decreasing chain of nonempty compact sets. Consequently $\bigcap_{n=0}^{\infty} \overline{U_{a\upharpoonright n}}$ contains a unique element for each $a \in 2^\omega$. Define $\phi : 2^\omega \to E$ by letting $\phi(a)$ be the unique element of $\bigcap_{n=0}^{\infty} \overline{U_{a\upharpoonright n}}$. If $x \in E$, then one can readily recursively construct an $a \in 2^\omega$ such that for all $n$, $x \in \overline{U_{a\upharpoonright n}}$. In particular, $\phi$ is onto.

We must now show that $\phi$ is continuous. Toward this end, let $a \in 2^\omega$ and $V \subseteq E$ be an open set containing $x$. Let $n$ be such that $x \in W_n \subseteq \overline{W_n} \subseteq V$. Since $\phi(a) \in \overline{U_{a\upharpoonright n}}$, it cannot be that $U_{a\upharpoonright n}$ is disjoint from $W_n$. It follows from the construction that $U_{a\upharpoonright n} \subseteq W_n$. Now if $b \in [a \upharpoonright n]$, then

$$\phi(b) \in \overline{U_{b\upharpoonright n}} \subseteq \overline{W_n} \subseteq V.$$  

Since $a$ and $V$ were arbitrary, $\phi$ is continuous.

Finally, observe that if $E$ is 0-dimensional and has no isolated points, then $U_{s^{-}(0)}$ and $U_{s^{-}(1)}$ are disjoint clopen sets. In particular, $\phi$ is one-to-one. We are now finished by recalling that any continuous bijection between compact spaces is in fact a homeomorphism. \[\Box\]
Let $X$ be a fixed Polish space. The Borel subsets of $X$ are the members of the minimum collection $\text{Borel}(X) \subseteq \mathcal{P}(X)$ such that:

- every open subset of $X$ is in $\text{Borel}(X)$;
- if $B \in \text{Borel}(X)$, then so is $X \setminus B$;
- if $\mathcal{B} \subseteq \text{Borel}(X)$ is countable, then $\bigcup \mathcal{B}$ is in $\text{Borel}(X)$.

Our first goal will be to prove the following theorem.

**Theorem 10.** If $X$ is a Polish space and $\mathcal{B}$ is a countable family of Borel subsets of $X$, then there is a stronger Polish topology on $X$ with the same Borel subsets in which each element of $\mathcal{B}$ is clopen.

This follows from two lemmas which are of interest in their own right. Recall that a regular open set is an open set which is the interior of its closure. Notice that any separable metric space has a countable base consisting of regular open sets.

**Lemma 2.** If $X$ is a Polish space and $U \subseteq X$ is open, then the stronger topology generated by declaring $U$ to be clopen is Polish. Moreover, if $U$ is a regular open set and $X$ has no isolated points, then the stronger topology has no isolated points as well.

**Proof.** Define $Y \subseteq X \times \{0, 1\}$ by $Y := U \times \{0\} \cup (X \setminus U) \times \{1\}$. It is readily checked that $Y$ is a $G_\delta$-subset of the Polish space $X \times \{0, 1\}$ and hence Polish. Furthermore projection onto the first coordinate defines a continuous bijection $\pi$ from $Y$ to $X$. Moreover the restrictions of $\pi$ to $U \times \{0\}$ and $(X \setminus U) \times \{1\}$ are homeomorphisms onto $U$ and $X \setminus U$. In particular, $\pi$ is a homeomorphism to the stronger topology on $X$ generated by declaring $U$ to be clopen. Finally, if $X$ has no isolated points and $U \subseteq X$ is a regular open set, then neither $U$ nor $X \setminus U$ have isolated points. Consequently $Y$ has no isolated points. \qed

**Lemma 3.** If $X$ is a set and $\langle \tau_n \mid n \in \omega \rangle$ is an increasing sequence of Polish topologies on $X$, then the topology $\tau$ generated by $\bigcup_{n=0}^\infty \tau_n$ is Polish.

**Proof.** By Theorem 2, $\prod_{n=0}^\infty (X, \tau_n)$ is Polish. Moreover, the constant sequences form a closed — and hence Polish — subspace of the product which is homeomorphic to $(X, \tau)$. \qed

In order to prove Theorem 10, consider the collection $\mathcal{C}$ of all Borel subsets of $X$ which are clopen in some stronger Polish topology which generates the same Borel sets. Clearly $\mathcal{C}$ is closed under taking complements. Lemma 2 implies that $\mathcal{C}$ contains all open subsets of $X$ and Lemma 3 implies that $\mathcal{C}$ is closed under taking countable unions. It
follows that $C$ coincides with the class of all Borel subsets of $X$. This establishes Theorem 10 in the special case in which $B$ is a singleton. The general case follows by iterating the singleton case and applying Lemma 3.

If $X$ and $Y$ are Polish spaces, a function $f : X \to Y$ is Borel (or Borel measurable) if whenever $B \subseteq Y$ is Borel, $f^{-1}(B)$ is Borel. Observe that in order to verify that $f$ is Borel measurable, it suffices to show that $f^{-1}(U)$ is Borel whenever $U \subseteq Y$ is open.

**Corollary 2.** If $X$ and $Y$ are Polish and $f : X \to Y$ is Borel measurable, then there is a stronger Polish topology on $X$ with the same Borel sets such that $f$ is continuous with respect to the stronger topology.

**Proof.** Apply Theorem 10 to $\{f^{-1}(W_n) \mid n \in \omega\}$ where $\{W_n \mid n \in \omega\}$ is a base for the topology on $Y$. □

**Corollary 3.** If $X$ is Polish, then there is a stronger 0-dimensional Polish topology on $X$ with the same Borel sets. Moreover, if $X$ has no isolated points, then the stronger topology also does not have isolated points.

**Proof.** Let $\{W_n \mid n \in \omega\}$ be a base for the topology $\tau$ on $X$ consisting of regular open sets. By Lemmas 2 and 3, the topology generated by $\tau \cup \{X \setminus W_n \mid n \in \omega\}$ Polish and 0-dimensional. Moreover, if $\tau$ has no isolated points, neither does the stronger topology. □

**Theorem 11.** If $X$ is Polish, then there is a continuous surjection $f : \omega^\omega \to X$ such that $f^{-1}(U)$ is Borel whenever $U \subseteq X$ is open. Moreover, if $X$ has no isolated points, then $f$ can be taken to be one-to-one.

**Proof.** If $X$ has isolated points, then observe that $X \times \mathbb{R}$ is Polish, has no isolated points, and maps continuously onto $X$. Thus by replacing $X$ by $X \times \mathbb{R}$ if necessary, we may assume that $X$ has no isolated points. We will be finished once we show that there is a stronger topology on $X$ which is homeomorphic to $\omega^\omega$.

Let $\tau_0$ denote the topology on $X$ and let $D \subseteq X$ be a countable dense set. Observe that if $U \subseteq X$ is a regular open set, then $D$ is still dense in the topology generated by $\tau_0 \cup \{X \setminus U\}$: this is because any open subset of $X$ which intersects $X \setminus U$ must intersect $X \setminus \overline{U}$. Let $\mathcal{W}$ be a countable collection of regular open sets which form a base for the topology on $X$ such that for each $x \in D$, there is a $W \in \mathcal{W}$ such that $x$ is in the boundary of $W$. By Lemmas 2 and 3, the topology $\tau$ generated by $\tau_0 \cup \{X \setminus W \mid W \in \mathcal{W}\}$ is Polish, 0-dimensional, and has no isolated points. Moreover, by our observation $D$ is still dense in $\tau$. 
By Theorem 8, it suffices to show that $\tau$ does not contain any compact open sets. In order to see this, suppose that $U \subseteq \tau$ is clopen in $\tau$ and let $x \in D \cap U$. Pick a $W \in \mathcal{W}$ such that $x$ is in the $\tau_0$-boundary of $W$. The collection of closed $\tau_0$-neighborhoods of $x$ together with $U \cap W$ form a family of $\tau$-closed sets with the finite intersection property but having empty intersection. Thus $U$ is not compact. \hfill \Box

**Corollary 4.** If $X$ is Polish and $B \subseteq X$ is a Borel set, then $B$ is a continuous image of $\omega^\omega$.

*Proof.* Let $\tau$ be the topology on $X$ and let $\tau'$ be a stronger Polish topology on $X$ in which $B$ is clopen. By the previous theorem, there is a continuous surjection $\phi : \omega^\omega \to (X, \tau')$. Since $\phi$ is still continuous with respect to $\tau$, we are done. \hfill \Box
Lecture 5

We begin by noting a few examples of explicit continuous surjections which are sometimes useful. The first is given by the traditional binary expansion of elements of $[0, 1]$: define $f : 2^\omega \to [0, 1]$ by

$$f(x) = \sum_{n=0}^{\infty} x(n)2^{-n-1}$$

If we equip $2^\omega$ with the lexicographic order, then this map a continuous order preserving surjection. Furthermore, the preimages of elements of $[0, 1]$ contain at most 2 elements and are singletons except for dyadic rationals.

Next recall that the real projective line $P^1(\mathbb{R})$ is the family of all lines in $\mathbb{R}^2$ which pass through the origin. An element of $P^1(\mathbb{R})$ can be identified with the $x$-coordinate of its intersection with the line $y = 1$; the horizontal line $y = 0$ corresponds to the point $\infty$. We can define a function $\phi : 2^\omega \to [0, \infty]$ by the implicit formula:

$$\phi(\langle 1 \rangle \upharpoonright x) = 1 + \phi(x)$$
$$\phi(\langle 0 \rangle \upharpoonright x) = \frac{1}{1 + \phi(x)}$$

In fact it can be shown that there is a unique continuous function $\phi$ which satisfies this formula. To see this, observe that if $x$ is constantly 0, then $\phi(x)$ must be 0. If $x$ is a sequence with only finitely many occurrences of 1, then the above definition is recursive. One can moreover verify that the image of the sequences which are eventually 0 is exactly the rationals and that $\phi$ is order preserving.

The function $\phi$ can be modified slightly to give a surjection from $2^\omega$ to $P^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$:

$$\Phi(\langle 1 \rangle \upharpoonright x) := \phi(x)$$
$$\Phi(\langle 0 \rangle \upharpoonright x) := -\phi(\bar{x})$$

where $\bar{x}$ denotes the bitwise complement of $x$ (i.e. $\bar{x}(n) = 1 - x(n)$ for each $n \in \omega$). This function preserves cyclic order.

It is possible to give a finer stratification of $\text{Borel}(X)$ by keeping track of how its elements are generated. Specifically, define $\Sigma^0_1(X)$ to be the collection of all open subsets of $X$. If $\alpha > 0$ is an ordinal, then $\Pi^0_\alpha(X)$ consists of all complements of sets in $\Sigma^0_\alpha(X)$. If $\alpha > 1$ is an ordinal, then $\Sigma^0_\alpha(X)$ consists of all sets which can be expressed as a countable union of sets in $\bigcup_{\gamma < \alpha} \Pi^0_\gamma$. In particular:

- $\Pi^0_1(X)$ consists of the closed subsets of $X$;
- $\Sigma^0_2(X)$ consists of the $F_\sigma$-subsets of $X$;
- $\Pi^0_2(X)$ consists of the $G_\delta$-subsets of $X$. 
Just as every open set is an $F_\sigma$-set, it can be shown than if $\alpha < \beta$, then
\[
\Sigma^0_\alpha(X) \cup \Pi^0_\alpha(X) \subseteq \Sigma^0_\beta(X) \cap \Pi^0_\beta(X)
\]
It can be easily verified that if $\omega_1$ is the first uncountable ordinal, then $\Sigma^0_{\omega_1}(X) = \text{Borel}(X)$ and thus if $\alpha \geq \omega_1$, then $\Sigma^0_\alpha(X) = \Pi^0_\alpha(X) = \Sigma^0_\alpha(X)$.

In the previous lecture, we saw that every Borel set in a Polish space is a continuous image of $\omega^\omega$. Continuous images of $\omega^\omega$ are of considerable interest in their own right and will play an important role in the remainder of the course. The next proposition shows that there are a number of equivalent formulations of being analytic.

**Proposition 3.** If $X$ is a Polish space and $A \subseteq X$, then the following are equivalent:

1. $A$ is a continuous image of $\omega^\omega$;
2. there is a Polish space $Y$, a Borel set $B \subseteq Y$, and a Borel measurable $f : B \to A$;
3. there is a closed set $E \subseteq X \times \omega^\omega$ such that $A = \{x \in X \mid \exists y \in \omega^\omega((x,y) \in E)\}$.

**Proof.** It is trivial that (3) implies (2). To see that (2) implies (1), let $B \subseteq Y$ and $f : B \to A$ be given. By increasing the topology on $Y$, we may assume that $f : B \to A$ is continuous. Let $g : \omega^\omega \to B$ be a continuous surjection and observe that $f \circ g : \omega^\omega \to A$ is a continuous surjection. In order to see that (1) implies (3), fix a continuous function $f : \omega^\omega \to X$ whose range is $A$ and define
\[
E := \{(x,y) \in X \times \omega^\omega \mid f(y) = x\}.
\]
Since $f$ is continuous, $E$ is closed. Clearly the projection of $E$ onto the first coordinate is $A$. ☐

The next theorem of Souslin is known as the *Separation Theorem* and gives a useful characterization of Borel sets.

**Theorem 12.** If $X$ is Polish and $A$ and $B$ are disjoint analytic subsets of $X$, then there is a Borel set $C$ such that $A \subseteq C$ and $C \cap B = \emptyset$. In particular, a subset $B$ of $X$ is Borel if and only if both $B$ and $X \setminus B$ are analytic.

**Proof.** Fix continuous surjections $f : \omega^\omega \to A$ and $g : \omega^\omega \to B$. Given $s \in \omega^{<\omega}$, define
\[
A_s := \{f(x) \mid x \in [s]\}, B_s := \{g(x) \mid x \in [s]\},
\]
noting that $A_s = \bigcup_{m=0}^{\infty} A_{s^{\prec}(m)}$ and $B_s = \bigcup_{n=0}^{\infty} B_{s^{\prec}(n)}$. 
Suppose for contradiction that there is no Borel set $C$ which separates $A$ from $B$. Construct sequences $\langle m_k \mid k \in \omega \rangle$ and $\langle n_k \mid k \in \omega \rangle$ so that if $s_k := \langle m_i \mid i < k \rangle$ and $t_k := \langle n_i \mid i < k \rangle$, then there is no Borel set $C$ such that $A_{s_k} \subseteq C$ and $B_{t_k} \cap C = \emptyset$. Notice that if $k = 0$, then $A_{s_k} = A_\varepsilon = A$ and $B_{t_k} = B_\varepsilon = B$. Suppose that $s_k$ and $t_k$ have been determined. If for each $m,n \in \omega$ there is a Borel set $C_{m,n}$ which separates $A_{s_k} \langle m \rangle$ from $B_{t_k} \langle n \rangle$, then $C = \bigcup_{m=0}^{\infty} \bigcap_{n=0}^{\infty} C_{m,n}$ separates $A_{s_k}$ from $B_{t_k}$. Thus there exist $m_k$ and $n_k$ such that $A_{s_k} \langle m_k \rangle$ is not separated from $B_{t_k} \langle n_k \rangle$ by a Borel set.

Finally, define $x := \langle m_k \mid k \in \omega \rangle$ and $y := \langle n_k \mid k \in \omega \rangle$. Since $f(x) \in A$ and $g(y) \in B$, $f(x) \neq g(y)$ and there are disjoint open sets $U$ and $V$ containing $f(x)$ and $g(y)$ respectively. Since $f$ and $g$ are continuous, there is a $k$ such that $[s_k] \subseteq f^{-1}(U)$ and $[t_k] \subseteq g^{-1}(V)$. But now $U$ is a Borel set which separates $A_{s_k}$ from $B_{t_k}$, which is a contradiction.

There are analytic sets which are not Borel. For example,

$$\{A \subseteq \mathbb{Q} \mid A \text{ contains an infinite decreasing sequence}\}$$

is the projection of the closed set

$$\{(A, \langle q_k \mid k \in \omega \rangle) \in \mathcal{P}(\mathbb{Q}) \times \mathbb{Q}^\omega \mid \forall k \in \omega (q_k \in A) \text{ and } (q_{k+1} < q_k)\}.$$  

We do not yet have the tools to see that this set is not Borel, however.
Lecture 6

If $X$ is a Polish space, then the collection of analytic subsets of $X$ is denoted $\Sigma^1_1(X)$. A subset of $X$ is co-analytic if it is the complement of an analytic set. The collection of co-analytic subsets of $X$ is denoted $\Pi^1_1(X)$.

In this lecture we will see two proofs that there is an analytic set which is not Borel. Each illustrates a technique which is of independent interest.

**Theorem 13.** If $X$ is a Polish space, then there is an analytic set $A \subseteq X \times \omega^\omega$ such that if $B \subseteq X$ is analytic, then there is a $y \in \omega^\omega$ such that

$$B = A^y := \{x \in X \mid (x, y) \in A\}.$$  

An analytic set satisfying the conclusion of this theorem is said to be a *universal analytic set* for $X$.

**Proof.** Let $\{U_n \mid n \in \omega\}$ enumerate a base for $X \times \omega^\omega$ and define

$$F := \{(x, y, z) \in X \times (\omega^\omega)^2 \mid \forall n((x, z) \notin U_{y(n)})\}.$$  

Observe that if $E \subseteq X$ is closed and $y \in \omega^\omega$ is an enumeration of the set $\{n \in \omega \mid E \cap U_n = \emptyset\}$, then

$$E = \{(x, z) \in X \times \omega^\omega \mid \exists y((x, y, z) \in F)\}.$$  

It suffices to check that

$$A = \{(x, y) \in X \times \omega^\omega \mid \exists z((x, y, z) \in F)\}$$

satisfies the conclusion. To see this, suppose that $B \subseteq X$ is analytic and let $E \subseteq X \times \omega^\omega$ be such that $B = \{x \in X \mid \exists y((x, z) \in E)\}$. Now if $y \in \omega^\omega$ is such that

$$E = \{(x, z) \in X \times \omega^\omega \mid \exists y((x, y, z) \in F)\},$$

then we have $B = \{x \in X \mid (x, y) \in A\}$. \qed

**Corollary 5.** If $A \subseteq \omega^\omega \times \omega^\omega$ is a universal analytic set, then $A$ is not Borel.

**Proof.** If $A$ were Borel, then

$$A_\Delta := \{x \in \omega^\omega \mid (x, x) \in A\}$$

would be Borel as well. In particular $\omega^\omega \setminus A_\Delta$ would be analytic. Since $A$ is universal, there would be an $y \in \omega^\omega$ such that $A^y = \omega^\omega \setminus A_\Delta$. But now $y \in A_\Delta$ if and only if $(y, y) \in A$ if and only if $y \in \omega^\omega \setminus A_\Delta$, which is a contradiction. \qed
Next we will show that
\[ \mathcal{W} := \{ A \in \mathcal{P}(\mathbb{Q}) \mid A \text{ is well ordered} \} \]
is co-analytic but not Borel. This proof centers on the boundedness principle. Recall that a binary relation \( \prec \) on a set \( X \) is well-founded if there does not exist an infinite sequence \( \langle x_n \mid n \in \omega \rangle \) of elements of \( X \) such that for all \( n \in \omega \), \( x_{n+1} \prec x_n \); otherwise we say \( \prec \) is ill-founded. Given a well-founded relation, we can recursively define the rank function \( \rho \) for \( (X, \prec) \). This function maps \( X \) into the ordinals so that \( \rho(x) \) is the least strict upper bound for \( \{ \rho(y) \mid y \in X \text{ and } y \prec x \} \) (in particular, \( \rho(x) = 0 \) if there is no \( y \in X \) such that \( y \prec x \)). The rank of \( (X, \prec) \) is the least strict upper bound for \( \{ \rho(x) \mid x \in X \} \). It can be shown that the rank of a well-founded relation is always an ordinal of cardinality at most \( |X| \). In particular, the rank of a well-founded relation on a countable set is always less than \( \omega_1 \). In the case of analytic relations on Polish spaces, this bounded can be improved.

**Theorem 14.** Let \( \prec \) be an analytic relation on a Polish space \( X \). If \( (X, \prec) \) is well-founded, then its rank is a countable ordinal.

**Proof.** Without loss of generality, we may assume that \( X = \omega^\omega \). It suffices to show that there is a countable set \( S \), a relation \( \prec^* \), and a function \( \phi \) from the finite \( \prec \)-decreasing sequences in \( X \) into \( S \) such that:

- if \( (S, \prec^*) \) is ill-founded, then \( (X, \prec) \) is ill-founded and
- if \( \vec{x} \) is an initial part of \( \vec{y} \), then \( \phi(\vec{x}) \prec^* \phi(\vec{y}) \).

Let \( E \subseteq (\omega^\omega)^3 \) be such that \( x \prec y \) if and only if there is a \( z \in \omega^\omega \) such that \( (x, y, z) \in E \). If \( x \prec y \), define \( \theta(x, y) \) to be the lexicographically least \( z \) such that \( (x, y, z) \in E \). Define a relation on define \( S \) to consist of all finite sequences \( \sigma = \langle (s_\tau^\sigma, t_\tau^\sigma, u_\tau^\sigma) \mid i < n^\sigma \rangle \) such that for all \( i < n^\sigma \),

\[ [s_\tau^\sigma] \times [t_\tau^\sigma] \times [u_\tau^\sigma] \cap E \neq \emptyset \]

and for \( i < n^\sigma - 1, t_{i+1}^\sigma = s_i^\sigma \). Define \( \prec^* \) on \( S \) by \( \tau \prec^* \sigma \) if \( n^\tau < n^\sigma \) and for all \( i < n^\sigma \),

\[ s_i^\sigma \subset s_i^\tau \quad \text{and} \quad u_i^\sigma \subset u_i^\tau. \]

First we will show that if \( (S, \prec^*) \) is ill-founded, then \( (X, \prec) \) is ill-founded. To this end, suppose that \( \langle \sigma_k \mid k \in \omega \rangle \) is a sequence in \( S \) such that \( \sigma_{k+1} \prec^* \sigma_k \) for all \( k \in \omega \). Define \( x_i = \bigcup_{k \in \omega} s_i^{\sigma_k} \) and \( y_i = \bigcup_{k \in \omega} u_i^{\sigma_k} \). Observe that \( (x_{i+1}, x_i, y_i) \in E \) and therefore that \( x_{i+1} \prec x_i \).

It remains to define a function \( \phi \) which maps the finite \( \prec \)-decreasing sequences in \( X \) into \( S \) so that \( \phi(\vec{x}) \prec^* \phi(\vec{y}) \) whenever \( \vec{x} \) is an initial
part of $\vec{y}$. Given a finite $\prec$-decreasing sequence $\vec{x} \in X^{<\omega}$ of length $n$, define

$$\phi(\vec{x}) = \langle (x_i \upharpoonright n, x_{i+1} \upharpoonright n, \theta(x_i, x_{i+1}) \upharpoonright n) \mid i < n \rangle.$$ 

It is routine to check that $\phi$ is as desired. \hfill \Box

**Theorem 15.** Suppose that $A \subseteq \mathcal{P}(\mathbb{Q})$ is analytic and if $A \in A$, then $A$ is well ordered by the usual order on $\mathbb{Q}$. Then there is a countable ordinal $\alpha$ such that if $A \in A$, then $\text{otp}(A) < \alpha$.

**Proof.** Define $\prec$ on $\mathcal{P}(\mathbb{Q})$ by $A \prec B$ if $A$ and $B$ are in $A$ and there is an $f : A \to B$ which is an order preserving function and whose range is bounded by an element of $B$. Notice that the collection of pairs $(A, B, f) \in \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q}^2)$ such that $f : A \to B$ is order preserving with range bounded by an element of $b$ is Borel. Consequently $\prec \subseteq A \times A$ is analytic. By the boundedness principle, the rank of $(A, \prec)$ is a countable ordinal $\alpha$. This implies that $\{\text{otp}(A) \mid A \in A\}$ has ordertype $\alpha$ and in particular is countable. Since any countable subset of $\omega_1$ is bounded, we are done. \hfill \Box

**Corollary 6.** The collection $\mathcal{W}$ of subsets of $\mathbb{Q}$ which are well ordered by the usual order on $\mathbb{Q}$ is co-analytic but not Borel.

**Proof.** Since every countable linear order embeds into $\mathbb{Q}$, every countable ordinal embeds into $\mathbb{Q}$. Thus $\omega_1$ is the least upper bounded for the ordertypes of the well ordered subsets of $\mathbb{Q}$. \hfill \Box
Lecture 7

As you may have noticed, the interesting examples of Polish spaces all have the same cardinality of \( \mathbb{R} \). We have seen already that every separable metric space is embeddable into \( \mathbb{R}^\omega \) and hence has cardinality at most

\[
|\mathbb{R}^\omega| = |(2^\omega)^\omega| = |2^\omega| = 2^{2^\omega}.
\]

We will now see that any Polish space which has cardinality less than \( 2^{\aleph_0} \) must in fact be countable and moreover satisfy strong topological constraints.

A subset of a topological space \( X \) is perfect if it is closed, nonempty, and has no isolated points. A topological space \( S \) is scattered if every nonempty subspace of \( S \) contains an isolated point. The assertion that \( E \) is perfect becomes a much stronger assumption if \( X \) is assumed to be, e.g., Polish.

**Theorem 16.** If \( X \) is a Polish space and \( E \subseteq X \) is perfect, then \( E \) contains a homeomorphic copy of the Cantor set.

If \( X \) is a topological space, then define the Cantor-Bendixon derivative of \( X \) to be the set \( X' \) consisting of all nonisolated points of \( X \). If \( \alpha \) is an ordinal, the define \( X^{(\alpha)} \) recursively by:

\[
X^{(\alpha)} = \begin{cases} 
X & \text{if } \alpha = 0 \\
(X^{(\beta)})' & \text{if } \alpha = \beta + 1 \\
\bigcap_{\beta < \alpha} X^{(\beta)} & \text{if } \alpha \text{ is a positive limit ordinal}
\end{cases}
\]

It is readily verified by induction on \( \alpha \) that \( X^{(\alpha)} \) is a closed subset of \( X \) for each \( \alpha \) and that if \( \alpha < \beta \), then \( X^{(\beta)} \subseteq X^{(\alpha)} \). In particular, there is a least \( \alpha \), called the Cantor-Bendixon rank of \( X \) such that \( X^{(\alpha)} \) has no isolated points (i.e. \( X^{(\alpha+1)} = X^{(\alpha)} \)). We say that \( X \) is scattered if there is an \( \alpha \) such that \( X^{(\alpha)} \) is empty. Notice that this is the same as saying that every subspace of \( X \) has an isolated point.

**Proposition 4.** Suppose that \( X \) is a topological space with a base of cardinality \( \theta \).

- Any strictly \( \subseteq \)-decreasing well ordered sequence of closed subsets of \( X \) has cardinality at most \( \theta \) (and hence length less than \( \theta^+ \)).
- Any discrete subset of \( X \) has cardinality at most \( \theta \).

In particular, the Cantor-Bendixon rank of \( X \) is an ordinal \( \alpha < \theta^+ \) and \( X \setminus X^{(\alpha)} \) has cardinality at most \( \theta \).

In particular, we have the following theorem.
Theorem 17. If $X$ is a separable metric space, then there is a perfect set $P \subseteq X$ such that $X \setminus P$ is scattered and countable.

Next we turn to the cardinality of analytic sets in Polish spaces. If $X$ is a separable metric space, then an open graph on $X$ is a graph (with no loops or multiple edges) on the vertex set $X$ in which the adjacency relation is open when viewed as a symmetric subset of $X \times X$. Let $\text{OCA}^*(X)$ be the assertion that whenever $G$ is an open graph on $X$, then either:

- there is an $H \subseteq X$ which is homeomorphic to the Cantor set which is a complete subgraph of $G$;
- there is a function $\chi : X \rightarrow \omega$ such that if $u, v \in X$ are adjacent, then $\chi(u) \neq \chi(v)$.

If $H$ witnesses the first alternative, we say that $H$ is a perfect clique of $G$. If the second alternative holds for $X$, then we say that $G$ is countably chromatic.

Theorem 18. If $A$ is an analytic set, then $\text{OCA}^*(X)$ is true.

Proof. First observe that if $\text{OCA}^*(X)$ is true and $f : X \rightarrow Y$ is a continuous surjection, then $\text{OCA}^*(Y)$ is true. To see this, suppose that $G$ is an open graph on $Y$ and define $\tilde{G}$ on $Y$ so that $u, v \in X$ are connected by an edge in $\tilde{G}$ if and only if $f(u) \neq f(v)$ are connected by an edge in $G$. Observe that if $E \subseteq X$ is such that no two elements of $E$ are connected by an edge in $\tilde{G}$, then no two elements of the $f$-image of $E$ are connected by an edge in $G$. In particular, if $\tilde{G}$ is countably chromatic, so is $G$. Also, if $H \subseteq X$ is a perfect clique, then the restriction of $f$ to $H$ is a one-to-one continuous function and hence an embedding; the $f$-image of $H$ is therefore a perfect clique of $G$.

It therefore suffices to show that $\text{OCA}^*(\omega^\omega)$ is true. To this end, let $G$ be a given open graph on $\omega^\omega$. We will say that a subset $E$ of $\omega^\omega$ is countably chromatic if the restriction of $G$ to this set is countably chromatic. If $\omega^\omega$ is countably chromatic, then we are done. Suppose not and construct $\langle u_s \mid s \in 2^{<\omega} \rangle$ so that:

1. $u_s$ is an initial part of $u_t$ if and only if $s$ is an initial part of $t$;
2. $[u_s]$ is not countably chromatic for each $s \in 2^{<\omega}$;
3. if $s \in 2^{<\omega}$ and $x, y \in \omega^\omega$ extend $u_{s \upharpoonright 0}$ and $u_{s \upharpoonright 1}$ respectively, then $x$ and $y$ are adjacent in $G$.

Start by setting $u_\varepsilon = \varepsilon$. Given $u_s$, let $X_s$ consist of all $x \in [u_s]$ such that no open neighborhood of $x$ is countably chromatic. Notice that $X_s$ is closed and that $[u_s] \setminus X_s$ is countably chromatic. In particular, $X_s$ is not countably chromatic and there are $x \neq y \in X_s$ such that $x$ and $y$
are adjacent in $G$. Since $G$ is open, there are initial parts $u_{s^{-0}} \subset x$ and $u_{s^{-1}} \subset y$ which properly extend $u_s$, which do not extend each other, and which satisfy that if $x' \in [u_{s^{-0}}]$ and $y' \in [u_{s^{-1}}]$, then $x'$ and $y'$ are adjacent. This finishes the construction.

Define $\phi : 2^\omega \to \omega^\omega$ by $\phi(a) = \bigcup_{n=0}^\infty u_a|_n$. It is easy to see that $\phi$ is a homeomorphism. Furthermore if $a <_{\text{lex}} b \in 2^\omega$ and $s$ is the maximum common initial segment of $a$ and $b$, then $\phi(a) \in u_{s^{-0}}$ and $\phi(b) \in u_{s^{-1}}$. In particular, $\phi(a)$ and $\phi(b)$ are adjacent in $G$. □

**Corollary 7.** If $A$ is an analytic set, then either $A$ is countable or else $A$ contains a homeomorphic copy of the Cantor set.

**Proof.** Let $G$ be the complete graph on $A$. If $G$ is countably chromatic, then $A$ is necessarily countable. If $H$ is a perfect clique of $G$, then $H \subseteq A$ and is homeomorphic to the Cantor set. □
In this lecture we will present some important consequences of the separation theorem for analytic sets.

**Theorem 19.** If $X$ and $Y$ are Polish spaces and $f : X \to Y$ is a function, then the following are equivalent:

1. $f$ is Borel measurable;
2. $\Gamma_f := \{(x, y) \in X \times Y \mid f(x) = y\}$ is Borel;
3. $\Gamma_f$ is analytic.

**Proof.** To see that (1) implies (2), let $\{y_n \mid n \in \omega\}$ enumerate a countable dense subset of $Y$ and let $U_{m,n}$ denote the $1/n$ ball about $y_m$. Observe that:

$$\Gamma_f = \bigcap_{n=0}^{\infty} \bigcup_{m=0}^{\infty} f^{-1}(U_{m,n}) \times U_{m,n}.$$ 

Since $f$ is Borel measurable, $f^{-1}(U_{m,n})$ is Borel and hence $\Gamma_f$ is a Borel, being a countable Boolean combination of Borel sets.

That (2) implies (3) is trivial.

To see that (3) implies (1), observe that by the separation theorem it suffices to show that if $U \subseteq Y$ is open, then $f^{-1}(U)$ and $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ are analytic. This is a trivial consequence of the observation that

$$f^{-1}(U) = \pi_X(\Gamma_f \cap X \times U) \quad f^{-1}(Y \setminus U) = \pi_X(\Gamma_f \cap X \times (Y \setminus U))$$

are projections of analytic sets and hence analytic. \hfill \Box

While continuous images of Borel sets need not be Borel, one-to-one continuous images of Borel sets are Borel.

**Theorem 20.** If $X$ and $Y$ are Polish and $f : X \to Y$ is a one-to-one Borel function, then the $f$-image of any Borel set is Borel.

**Proof.** First recall that, given any Borel subset of $X$, there is a stronger 0-dimensional Polish topology on $X$ in which the Borel set is clopen and with respect to which the function $f$ is continuous. Since any 0-dimensional Polish space is homeomorphic to a closed subspace of $\omega^\omega$, it suffices to show that if $E \subseteq \omega^\omega$ is closed, then any one-to-one continuous image of $E$ is Borel.

For each $t \in \omega^{<\omega}$, define $A_t$ to be the image of $E \cap [t]$ under $f$. Notice that, for each $t$, $\{A_{t \cup \langle n \rangle} \mid n \in \omega\}$ is a pairwise disjoint family of analytic sets which partitions $A_t$. Using the separation theorem, construct Borel sets $B_t$ such that the following are true for each $t \in \omega^{<\omega}$:

- $A_t \subseteq B_t \subseteq \overline{A_t}$;
• \( \{ B_{t^{-n}} \mid n \in \omega \} \) is a partition of \( B_t \).

For each \( n \), we have

\[
f(E) = \bigcup_{t \in \omega^n} A_t \subseteq \bigcup_{t \in \omega^n} B_t.
\]

It therefore suffices to show that

\[
\bigcap_{n=0}^{\infty} \bigcup_{t \in \omega^n} B_t \subseteq f(E).
\]

If \( y \in \bigcap_{n=0}^{\infty} \bigcup_{t \in \omega^n} B_t \), then for each \( n \) let \( t_n \) be the unique sequence of length \( n \) such that \( y \in B_{t_n} \). Let \( y_n \in A_{t_n} \) be such that the distance between \( y_n \) and \( y \) is at most \( 1/n \). Fix \( x_n \in E \) with \( f(x_n) = y_n \), noting that \( x_n \) extends \( t_n \). Observe that if \( m < n \), then \( t_m \) is an initial part of \( t_n \) and thus \( x_n \to x \) where \( x \in \omega^\omega \) is such that \( x \upharpoonright n = t_n \) for each \( n \). Since \( E \) is closed, \( x \) is in \( E \) and since \( f \) is continuous, \( f(x) = y \). □

When developing our “change of topology” machinery, we took care to ensure that the stronger topologies which we considered generated the same Borel sets as the original topology. It turns out that this happens automatically as the next corollary shows.

**Corollary 8.** If \( X \) is a Polish space, then any stronger topology on \( X \) has the same Borel sets.

**Proof.** Let \( \tau \) be the Polish topology on \( X \) and let \( \tau' \) be a stronger Polish topology on \( X \). The identity function from \( (X, \tau') \) to \( (X, \tau) \) is continuous and one-to-one. It follows that if \( U \) is \( \tau' \)-open, then \( U \) is \( \tau \)-Borel. □

If \( X \) is an uncountable set equipped with a \( \sigma \)-algebra of subsets, then we say that \( X \) is a **standard Borel space** if there is a Polish topology on \( X \) such that the \( \sigma \)-algebra consists of the Borel sets in this topology. The terminology is justified by the following theorem.

**Theorem 21.** If \( X \) and \( Y \) are standard Borel spaces, then there is a bijection between \( X \) and \( Y \) which is Borel measurable with respect to the witnessing Polish topologies.

**Proof.** It clearly suffices to prove this theorem when \( Y = \omega^\omega \oplus \omega \). The theorem now follows from the observation that any Polish topology can be strengthened to a Polish topology which is homeomorphic to \( \omega^\omega \oplus \omega \). □
Lecture 9

In this lecture we will finish our foundational study of analytic sets. Recall that if \( A \) is an analytic subset of a Polish space \( X \), then by strengthening the Polish topology on \( X \) if necessary, we may assume that \( A \subseteq X \subseteq \omega^\omega \).

If \( S \) is a set, then a tree on \( S \) is a set \( T \subseteq S^{<\omega} \) which is closed under taking initial segments. Let \( \text{Trees} \) denote the collection of all trees on \( \omega \), noting that this is a closed subspace of \( \mathcal{P}(\omega^{<\omega}) \) and hence a compact metric space. We will let \( \text{WF} \) and \( \text{IF} \) denote the set of all well founded and ill founded trees on \( \omega \), respectively. If \( S = A \times B \) is a cartesian product, then it is common to regard finite sequences \( \langle (a_i, b_i) \mid i < n \rangle \) on \( A \times B \) as the pair of sequences \( \langle a_i \mid i < n \rangle, \langle b_i \mid i < n \rangle \).

There is a natural correspondence between trees on \( S \) and closed subsets of \( S^{\omega} \), where \( S \) is given the discrete topology and \( S^{\omega} \) is given the product topology. Specifically, if \( T \) is a tree on \( S \), then \( B(T) := \{ x \in S^{\omega} \mid \forall n \in \omega (x \upharpoonright n \in T) \} \) is a closed subset of \( S^{\omega} \). If \( E \subseteq S^{\omega} \), then \( T_E = \{ x \upharpoonright n \mid (x \in E) \text{ and } (n \in \omega) \} \) is a tree. Notice that \( B(T) \) is always closed and in fact \( B(T_E) \) is the closure of \( E \). Similarly, \( T_E \) is always a pruned tree. Here a tree \( T \) is pruned if it does not contain any maximal elements. Every tree \( T \subseteq S^{<\omega} \) contains a maximal pruned subtree; this in fact coincides with \( T_{B(T)} \).

If \( T \subseteq S^{<\omega} \) is a tree, define \( T^{(\alpha)} \) for \( \alpha \) and ordinal by recursion. Set \( T^{(0)} = T \) and if \( T^{(\beta)} \) has been defined for all \( \beta < \alpha \), define \( T^{(\alpha)} \) to be all \( s \in T \) such that for every \( \beta < \alpha \) there is a \( t \in T^{(\beta)} \) which has \( s \) as a proper initial part.

**Lemma 4.** For each countable ordinal \( \alpha \), the map \( T \mapsto T^{(\alpha)} \) defined on the trees on \( \omega \) is a Borel function.

**Proof.** The proof is by induction on \( \alpha \). If \( \alpha = 0 \), then \( T \mapsto T^{(0)} \) is just the identity function. If \( \alpha > 0 \), then
\[
T^{(\alpha)} = \bigcap_{\beta < \alpha} \bigcup_{t \in \omega^{<\omega}} \{ s \in T \mid (s \subseteq t) \text{ and } (t \in T^{(\beta)}) \}
\]
which is a Borel set. \( \square \)

It is readily checked that each \( T^{(\alpha)} \) is a tree and if \( \alpha < \beta \), then \( T^{(\beta)} \subseteq T^{(\alpha)} \). In particular, there is an \( \alpha \) such that \( T^{(\alpha+1)} = T^{(\alpha)} \); set \( T^{(\omega)} := T^{(\alpha)} \) where \( T^{(\alpha+1)} = T^{(\alpha)} \). Observe that \( T^{(\omega)} \) is pruned and is the maximum pruned subtree of \( T \).
Lemma 5. If Trees is the Polish space consisting of all trees on $\omega$ and $\alpha$ is a countable ordinal, then
\[\WF_{\alpha} := \{T \in \text{Trees} \mid T^{(\alpha)} = \emptyset\}\]
\[\IF_{\alpha} := \{T \in \text{Trees} \mid T^{(\alpha+1)} = T^{(\alpha)} \neq \emptyset\}\]
are Borel sets.

Define $\rho$ on $T \setminus T^{(\infty)}$ by
\[\rho(s) = \min\{\xi \mid s \not\in T^{(\xi)}\}\]
Notice that if $T$ is well founded, then $T^{(\infty)}$ is empty and $\rho$ is a strictly decreasing function from $T$ into the ordinals.

If $T$ is a tree on $\omega \times S$ for some $S$, then define
\[pT := \{s \in \omega^\omega \mid \exists y \forall n \in \omega((x \upharpoonright n, y \upharpoonright n) \in T)\}\]
If $x \in \omega^\omega$, set $T(x) := \{t \in S^{<\omega} \mid (x \upharpoonright lh(t), t) \in T\}$. Observe that if $T$ is a tree on $\omega \times \omega$, the function $x \mapsto T(x)$ is continuous. Also, $x \in pT$ if and only if $T(x)$ is ill founded. It should be clear that if $A \subseteq \omega^\omega$, then $A = pT$ for some tree $T$ on $\omega \times \omega$.

Theorem 22. If $A \subseteq \omega^\omega$ is analytic, then there are sequences of Borel sets $\langle B_\xi \mid \xi \in \omega_1 \rangle$ and $\langle C_\xi \mid \xi \in \omega_1 \rangle$ such that
\[A = \bigcup_{\xi \in \omega_1} B_\xi \quad \omega^\omega \setminus A = \bigcup_{\xi \in \omega_1} C_\xi\]

Proof. Let $T$ be a tree on $\omega \times \omega$ such that $A = pT$. Set
\[B_\xi := \{x \in \omega^\omega \mid T(x) \in \IF_{\alpha}\}\]
\[C_\xi := \{x \in \omega^\omega \mid T(x) \in \WF_{\alpha}\}\]
By Lemma 5, these sets are Borel. Since $\WF = \bigcup_{\xi \in \omega_1} \WF_\xi$ and $\IF = \bigcup_{\xi \in \omega_1} \IF_\xi$, we have $pT = \bigcup_{\xi \in \omega_1} C_\xi$.

Corollary 9. If $C \subseteq \omega^\omega$ is a coanalytic set, then either $C$ contains a perfect subset or else $|C| \leq \aleph_1$.

Remark 1. This is the best result provable in ZFC. In fact if $\omega_1$ is not an inaccessible cardinal in Gödel's constructible universe, then there is an uncountable coanalytic set with no perfect subset.

Proof. By the previous theorem $C = \bigcup_{\xi \in \omega_1} C_\xi$ where each $C_\xi$ is a Borel set. If any $C_\xi$ is uncountable, then it must contain a perfect subset. Otherwise $|C| \leq \aleph_1 \cdot \aleph_0 = \aleph_1$. □
Lecture 10

Recall that a subset of a topological space is \textit{nowhere dense} if its closure does not contain a nonempty open set. A set which is a countable union of nowhere dense sets is said to be \textit{meager}. The complement of a meager set in a Polish space is said to be \textit{comeager}. The motivation for these definitions is the \textit{Baire category theorem}.

\textbf{Theorem 23.} Every Polish space is nonmeager.

\textit{Proof.} Let \( X \) be Polish and fix a complete metric \( d \) which generates the topology. Suppose that \( \langle E_n \mid n \in \omega \rangle \) is a sequence of closed nowhere dense subsets of \( X \). Construct a sequence of nonempty open sets \( \langle U_n \mid n \in \omega \rangle \) such that:

- the \( d \)-diameter of \( U_n \) is at most \( 1/n \);
- the closure of \( U_{n+1} \) is contained in \( U_n \setminus E_n \).

Notice that if \( U_n \) has been constructed, then \( U_n \) is not contained in \( E_n \) and thus \( U_n \setminus E_n \) is nonempty. We can then take \( U_{n+1} \) to be a suitably small open ball inside \( U_n \setminus E_n \). The proof is now finished by observing that, since \( d \) is complete,

\[
\bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} U_{n+1}
\]

is a singleton \( \{x\} \) such that \( x \) is not an element of \( E_n \) for any \( n \). \( \square \)

A subset \( A \) of a Polish space has the \textit{Baire property} (or \textit{property of Baire}) if there is an open set \( U \subseteq X \) such that \( A \triangle U \) is meager.

\textbf{Theorem 24.} If \( X \) is a Polish space, then the collection of subsets of \( X \) with the Baire property forms a \( \sigma \)-algebra which includes the open sets. In particular, every Borel set has the Baire property.

\textit{Proof.} Clearly every open set has the Baire property. If \( A \subseteq X \) has meager symmetric difference with an open set \( U \subseteq X \), then \( X \setminus A \) has meager symmetric difference with the interior of \( X \setminus U \):

\[
(X \setminus A) \triangle \text{int}(X \setminus U) \subseteq A \Delta U \cup (U \setminus U)
\]

In particular, the complement of a set with the Baire property also has the Baire property. Finally, if \( \langle A_n \mid n \in \omega \rangle \) is a sequence of sets with the Baire property and \( U_n \) is open for each \( n \) with \( A_n \triangle U_n \) meager, then

\[
(\bigcup_{n=0}^{\infty} A_n) \triangle (\bigcup_{n=0}^{\infty} U_n) \subseteq \bigcup_{n=0}^{\infty} (A_n \Delta U_n).
\]

In particular, a countable union of sets with the Baire property has the Baire property. \( \square \)
The next theorem of Kuratowski and Ulam characterizes meager subsets of the product of two Polish spaces. It has a wealth of applications in descriptive set theory.

**Theorem 25.** Suppose that $X$ and $Y$ are Polish spaces. If $E \subseteq X \times Y$ has the Baire property, then the following are equivalent:

1. $E$ is meager in the product topology;
2. $\{x \in X \mid \{y \in Y \mid (x, y) \in E\} \text{ is nonmeager}\}$ is meager;
3. $\{y \in Y \mid \{x \in X \mid (x, y) \in E\} \text{ is nonmeager}\}$ is meager.

**Proof.** By exchanging the roll of $X$ and $Y$, it clearly suffices to prove the equivalence of the first two items. Let $\{V_n \mid n \in \omega\}$ enumerate a basis for $Y$. Suppose first that $E \subseteq X \times Y$ is closed and define $A$ to be the set of all $x$ such that $\{y \in Y \mid (x, y) \in E\}$ is somewhere dense in $Y$. For each $n$, define

$$A_n := \{x \in X \mid \{y \in Y \mid (x, y) \in E_n\} \subseteq E\}.$$

Clearly $A_n$ is the maximal set such that $A_n \times \overline{V_n} \subseteq E_n$. Since $E$ is closed, it follows that $A_n$ is closed as well. Furthermore, $A = \bigcup_{n=0}^{\infty} A_n$. Also, $A_n$ has interior in $X$ if and only if $A_n \times \overline{V_n}$ has interior in $X \times Y$. It follows that $A$ is nonmeager if and only if some $A_n$ has interior if and only if $E$ has interior.

If $E \subseteq X \times Y$ is meager, then it is contained in $\bigcup_{n=0}^{\infty} E_n$ where each $E_n$ is a closed nowhere dense set. For each $n$,

$$A_n := \{x \in X \mid \{y \in Y \mid (x, y) \in E_n\} \text{ is somewhere dense}\}$$

is nowhere dense. Consequently

$$\{x \in X \mid \{y \in Y \mid (x, y) \in \bigcup_{n=0}^{\infty} E_n\} \text{ is nonmeager}\} \subseteq \bigcup_{n=0}^{\infty} A_n$$

is meager.

Finally, suppose that $E \subseteq X \times Y$ has the Baire property and is nonmeager. We must show that

$$\{x \in X \mid \{y \in Y \mid (x, y) \in E\} \text{ is nonmeager}\}$$

is nonmeager. Let $U$ and $V$ be nonempty open sets such that $E \cap U \times V$ is comeager in $U \times V$. Observe that both $U$ and $V$ are themselves Polish and also nonmeager. Thus $U \times V \setminus E$ is meager and

$$\{x \in U \mid \{y \in V \mid (x, y) \notin E\} \text{ is nonmeager}\}$$

is meager. Consequently

$$\{x \in U \mid \{y \in V \mid (x, y) \in E\} \text{ is nonmeager}\}$$

is nonmeager, as desired. □
The goal of this lecture will be to prove that analytic subsets of Polish spaces have the Baire property and that analytic subsets of $\mathbb{R}$ are Lebesgue measurable. We will actually prove a more general result.

Let $X$ be a Polish space. A collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is a Borel $\sigma$-ideal on $X$ if:

- $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$;
- $\mathcal{I}$ is closed under subsets and countable unions;
- every element of $\mathcal{I}$ is contained in a Borel set in $\mathcal{I}$.

We can think of $\mathcal{I}$ as specifying a notion of being “small” – where subsets of $X$ are small precisely when they are in $\mathcal{I}$. If $A, B \subseteq X$ are Borel, then we will write $A \subseteq^\mathcal{I} B$ to mean that $A \setminus B$ is in $\mathcal{I}$. A set $A \subseteq X$ is $\mathcal{I}$-measurable if there is a Borel set $B \subseteq X$ such that the symmetric difference $A \Delta B$ is in $\mathcal{I}$. It is not difficult to adapt the proof from the previous lecture to show that the collection of $\mathcal{I}$-measurable sets forms a $\sigma$-algebra which contains the Borel sets.

A prototypical example of a Borel $\sigma$-ideal is the collection of meager subsets of a Polish space $X$ (denoted $\mathcal{M}(X)$ or $\mathcal{M}$ if $X$ is clear from the context). Another is the collection $\mathcal{N}$ of Lebesgue measure zero subsets of $\mathbb{R}$ (these are sometimes referred to as null sets and denoted collectively by $\mathcal{N}$). More generally, if $\mu$ is a countably additive Borel measure on a Polish space, then the $\mu$-null sets $\mathcal{N}_\mu$ form a Borel $\sigma$-ideal. Observe that $\mathcal{M}$-measurability coincides with the Baire Property and that $\mathcal{N}_\mu$-measurability coincides with $\mu$-measurability.

A Borel $\sigma$-ideal $\mathcal{I}$ is said to have the c.c.c. if whenever $\mathcal{A}$ is a family of Borel sets which are not in $\mathcal{I}$ such that the intersection of every two distinct elements of $\mathcal{A}$ is in $\mathcal{I}$, then $\mathcal{A}$ is at most countable. Notice that $\mathcal{M}(X)$ is always a c.c.c. ideal and that $\mathcal{N}_\mu$ is a c.c.c. ideal provided that $\mu$ is a $\sigma$-finite measure.

**Theorem 26.** Suppose that $X$ is a Polish space and $\mathcal{I}$ is a c.c.c. Borel $\sigma$-ideal on $X$. If $\mathcal{F}$ is a collection of Borel sets, then there exists a Borel sets $A$ such that if $B$ is in $\mathcal{F}$, then $A \subseteq^\mathcal{I} B$ and if $A'$ is any other Borel set with this property, then $A' \subseteq^\mathcal{I} A$.

**Proof.** It suffices to show that there is a countable subset $\mathcal{F}' \subseteq \mathcal{F}$ such that if $A = \bigcup \mathcal{F}'$, then $B \subseteq^\mathcal{I} A$. If this is not true, then construct a sequence $\langle B_\xi \mid \xi \in \omega_1 \rangle$ of elements of $\mathcal{F}$ such that $B_\xi \not\subseteq^\mathcal{I} \bigcup_{\eta \in \xi} B_\eta$. If we set $A_\xi := B_\xi \setminus \bigcup_{\eta \in \xi} B_\eta$, then $A = \{A_\xi \mid \xi \in \omega_1\}$ is pairwise disjoint and consists of Borel sets not in $\mathcal{I}$, which contradicts our assumption that $\mathcal{I}$ is c.c.c.

We are now ready to state the main result of this lecture.
Theorem 27. Suppose that $X$ is a Polish space and that $\mathcal{I}$ is a c.c.c. Borel $\sigma$-ideal on $X$. Every analytic subset of $X$ is $\mathcal{I}$-measurable.

In fact this theorem will be an immediate consequence of a more general closure property which holds of $\mathcal{I}$-measurability. If $\{A_s \mid s \in \omega^{<\omega}\}$ is a family of subsets of a Polish space $X$, define

$$\mathcal{A}(A_s \mid s \in \omega^{<\omega}) := \bigcup_{x \in \omega^\omega} \bigcap_{n=0}^{\infty} A_{x|n}.$$ 

This operation is known as Souslin’s $\mathcal{A}$-operation.

Fact 1. A subset of a Polish space is analytic if and only if it can be obtained from a family of closed sets by the $\mathcal{A}$-operation.

Proof. If $A \subseteq X$ is the image of a continuous $f : \omega^\omega \to X$, then observe that $A = \mathcal{A}(f([s]) \mid s \in \omega^{<\omega})$. To see this, notice that if $x \in \omega^\omega$, then the continuity of $f$ implies that

$$\bigcap_{k=0}^{\infty} f([x \upharpoonright k]) = \{f(x)\}.$$ 

Conversely, if $A = \mathcal{A}(A_s \mid s \in \omega^{<\omega})$, then $A$ is the projection of the closed set

$$\{(x,y) \in X \times \omega^\omega \mid \forall k \in \omega (x \in A_{y|k})\}$$

and hence is analytic. \qed

Theorem 28. If $X$ is a Polish space and $\mathcal{I}$ is a c.c.c. Borel $\sigma$-ideal in $X$, then the $\mathcal{A}$-operation preserves $\mathcal{I}$-measurability.

Proof. Let $A = \mathcal{A}(A_s \mid s \in \omega^{<\omega})$ where each $A_s$ is $\mathcal{I}$-measurable. It suffices to show that $A$ differs from an $\mathcal{I}$-measurable set by an element of $\mathcal{I}$. For each $s \in \omega^{<\omega}$, define

$$A^s := \bigcup_{x \in [s]} \bigcap_{k=0}^{\infty} A_{x|k},$$

observing that $A^s \subseteq A_s$.

Let $\hat{A}^s$ be an $\mathcal{I}$-measurable set such that $A^s \subseteq \hat{A}^s \subseteq A_s$ and such that if $B$ is $\mathcal{I}$-measurable and $A^s \subseteq B$, then $\hat{A}^s \subseteq \mathcal{I} B$. Define

$$I_s = \hat{A}^s \bigtriangleup \bigcup_{n=0}^{\infty} \hat{A}^{s\prec(n)}.$$
Observe that $I_s$ is $\mathcal{I}$-measurable. Furthermore, since both $\hat{A}^s$ and $\bigcup_{n=0}^{\infty} \hat{A}^{s^{-}(n)}$ are $\mathcal{I}$-minimal sets containing $A^s = \bigcup_{n=0}^{\infty} A^{s^{-}(n)}$, it follows that $I_s$ is in $\mathcal{I}$ for each $s \in \omega^{<\omega}$. Since $\mathcal{I}$ is a $\sigma$-ideal, $I := \bigcup_{s \in \omega^{<\omega}} I_s$ is in $\mathcal{I}$ as well.

It suffices to show that the symmetric difference of $A = A^\varepsilon$ and $\hat{A}^\varepsilon$ is $I$. Since $I$ and $A$ are disjoint and since both are contained in $\hat{A}^\varepsilon$ it is sufficient to show that $\hat{A}^\varepsilon \setminus I \subseteq A$. Let $x \in \hat{A}^\varepsilon \setminus I$ and construct a sequence $y = \langle n_i \mid i \in \omega \rangle$ by recursion so that for all $k$, $x \in \hat{A}^{y|k}$. This is always possible since for all $s \in \omega^{<\omega}$

$$\hat{A}^s \setminus I = \bigcup_{n=0}^{\infty} \hat{A}^{s^{-}(n)} \setminus I.$$  

We now have that

$$y \in \bigcap_{k=0}^{\infty} \hat{A}^{y|k} \subseteq \bigcap_{k=0}^{\infty} A^{y|k} \subseteq A.$$  

This completes the proof. \qed
A construction in harmonic analysis which plays a recurring role in descriptive set theory is Haar measure. Recall that a topological group is a group $G$ equipped with a topology which makes both the group operation and the inversion map $g \mapsto g^{-1}$ continuous. A Polish group is a topological group in which the underlying topology is Polish. The next remarkable theorem of Haar is very useful in producing examples of Borel measures on Polish spaces.

**Theorem 29.** If $G$ is a locally compact group, then there is a Borel measure $\mu$ on $G$ such that:

- $\mu(K) < \infty$ if $K \subseteq G$ is compact;
- $\mu(U) > 0$ if $U$ is a nonempty open set;
- $\mu(gB) = \mu(B)$ whenever $B \subseteq G$ is measurable.

Moreover any two measures which satisfy these two conditions are multiples of one another.

The measures which satisfy the conclusion of this theorem are referred to as Haar measures on $G$. If $G$ is compact, then it is conventional to use the phrase the Haar measure on $G$ to refer to the unique Haar measure $\mu$ which satisfies $\mu(G) = 1$.

For now we note some important examples. If $G$ is a discrete group, then the counting measure is a Haar measure on $G$. The Haar measure $\lambda$ on $(\mathbb{R}, +)$ which satisfies $\lambda([0, 1]) = 1$ is Lebesgue measure. If $\langle G_i \mid i \in I \rangle$ is a sequence of compact groups, then the product group $\prod_{i \in I} G_i$ is also a compact group; the Haar measure on the product is the product of the Haar measures of the $G_i$’s. In particular, the Haar measure on $2^\omega$ is the product measure.

Before proceeding, we’ll prove the following proposition, which whose proof requires techniques which are of independent interest to us. Recall that the map $A \mapsto \chi_A$ defines a homeomorphism between $\mathcal{P}(\omega)$ and $2^\omega$. Moreover $\chi_{A \triangle B} = \chi_A + \chi_B$, where $+$ is coordinatewise addition modulo 2. It follows that $A \mapsto \omega \setminus A = \omega \triangle A$ preserves the Haar measure on $(\mathcal{P}(\omega), \triangle)$.

**Proposition 5.** If $\mathcal{U}$ is a nonprinciple ultrafilter on $\omega$, then $\mathcal{U}$ does not have the Baire property and $\mathcal{U}$ is not Haar measurable.

Observe that if $\mathcal{U}$ is an ultrafilter, then $A \mapsto \omega \setminus A$ sends $\mathcal{U}$ bijectively to $\mathcal{P}(\omega) \setminus \mathcal{U}$. Since $\mathcal{P}(\omega)$ is nonmeager and measure 1, it follows that $\mathcal{U}$ is nonmeager and has positive measure. Since any nonprinciple ultrafilter is invariant under finite modifications, the proposition will follow from the next theorems.
\textbf{Theorem 30.} If $X \subseteq 2^\omega$ is a Borel set which is invariant under finite modifications, then is either meager or comeager.

\textit{Proof.} If $X$ is meager, there is nothing to show. Suppose now that $t \in 2^{<\omega}$ is such that $[t] \setminus X$ is meager. If $s \in 2^{<\omega}$ has the same length as $t$, then since $X$ is invariant under finite modifications $s^\updownarrow z \mapsto t^\updownarrow z$ defines a homeomorphism between $[s] \cap X$ and $[t] \cap X$. In particular, $[s] \setminus X$ is meager. But now if $n = lh(t)$, we have that $2^\omega \setminus X = \bigcup_{s \in 2^n} [s] \setminus X$ is meager. \hfill \Box

\textbf{Theorem 31.} If $X \subseteq 2^\omega$ is a Borel set which is invariant under finite modifications, then has either measure 0 or 1 with respect to the product measure on $2^\omega$.

\textit{Proof.} Let $X$ be given. If $X$ has measure 0, there is nothing to show so suppose that $X$ has positive measure. It is sufficient to show that for every $\varepsilon > 0$, the measure of $X$ is at least $1 - \varepsilon$. Let $E \subseteq X$ be a compact set of positive measure. Observe that

$$\mu(E) = \lim_{n \to \infty} 2^{-n}|\{t \in 2^n \mid [t] \cap E \neq \emptyset\}|.$$ 

Let $n$ be sufficiently large such that

$$(1 - \varepsilon)2^{-n}|\{t \in 2^n \mid [t] \cap E \neq \emptyset\}| \leq \mu(E).$$

Since $\mu(E) = \sum_{t \in 2^n} \mu(E \cap [t])$, it follows that there must be an $t \in 2^n$ such that $\mu(E \cap [t]) \geq 2^{-n}(1 - \varepsilon)$. Since $X$ is invariant under finite modifications, it follows that

$$\bigcup_{s \in 2^n} \{s^\updownarrow x \mid t^\updownarrow x \in E\} \subseteq X$$

Since $\mu(\{s^\updownarrow x \mid t^\updownarrow x \in E\}) = \mu(E \cap [t]) \geq r2^{-n}$, we have that $\mu(X) \geq r$. \hfill \Box
Lecture 13

We will now present the construction of Haar measure. Let $G$ be a locally compact group with identity 1 and fix a compact open neighborhood $E$ of the identity. If $A, B \subseteq G$ are nonempty, define $[A : B]$ to be the minimum number of translates of $A$ needed to cover $B$ (if this is not finite, we just set $[A : B] := \infty$). If $W$ is a neighborhood of the identity and $K \subseteq G$ is compact, define $\mu_W(K) = [W : K]/[W : E]$. Observe that $\mu_W(gK) = \mu_W(K)$ for all $g \in G$ and compact $K \subseteq G$. Furthermore, if $A, B \subseteq G$ are compact, $W$ is open, and for all $g \in G$ either $gW \cap A = \emptyset$ or $gW \cap B = \emptyset$, then $\mu_W(A \cup B) = \mu_W(A) + \mu_W(B)$.

Now let $\mathcal{O}$ be the collection of all open neighborhoods of 1 and $\mathcal{W}$ be an ultrafilter on $\mathcal{O}$ such that if $W \in \mathcal{W}$, then $W$ is a neighborhood base of 1. If $K \subseteq G$ is compact, define

$$\mu(K) = \lim_{W \to \mathcal{W}} \mu_W(K).$$

That is, if $a \in [0, 1]$, then $\mu(K) < a$ if and only if $\{W \in \mathcal{O} \mid \mu_W(K) < a\} \in \mathcal{W}$. If $B \subseteq G$ is Borel, define

$$\mu(B) = \sup\{\mu(K) \mid K \subseteq B \text{ is compact}\}.$$

Observe that this extends the definition in the compact case.

**Theorem 32.** $\mu$ is a countably additive left invariant Borel measure on $G$ such that $\mu(E) = 1$.

We also note the following generalization of the 0-1 law for $2^\omega$.

**Theorem 33.** Suppose $G$ is a compact metric group and $H \leq G$ is a dense subgroup. If $B \subseteq G$ is Borel and $HB = B$, then either $\mu(B) = 0$ or $\mu(B) = 1$ and either $B$ is meager or comeager.

**Proof.** The details of the proof will be outlined here and left as a set of exercises. First one proves that if $A, B \subseteq G$ are Borel, then

$$\mu(A)\mu(B) = \int \mu(A \cap gB) \, d\mu(g).$$

This is established by showing that, for a fixed $B$, $\nu(A) = \frac{1}{\mu(B)} \int \mu(A \cap gB) \, d\mu(g)$ is a left invariant probability measure on $G$ and thus equal to Haar measure.

This identity is then used to prove that if $E \subseteq G$ is closed and $r > 1$, then there is a $h \in H$ and an $\varepsilon > 0$ such that

$$\mu(E \cap B_\varepsilon(h)) > r\mu(B_\varepsilon(h)).$$

This is then used to show that $HE$ has measure at least $r$.  \qed
Next we will turn to proving some rigidity results for Polish groups. We will need the following result, known as Pettis’s Lemma.

**Lemma 6.** If $G$ is a topological group and $A \subseteq G$ is a nonmeager set with the Baire property, then $A^{-1} A$ contains an neighborhood of the identity.

**Proof.** Let $U \subseteq G$ be open such that $U \Delta A$ is meager. Since $(x, y) \mapsto xy^{-1}$ is continuous, there is an open set $V$ containing the identity and a $g \in G$ such that $gV V^{-1} \subseteq U$. Observe that if $h \in V$, then $gV \subseteq U \cap U_h$. Since $U \Delta A$ is meager, $Uh \Delta Ah$ is meager. It follows that $gV \setminus (A \cap Ah)$ is meager for all $h \in V$. Since $gV$ is nonmeager, $A \cap Ah \neq \emptyset$ for all $h \in V$. But this implies that if $h \in V$, then there are $a, b \in A$ such that $a = bh$ and hence $b^{-1}a = h$. Thus $V \subseteq A^{-1} A$. □

Our main interest in the next theorem is when $G$ and $H$ are Polish groups.

**Theorem 34.** Suppose that $G$ and $H$ are topological groups such that $G$ is nonmeager and $H$ is separable. If $\phi : G \to H$ is a homomorphism which is Baire measurable, then $\phi$ is continuous.

**Proof.** Since $G$ is a topological group, it suffices to show that $\phi$ is continuous at $1$. Let $U \subseteq H$ be an open set containing $1 = \phi(1)$ and let $V \subseteq H$ be an open set containing $1$ such that $V^{-1} V \subseteq U$. Let $D \subseteq H$ be a countable dense set. Observe that $H \subseteq DV$: if $h \in H$ then $hV^{-1} \cap D$ contains some $h_0$, in which case $h \in h_0 V$. Since $G = \bigcup_{h \in D} \phi^{-1}(hV)$, there is an $h \in D$ such that $\phi^{-1}(hV)$ is nonmeager. By Pettis’s Lemma, there is an open set $W \subseteq G$ containing $1$ such that $W \subseteq (\phi^{-1}(hV))^{-1} \phi^{-1}(hV) = \phi^{-1}(V^{-1} h^{-1} h V) = \phi^{-1}(V^{-1} V) \subseteq \phi^{-1}(U)$. Since $U$ was arbitrary, $\phi$ is continuous at $1$ and hence continuous. □

**Corollary 10.** If $G$ is a group and $\tau_0$ and $\tau_1$ are two Polish group topologies on $G$ which yield the same Borel sets, then $\tau_0 = \tau_1$.

Notice that $(\mathbb{R}, +)$ and $(\mathbb{R} \oplus \mathbb{R}, +)$ are Polish groups which are isomorphic (in fact they are isomorphic as $\mathbb{Q}$-vector spaces). The theorem implies, however, that there is no isomorphism which is Baire measurable.
Lecture 14

Observe that if $A$ is the set of all $a \in 2^{\omega}$ such that $a$ is not eventually 0, then every compact subset of $A$ is nowhere dense. In particular, $A$ is not an $F_\sigma$-set. In fact this is the only obstruction to when an analytic set $A$ can be separated from a set $B$ by an $F_\sigma$-set.

**Theorem 35.** Suppose that $X$ is a Polish space. If $A \subseteq X$ is analytic and $B \subseteq X$ is disjoint from $A$, then exactly one of the following is true:

1. there is an $F_\sigma$-set which contains $A$ and is disjoint from $B$;
2. there is a continuous function $\phi : 2^{\omega} \to A \cup B$ such that $\phi(a) \in A$ if and only if $a$ is not eventually 0.

Stated in this generality, this theorem is due to Kechris, Louveau, and Woodin. Hurewicz proved the special case in which $B = X \setminus A$. Notice that in the second alternative, the range of $\phi$ is a Cantor set $P$ such that $P \cap B$ is a countable dense subset of $P$.

We will establish this result first when $A$ is a $G_\delta$-set and then use a result of Solecki to establish the full strength of the theorem. Solecki’s result can be stated as follows.

**Theorem 36.** Suppose that $A$ is an analytic set and $\mathcal{F}$ is a family of closed subsets of $A$. Either $A$ can be covered by countably many elements of $\mathcal{F}$ or else there is a Polish space $G \subseteq A$ such that $G$ can not be covered by countably many elements of $\mathcal{F}$.

More generally, we say that a separable metric space $X$ has the closed covering property if whenever $\mathcal{F}$ is a collection of closed subsets of $X$, either $X$ is a countable union of elements of $\mathcal{F}$ or else there is a Polish space $G \subseteq X$ which can not be covered by countably many elements of $\mathcal{F}$. Solecki’s theorem asserts that analytic sets have closed covering property. We will prove Theorem 36 in the next lecture.

**Proof of Theorem 35.** Let $X$ be a Polish space and $A,B \subseteq X$ be disjoint sets such that $A$ is analytic. Suppose that $A$ can not be separated from $B$ by an $F_\sigma$-set. Let $\mathcal{F}$ consist of those closed subsets of $X$ which are disjoint from $B$. By Solecki’s theorem, there is a Polish $G \subseteq A$ such that $G$ is not covered by countably many elements of $\mathcal{F}$. By replacing $G$ with a subset if necessary, we may assume that if $U \subseteq X$ is open and $U \cap G$ is nonempty, then $U \cap G$ can not be covered by countably many elements of $\mathcal{F}$. Fix complete metrics $d$ and $d'$ on $X$ and $G$ respectively which generate their topologies.

Construct a scheme of open sets $\{U_s \mid s \in 2^{<\omega}\}$ and points $\{y_s \mid s \in 2^{<\omega}\}$ such that for all $s \in 2^{<\omega}$:
(1) the $d$-diameter of $U_s$ is at most $2^{-lh(s)}$ and the $d'$-diameter of $U_s \cap G$ is at most $2^{-m}$ where $m$ is the number of occurrences of 1 in $s$;

(2) $y_s \in \overline{U_s \cap G} \cap B$ and $y_{s-0} = y_s$.

(3) the closures of $U_{s-0}$ and $U_{s-1}$ are disjoint and contained in $U_s$;

Start by picking $y_\varepsilon \in \overline{G} \cap B$ and letting $U_\varepsilon \subseteq X$ be an open set satisfying (1). Given $U_s$ and $y_s$ satisfying the conditions above, let $U_{s-1}$ be a $d$-ball is diameter at most $2^{-lh(s)-1}$ about $y_s$ such that $\overline{U_{s-1}} \subseteq U_s$ and such that $U_s \setminus \overline{U_{s-1}}$ intersects $G$. Notice that this is possible since $y_s$ is a limit point of $U_s \cap G$. Let $W \subseteq U_s \setminus \overline{U_{s-1}}$ be an open set such that $W \cap G$ is nonempty and of the $d'$-diameter at most $2^{-m-1}$ where $m$ is the number of occurrences of 1 in $s$. Since $W$ is open and intersects $G$, $W \cap G$ can not be separated from $B$ by an $F_\sigma$-set. In particular, there is a $y_{s-1} \in B$ which is a limit point of $W \cap G$. Let $U_{s-1}$ be open such that $y_{s-1} \in U_{s-1}$, $\overline{U_{s-1}} \subseteq W$, and such that (1) is satisfied. Observe that the conditions of the recursion are satisfied.

If $a \in 2^\omega$, then the completeness of the metric $d$ and condition (1) imply that

$$\bigcap_{n=0}^{\infty} U_{a|n} = \bigcap_{n=0}^{\infty} \overline{U_{a|n}} = \{\phi(a)\}$$

for some $\phi(a) \in X$. Condition (3) implies that $\phi : 2^\omega \to X$ is one-to-one. It follows immediately from the method of construction that $\phi$ is continuous. Moreover, if $a$ is not eventually 0, then the $d'$ diameter of $U_{a|n}$ tends to 0 as $n$ tends to $\infty$ and thus $\phi(a) \in G$. If $a$ is eventually 0, then $\phi(a) = y_s$ for some $s$, where $s$ is an initial part of $a$ which contains all the nonzero entries of $a$. Thus the range of $\phi$ is contained in $G \cup B$ and $a$ is eventually 0 if and only if $\phi(a) \in B$. \hfill \square

**Corollary 11.** Suppose that $Z$ is coanalytic. Exactly one of the following is true:

- $Z$ is Polish;
- $Z$ contains a closed copy of $\mathbb{Q}$.

**Proof.** If $Z$ contains a closed copy of $\mathbb{Q}$, then clearly $Z$ is not Polish. To see the other direction, let $X$ be a Polish space which contains $Z$ as a subspace. If $Z$ is not Polish, then $X \setminus Z$ is not an $F_\sigma$-set. By Hurewicz’s theorem, there is a copy of the cantor set $P \subseteq X$ such that $P \cap Z$ is a countable dense subset of $P$. Since $P \cap Z$ is a countable metric space with no isolated points, it is homeomorphic to $\mathbb{Q}$. Since $P \cap Z$ is clearly closed in $Z$, we’re done. \hfill \square
Lecture 15

**Theorem 37** (Solecki). *Let X be a Polish space. If A ⊆ X is an analytic set, then A has the closed covering property.*

*Proof.* Let A ⊆ X be a given analytic set and \( \mathcal{F} \) be a family of closed sets such that no countable subset of \( \mathcal{F} \) covers A. Define \( \mathcal{I} \) to be all \( I \subseteq X \) such that I is contained in the union of countably many elements of \( \mathcal{F} \).

First suppose that there is an analytic set \( B \subseteq A \) which is not in \( \mathcal{I} \) such that every relatively nowhere dense subset of \( B \) is in \( \mathcal{I} \). Since \( \overline{B} \) is Polish and \( B \) is analytic it has the Baire Property in \( \overline{B} \). Consequently there is a Polish space \( G \subseteq B \) such that \( B \setminus G \) is meager in \( \overline{B} \). Since every nowhere dense subset of \( \overline{B} \) is nowhere dense in \( B \), \( B \setminus G \) is in \( \mathcal{I} \). Since \( B \) is not in \( \mathcal{I} \), neither is \( G \).

Now suppose that whenever \( B \subseteq A \) is analytic and not in \( \mathcal{I} \), there is a relatively nowhere dense \( E \subseteq B \) such that \( E \) is not in \( \mathcal{I} \). If \( C \subseteq A \), define

\[
C' := C \setminus \bigcup\{U \in \mathcal{I} \mid U \text{ is open}\}.
\]

Observe that if \( C = \bigcup_{n=0}^{\infty} C_n \) and \( U \subseteq X \) is open, then \( U \cap C \) is in \( \mathcal{I} \) if and only if \( U \cap C_n \) is in \( \mathcal{I} \) for all \( n \). It follows that \( C' = \bigcup_{n=0}^{\infty} C'_n \).

Fix a compatible complete metric \( d \) on \( X \) bounded by 1 and let \( f : \omega^\omega \to X \) be continuous such that \( A \) is the range of \( f \). Define \( A_s \) to be the closure of \( f([s])' \), noting that \( \bigcup_{n=0}^{\infty} A_s \setminus (n) \) is dense in \( A_s \). Observe that \( A(A_s \mid s \in \omega^\omega) \subseteq A \). Recursively construct a function \( \psi : \omega^\omega \to \omega^\omega \) and a scheme of open sets \( \{U_s \mid s \in \omega^\omega\} \) such that for all \( s \in \omega^\omega \):

1. the \( d \)-diameter of \( U_s \) is at most \( 2^{-lh(s) - \max(s)} \);
2. if \( s \) is a proper initial part of \( t \), then \( U_t \subseteq U_s \);
3. if \( i \neq j \), then \( U_{s \setminus \langle i \rangle} \cap U_{s \setminus \langle j \rangle} = \emptyset \);
4. \( \psi \) is monotone and length preserving;
5. \( U_s \cap A_{\psi(s)} \neq \emptyset \);
6. \( N_s := \bigcup_{n=0}^{\infty} U_{s^{\setminus \langle n \rangle}} \setminus \bigcup_{n=0}^{\infty} U_{s^{\setminus \langle n \rangle}} \) is not in \( \mathcal{I} \) but is a subset of \( U_s \cap A_{\psi(s)} \).

Start by setting \( \psi(\varepsilon) := \varepsilon \) and \( U_\varepsilon := X \). Suppose now that \( U_s \) and \( \psi(s) \) have been defined. By construction \( U_s \cap A_{\psi(s)} \neq \emptyset \) and thus \( U_s \cap f([\psi(s)]) \) is not in \( \mathcal{I} \). Since \( U_s \cap f([\psi(s)]) \) is analytic, our assumption implies it contains a nowhere dense set which is not in \( \mathcal{I} \). Since \( \mathcal{I} \) is a \( \sigma \)-ideal, this nowhere dense set can be taken so that its closure, which we will denote \( N_s \), is contained in \( U_s \cap A_{\psi(s)} \). Pick a discrete set \( D_s \subseteq U_s \cap A_{\psi(s)} \setminus N_s \) such that \( N_s \) is the set of limit points of \( D_s \). Choose open sets \( U_s^{\setminus \langle n \rangle} \) which satisfy conditions (1)–(3), which
cover $D_s$, and which satisfy that $U_{s^\prec(n)} \cap D_s$ is a singleton for each $n$. It follows that condition (6) holds. Since $\bigcup_{n=0}^{\infty} A_{\psi(s^\prec(n))}$ is dense in $A_{\psi(s)}$, for each $n$ there is a $k_n$ such that $A_{\psi(s^\prec(k_n))} \cap U_{s^\prec(n)} \neq \emptyset$. Define $\psi(s^\prec(n)) := \psi(s^\prec(k_n))$. This completes the construction.

We now claim that $G := A(A_{\psi(s^\prec)} \mid s \in \omega^{<\omega})$ is a $G_\delta$-set which is contained in $A$ and which is not in $I$. First observe that if $x \in \omega^\omega$, then $\bigcap_{n=0}^{\infty} A_{\psi(x|n)} = \bigcap_{n=0}^{\infty} U_{x|n} = \bigcap_{n=0}^{\infty} U_{x|n}$.

Furthermore,

$$\bigcup_{x \in \omega^\omega} \bigcap_{n=0}^{\infty} A_{\psi(x|n)} = \bigcup_{x \in \omega^\omega} \bigcap_{n=0}^{\infty} U_{x|n} = \bigcap_{n=0}^{\infty} \bigcap_{s \in \omega^n} U_s$$

is a $G_\delta$-set. Now suppose that $G$ is in $I$. By the Baire category theorem, there is an $F \in F$ such that $F \cap G$ has interior and hence contains $U_s \cap G$ for some $s \in \omega^{<\omega}$. For such an $s$, $U_{s^\prec(n)} \cap G$ contains a point $x_n$, for each $n$. By (1), the set of limit points of $\{x_n \mid n \in \omega\}$ coincides with the set of limit points of $D_s$, which is $N_s$ by construction. But now this implies $G \cap U_s \subseteq F$ contains $N_s$, which is assumed not to be in $I$, a contradiction. \[\square\]

**Corollary 12.** If $X$ is Polish, $A \subseteq X$ is analytic, and $F$ is a collection of closed subsets of $X$, then either $A$ is covered by countably many elements of $F$ or else there is a $G_\delta$-set $G \subseteq A$ such that $F \cap G$ is nowhere dense in $G$ for every $F \in F$.

**Proof.** Let $A \subseteq X$ be given and suppose that $A$ is not covered by countably many elements of $F$. By Solecki’s theorem, there is a $G_\delta$-set $G_0 \subseteq A$ which is not covered by countably many elements of $F$. Define

$$G := G_0 \setminus \bigcup \{U \in \mathcal{I} \mid U \text{ is open}\}$$

where $\mathcal{I}$ is the $\sigma$-ideal generated by $F$. Observe that $G$ is also a $G_\delta$-set, $G_0 \setminus G \in \mathcal{I}$, and if $U \subseteq X$ is open with $U \cap G \neq \emptyset$, then $U \cap G \notin \mathcal{I}$. In particular, $F \cap G$ is nowhere dense in $G$ for every $F \in F$. \[\square\]
A tree $T$ on $\omega$ is **perfect** if every $t$ in $T$ has two incompatible extensions. Equivalently, there is a $u$ in $T$ extending $t$ such that $u \upharpoonright \langle i \rangle$ and $u \upharpoonright \langle j \rangle$ are in $T$ for $i \neq j$. A tree $T$ on $\omega$ is **superperfect** if it is perfect and for every $t \in T$

$$\{ i \in \omega \mid t \upharpoonright \langle i \rangle \in T \}$$

is either infinite or has a single element. A subset $S$ of $\omega^\omega$ is a **superperfect set** if it is the set of branches through a superperfect tree. Observe that superperfect sets are closed and that every compact subset of a superperfect set is nowhere dense. The following consequence of Solecki’s theorem can be regarded as an analog of the perfect set property for analytic sets.

**Corollary 13.** If $A \subseteq \omega^\omega$ is an analytic set, then either $A$ is contained in a $\sigma$-compact set or else $A$ contains a superperfect set.

**Proof.** By Solecki’s theorem, it suffices to show that every $G_\delta$ subset $A$ of $\omega^\omega$ which is not contained in a $\sigma$-compact set contains a superperfect subset. Suppose that $A$ is given and assume without loss of generality that $A$ is nonempty and if $U$ is open then either $U \cap A$ is empty or else not contained in a $\sigma$-compact set. Fix open sets $U_n \subseteq \omega^\omega$ such that $A = \bigcap_{n=0}^{\infty} U_n$. A $t \in \omega^{<\omega}$ is a **splitting node** for $A$ if

$$\{ i \in \omega \mid [t \upharpoonright \langle i \rangle] \cap A \neq \emptyset \}$$

is infinite. Observe that if $[s] \cap A \neq \emptyset$ then there is a $t$ extending $s$ such that $t$ is a splitting node of $A$. Otherwise the set of elements of $\omega^{<\omega}$ which are compatible with $s$ forms a finitely branching tree whose branches would be a compact set containing $[s] \cap A$. Recursively define a function $f : \omega^{<\omega} \to \omega^{<\omega}$ as follows. Let $f(\varepsilon)$ be any splitting node $t$ of $A$ such that $[t] \subseteq U_0$. If $f$ has been defined for sequences of length $n$ and $s \in \omega^n$, define $f(s \upharpoonright \langle k \rangle)$ to be a splitting node $t$ such that:

- $t$ extends $f(s) \upharpoonright \langle i_k \rangle$ where $i_k$ is the $k$th element of $\{ i \in \omega \mid [f(s) \upharpoonright \langle i \rangle] \cap A \neq \emptyset \}$
- $[t] \subseteq U_{n+1}$.

Observe that $T = \{ f(s) \upharpoonright k \mid s \in \omega^{<\omega} \text{ and } k \leq lh(s) \}$ is a superperfect tree and that all branches through $T$ are contained in $A = \bigcap_{n=0}^{\infty} U_n$. □

**Theorem 38** (Mycielski). If $X$ is an uncountable Polish space and $E \subseteq X^n \setminus \Delta$ is meager, there is a perfect set $P \subseteq X$ such that $P^n \cap E = \emptyset$. Here $\Delta \subseteq X^n$ denotes all $n$-tuples in which not all coordinates are distinct.
Proof. Let $X$ and $E \subseteq X^n$ be as in the statement of the theorem and fix a compatible complete metric $d$ on $X$ which is bounded by 1. By replacing $X$ with a perfect subset if necessary, we may assume that $X$ has no isolated points. Fix closed nowhere dense sets $E_k \subseteq X^n$ such that $E \subseteq \bigcup_{k=0}^{\infty} E_k$. Construct a scheme of nonempty open sets $\langle U_s \mid s \in 2^{\omega} \rangle$ such that for each $s \in 2^{\omega}$:

1. $\overline{U_{s^+(0)}}$ and $\overline{U_{s^+(1)}}$ are disjoint and contained in $U_s$;
2. the diameter of $U_s$ is at most $2^{-lh(s)}$;
3. if $\langle s_i \mid i < n \rangle$ are distinct sequences of length $m$ for some $m$, then $\prod_{i<n} U_{s_i}$ is disjoint from $\bigcup_{k<m} E_k$.

Suppose that we’ve constructed $U_s$ for each $s \in 2^{\omega}$ of length at most $m$. Define $\langle U^p_s \mid s \in 2^{m+1} \rangle$ for each $p \leq n! \binom{2^m}{n}$ as follows. Start by fixing, for each $s \in 2^m$, open sets $U^0_{s^+(0)}$ and $U^0_{s^+(1)}$ which have disjoint closures of diameter less than $2^{-m-1}$ which are contained in $U_s$. Given $\langle U^p_s \mid s \in 2^{m+1} \rangle$, pick $U^p_{s^{+1}} \subseteq U^p_s$ such that if $\langle s_i \mid i < n \rangle$ is the $p$th sequence of distinct elements of $2^{m+1}$ then $\prod_{i<n} U^p_{s_i} \cap \bigcup_{k<m} E_k = \emptyset$. This is possible since $\bigcup_{k<m} E_k$ is nowhere dense in $X^n$. Finally, set $U_s = U^p_s$ where $p = n! \binom{2^{m+1}}{n}$. This finishes the construction of $\langle U_s \mid s \in 2^{\omega} \rangle$.

Set

$$P = \bigcap_{m=0}^{\infty} \bigcup_{s \in 2^m} U_s = \bigcap_{m=0}^{\infty} \bigcup_{s \in 2^m} U_s.$$

It is immediate from the requirements of the construction that $P$ is homeomorphic to $2^{\omega}$ and satisfies that $P^n \cap E = \emptyset$. \hfill \Box

Corollary 14. If $A \subseteq \mathbb{R}^2$ is analytic, then either $A$ can be covered by countably many lines or else $A$ contains a perfect set $P$ which intersects any line in at most 3 points.

Proof. Let $A$ be given. By Solecki’s theorem, either $A$ is covered by countably many lines or else there is a there is a $G_\delta$-set $G \subseteq A$ which is not covered by countably many lines. Suppose the latter and, by replacing $G$ by a subset if necessary, assume that no nonempty relatively open subset of $G$ can be covered by countably many lines. Define

$$E = \{(x, y, z) \in G^3 \mid x, y, z \text{ are collinear}\}$$

and observe that $E$ is closed. Furthermore, for each $x \neq y \in G$, $\{z \in G \mid x, y, z \text{ are collinear}\}$ is on the line determined by $x$ and $y$ and hence is nowhere dense. By the Kuratowski-Ulam theorem, $E$ is meager in $G^3$ and by Mycielski’s theorem, there is a perfect set $P \subseteq G$ such that no three points of $P$ are collinear. \hfill \Box
Lecture 17

In this section we will consider an important subclass of the Borel functions on a Polish space and which includes the continuous functions.

Let $X$ and $Y$ be separable metric spaces. A function $f : X \to Y$ is said to be Baire class 1 if whenever $U \subseteq Y$ is open, $f^{-1}(U)$ is an $F_\sigma$-set. Notice if $S$ is a sub-base for the topology on $Y$, then a function is Baire class 1 provided that $f^{-1}(U)$ is $F_\sigma$ whenever $U \in S$.

Since open sets are $F_\sigma$, every continuous function is Baire class 1. The class is much richer, however. For instance, every monotone increasing function $f$ from $\mathbb{R}$ to $\mathbb{R}$ is Baire class 1: the preimage of an open interval is a (not necessarily open) interval. Also the function $\delta_r : \mathbb{R} \to \mathbb{R}$ defined by

$$\delta_r(t) := \begin{cases} 1 & \text{if } t = r \\ 0 & \text{otherwise} \end{cases}$$

is readily seen to be Baire class 1. On the other hand, the characteristic function of $\mathbb{Q}$ is not Baire class 1: $\chi_{\mathbb{Q}}^{-1}(-\infty, 1/2) = \mathbb{R} \setminus \mathbb{Q}$ which is not $F_\sigma$.

Our goal will be to prove the following characterization of Baire class 1 functions.

**Theorem 39** (Baire). Let $X$ be a Polish spaces. The following are equivalent for a function $f : X \to \mathbb{R}$:

1. $f$ is Baire class 1;
2. there is a sequence $\langle f_n \mid n \in \omega \rangle$ of continuous functions from $X$ to $\mathbb{R}$ which converges to $f$ pointwise;
3. whenever $P \subseteq X$ is a Cantor set, $f \upharpoonright P$ has a point of continuity.

We will first need to prove an additional result about separating sets by $F_\sigma$-sets. We will need the following proposition, which will be left as an exercise.

**Proposition 6.** Suppose that $X$ is a Polish space and that $A_0, A_1 \subseteq X$ are two disjoint countable dense subsets of $X$. There is an embedding $\phi : 2^\omega \to X$ such that for all $a \in 2^\omega$, $a$ is eventually $i$ if and only if $\phi(a) \in A_i$.

**Theorem 40.** Suppose that $X$ is a Polish space and $A_0, A_1 \subseteq X$ are disjoint. Exactly one of the following is true:

- there is a set $C \subseteq X$ which is both $F_\sigma$ and $G_\delta$ such that $A_0 \subseteq C$ and $C \cap A_1 = \emptyset$;
• there is a Cantor set $P \subseteq X$ such that $P \cap A_i$ is dense in $P$ for each $i \in \{0, 1\}$.

Proof. To see that the two conclusions can not both occur, observe that if $P \subseteq X$ is a Cantor set and $P \cap A_i$ is dense in $P$ for each $i$, then any closed subset of $P$ which is disjoint from either $A_0$ or $A_1$ is nowhere dense. The Baire Category Theorem therefore implies that $A_0 \cap P$ and $A_1 \cap P$ can not be separated by a set which is both $F_\sigma$ and $G_\delta$.

Now we will argue that at least one of the conclusions of the theorem must occur. By replacing $X$ by the closure of $A_0 \cup A_1$, we may assume without loss of generality that $A_0 \cup A_1$ is dense in $X$. Define $X(\alpha)$ for each ordinal $\alpha$ by recursion as follows:

- $X(0) := X$;
- $X(\alpha+1) := X(\alpha) \setminus \bigcup (V_{\alpha,0} \cup V_{\alpha,1})$ where $V_{\alpha,i}$ consists of all open sets $V \subseteq X$ such that $V \cap A_i \cap X(\alpha) = \emptyset$.
- $X(\beta) := \bigcap_{\alpha \in \beta} X(\alpha)$ if $\beta$ is a limit ordinal.

Observe that $X(\alpha)$ is always closed and if $\alpha \in \beta$, then $X(\beta) \subseteq X(\alpha)$. In particular, there is a countable $\gamma$ such that $X(\gamma) = X(\alpha)$. If $X(\gamma)$ is nonempty, then observe that $A \cap X(\gamma)$ and $B \cap X(\gamma)$ are both dense in $X(\gamma)$. In this case, we are done by Proposition 6.

Otherwise, observe that for each $\alpha \in \gamma$,

$$E_{\alpha,i} := X(\alpha) \cap \bigcup V_{\alpha,i}$$

is the intersection of an open set and a closed set and hence is $F_\sigma$. Furthermore, if $(\alpha, i) \neq (\beta, j)$, then $E_{\alpha, i} \cap E_{\beta, j} = \emptyset$ (that $E_{0,0} \cap E_{0,1} = \emptyset$ utilizes that $A_0 \cup A_1$ is dense in $X$). Furthermore, $A_i \subseteq \bigcup_{\alpha \in \gamma} E_{\alpha,i}$ and $X = \bigcup_{i=0,1} \bigcup_{\alpha \in \gamma} E_{\alpha,i}$. In particular $C = \bigcup_{\alpha \in \gamma} E_{\alpha,0}$ is as desired. □

Theorem 41. Let $X$ be a Polish space and $Y$ be a separable metric space. If $f : X \to Y$ is a Baire class 1 function, then the set of points of continuity of $f$ are a dense $G_\delta$-set.

Proof. Let $\{W_n : n \in \omega\}$ be a base for the topology on $Y$. Define

$$E_n := f^{-1}(W_n) \setminus \text{int}(f^{-1}(W_n)).$$

Since $f^{-1}(W_n)$ is $F_\sigma$, $E_n$ is a meager $F_\sigma$-set. Observe that $f$ is continuous at $x$ if and only if $x$ is not in $E_n$ for any $n$. □

We finish the lecture by noting the following deep theorem.

Theorem 42 (Bourgain-Fremlin-Talagrand). If $\{f_n\}_{n=0}^\infty$ is a uniformly bounded sequence of continuous real valued functions on a Polish space $X$, then exactly one of the following is true:
• the closure of \( \{f_n \mid n \in \omega \} \) in \( \mathbb{R}^X \) is contained in the Baire class 1 functions;
• there is an infinite \( I \subseteq \omega \) such that the closure of \( \{f_n \mid n \in I\} \) in \( \mathbb{R}^X \) is homeomorphic to \( \beta \omega \), the Čech-Stone compactification of \( \omega \).

Compact topological spaces which are embeddable into \( BC_1(X) \subseteq \mathbb{R}^X \) are known as Rosenthal compacta. This class of compact spaces is both rich and amenable to analysis. One one hand, it includes all compact metric spaces and is closed under countable products and closed subspaces. It also includes nonmetrizable compacta such as the space of all monotonic increasing functions from \([0,1]\) to \([0,1]\). On the other hand, Rosenthal compacta enjoy many properties of metric spaces.

**Theorem 43** (Grothendick). If \( K \) is a Rosenthal compacta, \( A \subseteq K \) and \( x \in \overline{A} \), then there is countable \( A_0 \subseteq A \) such that \( x \in \overline{A_0} \).

**Theorem 44** (Todorcevic). If \( K \) is a Rosenthal compactum, then \( K \) has a dense metrizable subspace.
Theorem 45 (Baire). If $X$ is a Polish space and $f : X \to \mathbb{R}$ is a function, the following are equivalent:

1. $f$ is Baire class 1;
2. there is a sequence $\langle f_n \mid n \in \omega \rangle$ of continuous functions from $X$ to $\mathbb{R}$ which converges pointwise to $f$;
3. if $P \subseteq X$ is a Cantor set, then there is a point of continuity of $f \upharpoonright P$;
4. if $P \subseteq X$ is a Cantor set and $a < b$, then either \{ $x \in P \mid f(x) \leq a$ \} or \{ $x \in P \mid f(x) \geq a$ \} is not dense in $P$.

Proof. (1) implies (2): to be added.

(2) implies (1): Let $f_n : X \to \mathbb{R}$ be continuous for $n \in \omega$ with $f_n \to f$ pointwise. It suffices to prove that for all $a < b$, $f^{-1}((a,b))$ is $F_\sigma$. Observe that $f(x) \in (a,b)$ if and only if there is a $k$ such that $f(x) \in [a + 2^{-k}, b - 2^{-k}]$ if and only if there is a $m$ such that for all $n > m$, $f_n(x) \in [a + 2^{-k}, b - 2^{-k}]$. Thus

$$f^{-1}((a,b)) = \bigcup_{m,k} \bigcap_{n>m} \{ x \in X \mid f_n(x) \in [a + 2^{-k}, b - 2^{-k}] \}$$

which is an $F_\sigma$-set.

(3) implies (4): this is trivial.

(4) implies (1): Let $a < b$ and fix $n \in \omega$. By our assumption and the separation theorem for $\Delta_0^2$-sets, there is a set $E_n \subseteq X$ which is both $F_\sigma$ and $G_\delta$ such that $E_n$ contains $f^{-1}([a + 2^{-n}, b - 2^{-n}])$ and is disjoint from $f^{-1}((-\infty, a] \cup [b, \infty))$. It follows that $f^{-1}((a,b)) = \bigcup_{n=0}^{\infty} E_n$.

We will now turn our attention to gaps in the quotient $\mathcal{P}(\omega)/\text{fin}$ and in particular gaps in which one or both sides of the gap are analytic. Suppose that $A, B \subseteq \mathcal{P}(\omega)$. We say that $A$ and $B$ are orthogonal if $a \cap b$ is finite whenever $a \in A$ and $b \in B$. A family $C \subseteq \mathcal{P}(\omega)$ separates $A$ from $B$ if for every $a \in A$ and $b \in B$, there is a $c \in C$ such that $a \subseteq^* c$ and $b \cap c$ is finite. Here $a \subseteq^* c$ means that there is a finite $e \subseteq \omega$ such that $a \subseteq c \cup e$. A gap is a pair $A, B \subseteq \mathcal{P}(\omega)$ which is orthogonal but not separated by a single set. An orthogonal pair $A, B \subseteq \mathcal{P}(\omega)$ is countably separated if there is a countable family of sets which separates $A$ from $B$. Of course all of these definitions apply equally well when $\omega$ is replaced by some other countable set.
Example 1. Let $A$ denote all $a \subseteq \omega \times \omega$ such that for all $i$ there is at most one $j$ such that $(i, j) \in A$ (i.e. $A$ is the set of graphs of partial functions from $\omega$ to $\omega$). Set $B$ to be all $b \subseteq \omega \times \omega$ which are contained in $\{i\} \times \omega$ for some $i$. Clearly $A$ is orthogonal to $B$. Moreover $A$ and $B$ for a gap. Notice that $\{\{i\} \times \omega \mid i \in \omega\}$ is a countable family which separates $B$ from $A$.

Example 2 (Lusin gap). For each $r \in 2^{\omega}$, define

$$a_r := \{r \upharpoonright n \mid r(n) = 0\} \quad b_r := \{r \upharpoonright n \mid r(n) = 1\}.$$ 

Set $A := \{a_r \mid r \in 2^{\omega}\}$ and $B := \{b_r \mid r \in 2^{\omega}\}$. Observe that $A$ is orthogonal to $B$. Moreover if $r \neq s$, then

$$(a_r \cap b_s) \cup (a_s \cap b_r)$$

is empty if and only $r = s$ and consists of the longest common initial segment of $r$ and $s$ otherwise.

Theorem 46. Suppose that $I$ is an uncountable set and $A = \{a_i \mid i \in I\}$ and $B = \{b_i \mid i \in I\}$ are orthogonal families of subsets of $\omega$ such that

$$(a_i \cap b_j) \cup (a_j \cap b_i)$$

is empty if and only if $i = j$. Then $A$ and $B$ are not countably separated.

Proof. Suppose not. By the pigeon hole principle there is a single $c \subseteq \omega$, an $n \in \omega$ and $\alpha, \beta \subseteq n$ such that for uncountably many $i \in I$:

1. $a_i \subseteq c \cup m$ and $b_i \cap c \subseteq m$;
2. $a_i \cap m = \alpha$ and $b_i \cap m = \beta$.

But now if $i \neq j$ are in this uncountable set, $(a_i \cap b_j) \cup (a_j \cap b_i)$ can not contain an element which is at least $m$ because of (1). On the other hand, $(a_i \cap b_j) \cup (a_j \cap b_i)$ can not contain an element which is less than $m$ because of (2). Thus we have a contradiction and it must have been that $A$ and $B$ were not countable separated. $\square$
Lecture 19

In the last lecture, we gave two examples of analytic gaps in \( \mathcal{P}(\omega)/\text{fin} \). The goal of this lecture is to present two results of Todorcevic which show that these examples play important roles with respect to all other analytic gaps. The first characterizes when gaps are countably separated. Define \( a =^* b \) to mean that \( a \Delta b \) is finite and define \( a \in^* A \) to mean that there is an \( a' \in A \) with \( a =^* a' \).

**Theorem 47** (Todorcevic). Suppose that \( A, B \subseteq [\omega]^{\omega} \) are orthogonal. Either \( A \) is countably separated from \( B \) or else there is a continuous functions \( r \mapsto a_r \) and \( r \mapsto b_r \) from \( \omega^\omega \) to \( [\omega]^{\omega} \) such that:

- for all \( r \in \omega^\omega \), \( a_r \in^* A \) and \( b_r \in^* B \);
- for all \( r, s \in \omega^\omega \),
  \[
  (a_r \cap b_s) \cup (a_s \cap b_r) = \emptyset
  
  \]
  if and only if \( r = s \).

**Proof.** Define \( X \subseteq [\omega]^{\omega} \times [\omega]^{\omega} \) to be all pairs \((a, b)\) such that \( a \cap b = \emptyset \), \( a \in^* A \), and \( b \in^* B \). Observe that since \( A \) and \( B \) are analytic, so is \( X \). Define a graph \( G \) on \( X \) by connecting \((a_0, b_0)\) and \((a_1, b_1)\) by an edge if

\[
(a_0 \cap b_1) \cup (a_1 \cap b_0) \neq \emptyset.
\]

Notice that \( G \) is an open graph. For instance, if \( n \in a_0 \cap b_1 \), then whenever \((a'_0, b'_0), (a'_1, b'_1) \in X \) with \( n \in a'_0 \cap b'_1 \), it follows that \((a'_0, b'_0)\) and \((a'_1, b'_1)\) are connected by an edge.

Since \( \text{OCA}^*(X) \) is true, either \( G \) is countably chromatic or else \( G \) contains a clique \( P \) which is a cantor set. If \( G \) is countably chromatic, then \( X = \bigcup_{n=0}^{\infty} X_n \) such that \([X_n]^2 \cap G = \emptyset\) for all \( n \). Set

\[
c_n := \bigcup \{ a \subseteq \omega \mid \exists b((a, b) \in X_n) \}
\]

Observe that if \( (a, b) \in X_n \), then \( a \subseteq c_n \) and \( b \cap c_n = \emptyset \). If \( a \in A \) and \( b \in B \), then \((a, b \setminus a) \in X \) and there is an \( n \) such that \((a, b \setminus a) \in X_n \). Since \( b \setminus a =^* b \), we have that \( c_n \) separates \( a \) from \( b \).

If \( \phi : 2^\omega \to X \) is a continuous injection whose range is a perfect clique, then let \( \phi(r) = (a_r, b_r) \). It follows immediately that \( r \mapsto a_r \) and \( r \mapsto b_r \) satisfy the second alternative of the theorem. \(\square\)

A collection of sets is hereditary if it is closed under taking subsets. If \( A \subseteq [\omega]^{\omega} \), define \( A^\downarrow \) to be the set of all subsets \( b \) of \( \omega \) which have finite intersection with every element of \( A \). Observe that \( A \subseteq A^{\downarrow\downarrow} \) but that even for hereditary \( A \), the containment may be proper For instance if \( A = \{ a \subseteq \omega \mid \sum_{n \in a} 1/n < \infty \} \), then \( A^\downarrow \) is just the finite subsets of \( \omega \) and \( A^{\downarrow\downarrow} = \mathcal{P}(\omega) \).
Returning to the first example of an analytic gap from the previous lecture, set \( B_0 = \{ \langle n \rangle \times \omega \mid n \in \omega \} \) and set \( A_0 \) to be the collection of all graphs of functions from \( \omega \) to \( \omega \). Observe that while \( A_0 \) is countable separated from \( B_0 \), there is no countable subset of \( B_0^\perp \) which separates \( A_0 \) from \( B_0 \). Our next theorem will show that this is a general phenomenon.

If \( C \subseteq [\omega]^\omega \), a tree \( T \) on a countable set \( I \) is a \( C \)-tree if elements of \( T \) are one-to-one sequences and whenever \( t \in T \), \( \{ i \in I \mid t^\frown \langle i \rangle \in T \} \) is an infinite set which is contained in an element of \( C \). If we let \( I = \omega \times \omega \), then the collection of all sequences \( t \) such that the \( i \)th entry has first coordinate \( i \) is a \( B \)-tree all of whose branches are in \( A \).

**Proposition 7.** If \( A, B \subseteq \mathcal{P}(\omega) \) are orthogonal and there is a \( B \)-tree all of whose branches have ranges which lie in \( A \), then \( A \) is not separated from \( B \) by a countable subset of \( B^\perp \).

**Proof.** Let \( A \) and \( B \) be given and let \( T \) be a \( B \)-tree all of whose branches are in \( A \). Let \( \{ a_n \mid n \in \omega \} \) be a collection of elements of \( B^\perp \). Recursively construct a sequence \( \langle n_k \mid k \in \omega \rangle \) such that for each \( k \), \( \langle n_i \mid i < k \rangle \) is in \( T \) and such that if \( i < j \), then \( n_j \notin a_i \). Given that \( \langle n_i \mid i < k \rangle \) has been constructed, let \( n_k \) be minimal such that \( \langle n_i \mid i < k + 1 \rangle \in T \) and \( n_k \notin a_i \) for all \( i \leq k \). This is possible since

\[
b = \{ p \in \omega \mid \langle n_i \mid i < k \rangle^\frown \langle p \rangle \}
\]

is contained in an element of \( B \) and hence has finite intersection with every element of \( B^\perp \), including \( \bigcup_{i \leq k} a_i \). Since \( a := \{ n_k \mid k \in \omega \} \) is in \( A \) and has finite intersection with each \( a_k \), we are done. \( \Box \)

**Theorem 48** (Todorcevic). Suppose that \( A, B \subseteq \mathcal{P}(\omega) \) are orthogonal families and that \( A \) is analytic and hereditary. Either \( A \) is separated from \( B \) by a countable subset of \( B^\perp \) or else there is a \( B \)-tree all of whose branches are in \( A \).

**Remark 2.** Here we are abusing language and identifying elements of \( A \) with their enumerations.
Proof of Theorem 48. Define $\mathcal{F} = \{[x]^\omega \mid x \in B^\perp\}$. Clearly $\mathcal{F}$ is a family of closed subsets of $[\omega]^\omega$ and $A$ is separated from $B$ by a countable subset of $B^\perp$ precisely when $A$ can be covered by countably many elements of $\mathcal{F}$. Suppose this is not the case. By Solecki’s theorem, there is a nonempty $G$-set $G \subseteq A$ such that if $U$ is open and $U \cap G$ is nonempty, then $U \cap G$ can not be separated from $B$ by a countable subset of $B^\perp$. Fix a $\subseteq$-decreasing sequence $\langle U_n \mid n \in \omega \rangle$ of open sets such that $G = \bigcap_{n=0}^{\omega} U_n$.

Construct a tree $T \subseteq [\omega]^{<\omega}$ by recursion as well as a function $t \mapsto \hat{t}$ from $T$ to $[\omega]^{<\omega}$ so that:

1. the emptyset is in $T$ and if $t \in T$, then $\{k \in \omega \mid t \cup \{k\} \in T\}$ is an infinite set which is contained in an element of $B$;
2. if $s,t \in T$ and $s$ is an initial part of $t$, then $\hat{s}$ is an initial part of $\hat{t}$;
3. if $s \in T$, then $\hat{s} \subseteq U_{|s|}$ and $\hat{s} \cap G \neq \emptyset$.

Suppose that $t \in T$ and that $\hat{t}$ has been defined. We need to determine for which $k > \max(t)$, $t \cup \{k\}$ is in $T$ and also to define $t \cup \{k\}$ for those $k$ with $t \cup \{k\} \in T$. Set $\Sigma$ to be the set of those $u \in [\omega]^{<\omega}$ such that $\hat{t}$ is an initial part of $u$, $[u] \subseteq U_{|t|+1}$ and $[u] \cap G \neq \emptyset$. Observe that $x := \bigcup \{u \setminus \hat{t} \mid u \in \Sigma\}$

can not be in $B^\perp$ since every element of $G \cap \hat{t}$ is contained in $\hat{t} \cup x$.

Thus there is a $b \in B$ such that $b \cap x$ is infinite. Put $t \cup \{k\} \in T$ if and only if $k \in b \cap x$. For each $k \in b \cap x$, fix a $u \in \Sigma$ such that $k \in u$ and set $t \cup \{k\} = u$. This completes the construction.

Clearly $T$ is a $B$-tree. We need to show that if $x \in [\omega]^\omega$ and $x \cap n \in T$ for all $n$, then $x \in A$. Observe that if $m \leq n$, then $x \cap m$ is an initial part of $\hat{x} \cap n$ (possibly they are equal). Thus $\hat{x} := \bigcup_{n=0}^{\infty} \hat{x} \cap n$ is in $G$ and $x \subseteq \hat{x}$. Since $G \subseteq A$ and $A$ is hereditary, $x \in A$. \hfill \Box

We will now turn to a different class of results in descriptive set theory: uniformization theorems. Given sets $X$ and $Y$ and a subset $F \subseteq X \times Y$, the axiom of choice implies that there is a function $f$ such that $\text{dom}(f) = \{x \in X \mid \exists y \in Y((x,y) \in F)\}$ and such that $(x,f(x)) \in F$ whenever $x \in \text{dom}(f)$. Such an $f$ is said to uniformize the relation $F$. Notice that we can think of $F$ as encoding a set valued function

$$x \mapsto F(x) := \{y \in Y \mid (x,y) \in F\}.$$
The uniformizing function \( f \) is required to be defined on the set of \( x \in X \) for which \( F(x) \neq \emptyset \) and must satisfy \( f(x) \in F(x) \).

In descriptive set theory, one is usually interested in the restricted setting in which \( X \) and \( Y \) are Polish spaces and in which \( F \) is Borel or perhaps analytic but with the additional requirement that \( f \) is regular in some sense. Naïvely one would like to show that if \( X \) and \( Y \) are Polish and \( F \) is Borel, then one can find a uniformizing function which is Borel. This is in general not possible. In fact there are closed relations which do not admit Borel uniformizations.

If we expand the \( \sigma \)-algebra of Borel sets slightly, however, then one can prove a uniformization result of this sort. If \( X \) is a Polish space, then the Souslin measurable subsets of \( X \) are the elements of the \( \sigma \)-algebra generated by the analytic subsets of \( X \). Notice in particular that every Souslin measurable set is Baire measurable and \( \mu \)-measurable for any \( \sigma \)-finite Borel measure \( \mu \) on \( X \).

A function \( f : X \to Y \) is Souslin measurable if the preimage of every open set is Souslin measurable.

**Theorem 49** (Jankov-von Neumann). If \( X \) and \( Y \) are Polish spaces and \( F \subseteq X \times Y \) is analytic, then there is a uniformization of \( F \) which is Souslin measurable. In particular, the uniformization is both Baire measurable and \( \mu \)-measurable for any \( \sigma \)-finite Borel measure on \( X \).

**Proof.** We will finish this lecture with a proof that we may assume without loss of generality that \( X = Y = \omega^\omega \) and that \( F \) is closed.

To see that we may assume that \( X = \omega^\omega \), first observe if we prove the result for a stronger Polish topology on \( X \), then this suffices: the Souslin measurable subsets of \( X \) are the same for both topologies. By enlarging \( X \) if necessary, we may assume that it is uncountable and, by increasing its topology if necessary, that it is homeomorphic to \( \omega^\omega \oplus \omega \).

If \( f_0 \) is a Souslin measurable uniformization of the restriction of \( F \) to the copy of \( \omega^\omega \) and \( f_1 \) is any uniformization of the restriction of \( F \) to the copy of \( \omega \), then \( f_0 \cup f_1 \) is Souslin measurable and uniformizes \( F \).

Thus we may assume that \( X = \omega^\omega \).

Next we will see that we may assume without loss of generality that \( Y = \omega^\omega \) and \( F \) is closed. Let \( \phi : \omega^\omega \to X \times Y \) be such that \( F \) is the range of \( \phi \). Define

\[
R = \{(x, z) \in X \times \omega^\omega \mid x = \pi_X(\phi(z))\}
\]

and observe that \( R \) is closed. Furthermore, if \( g \) uniformizes \( R \) and \( f(x) = \pi_Y(g(\phi(x))) \), then \( f \) uniformizes \( F \); if \( g \) is Souslin measurable, so is \( f \). This finishes the proof that we may assume \( X = Y = \omega^\omega \) and that \( F \) is closed. \( \square \)
Lecture 21

We will now continue with the proof of the Jankov-von Neumann Uniformization Theorem. In the last lecture, we showed that we may assume without loss of generality that $X = Y = \omega^\omega$ and $F$ is closed. Let $A := \pi_X(F) = \{x \in X \mid \exists y \in Y((x, y) \in F)\}$. Define $f : A \to Y$ by $f(x) = y$ for every $n$:

- $x$ is in the projection of $F \cap ([x \upharpoonright n] \times [y \upharpoonright n])$ and
- if $t \in \omega^n$ is such that $t <_{\text{lex}} y \upharpoonright n$, then $x$ is not in the projection of $F \cap ([x \upharpoonright n] \times [t])$.

Clearly $(x, f(x)) \in F$ whenever $x \in A$. Furthermore, if $u \in \omega^{<\omega}$ has length $n$, then $f^{-1}([u])$ is the set of all $x$ such that:

- $x$ is in the projection of $F \cap ([x \upharpoonright n] \times [u])$ and
- if $t \in \omega^n$ is such that $t <_{\text{lex}} u$, then $x$ is not in the projection of $F \cap ([x \upharpoonright n] \times [t])$.

This is a difference of an analytic set and a coanalytic set and thus us Souslin measurable.

To see this construction in a different way, let $T$ be a tree on $\omega \times \omega$ which represents $F$. The value $f(x)$ is the leftmost path through $T(x)$. Under some circumstances, this idea can be modified to produce a Borel function. For this we need the following definition. Let $X$ and $Y$ be Polish spaces and $x \mapsto F_x$ be a function from $X$ to $\mathcal{P}(\mathcal{P}(Y))$.

We say that this mapping is Borel on Borel if whenever $W$ is Polish and $A \subseteq W \times X \times Y$ is Borel, $\{(w, x) \in W \times X \mid A_w, x \in F_x\}$ is Borel. Typically our interest will be in Borel on Borel mappings where $F_x$ is a $\sigma$-ideal on $Y$.

Notice that the mapping $x \mapsto \{\emptyset\}$ is not Borel on Borel. To see this, let $E \subseteq (\omega^\omega)^3$ be a closed set such that the projection of $E$ onto the first two coordinates is not Borel. It follows that $\{(w, x) \in \omega^\omega \times \omega^\omega \mid E_{w, x} = \emptyset\}$ is coanalytic but not Borel. Never-the-less, there are many natural examples of Borel on Borel families of ideals. For instance if $x \mapsto \mu_x$ is a Borel map from $X$ into the Borel probability measures on $Y$, then $x \mapsto \mathcal{N}(\mu_x)$ is Borel on Borel where $\mathcal{N}(\mu_x)$ is the ideal of $\mu_x$-measure 0 sets. Also, the constant map $x \mapsto M(Y)$, where $M(Y)$ is the meager subsets of $Y$ is Borel on Borel.

The next result allows us to find Borel uniformizations in circumstances in which the sections of a set $F \subseteq X \times Y$ are “large.”

**Theorem 50.** Suppose that $X$ and $Y$ are Polish and $x \mapsto I_x$ is a Borel on Borel assignment of $\sigma$-ideals on $Y$. If $F \subseteq X \times Y$ is Borel and for every $x$, either $F_x = \emptyset$ or else $F_x \notin I_x$, then $F$ admits a Borel uniformization.
Proof. Fix a closed set \( E \subseteq \omega^\omega \) and a continuous bijection from \( E \) to \( F \). Define \( f \subseteq X \times Y \) so that if \( x \in X \) and \( u \in \omega^{<\omega} \) has length \( n \), then \( \{x\} \times [u] \cap f \neq \emptyset \) if and only if:

- \( F_x \cap (X \times [u]) \) is not in \( \mathcal{I}_x \);
- if \( t \in \omega^n \) satisfies \( t <_{\text{lex}} u \), then \( F_x \cap (X \times [t]) \) is in \( \mathcal{I}_x \).

Notice that for each \( x \) and \( n \), there is atmost one \( u \) in \( \omega^n \) such the above two clauses are satisfied. Moreover, if \( F_x \neq \emptyset \) then \( F_x \) is not in \( \mathcal{I}_x \) and is contained in \( \bigcup_{u \in \omega^n} X \times [u] \). It follows that there is at least one such \( u \in \omega^n \) satisfying the above clauses. Consequently, \( f \) is a uniformization of \( F \). Furthermore, for a fixed \( u \), the set of \( x \) satisfying both clauses is Borel since \( x \mapsto \mathcal{I}_x \) is assumed to be Borel on Borel. It follows that \( f \) is Borel. \( \square \)

Next we turn to a theorem which can be used to find uniformizations for Borel sets with “small” sections.

**Theorem 51** (Luzin-Novikov). Suppose that \( X \) and \( Y \) are Polish and \( F \subseteq X \times Y \) is a Borel set such that for each \( x \), \( F_x \) is (at most) countable. Then \( F \) can be covered by the graphs of countably many Borel functions and admits a Borel uniformization.

With some irony, we will derive this “small section” uniformization theorem from the “large section” uniformization theorem. To see that the latter conclusion follows from the former, suppose that \( \{h_n \mid n \in \omega\} \) is a family of Borel functions whose graphs cover \( F \). Set \( D_n := \{x \in X \mid (x, h_n(x)) \in F\} \). Observe that this set is Borel: it is the domain of the Borel function \( h_n \cap F \). Define \( f(x) = h_n(x) \) if \( n \) is minimal such that \( x \in D_n \). It follows that \( f \) is Borel.

With the goal now of being to cover \( F \) by the graphs of countably many Borel functions, we may assume without loss of generality that \( F_x \) is countably infinite. Now define \( Z = Y^\omega \) and let \( P \subseteq X \times Z \) consist of all \((x, z)\) such that the range of \( z \) is exactly \( F_x \). In particular, \( P_x \subseteq (F_x)^\omega \). Let \( \mathcal{I}_x \) be the collection of all subsets \( A \) of \( Z \) such that \( A \cap (F_x)^\omega \) is meager in \( (F_x)^\omega \) when \( F_x \) is given the discrete topology. Observe that a Borel uniformization of \( P \) gives a cover of \( F \) by the graphs of countably many Borel functions. Thus, by Theorem 50, our task is to prove that \( x \mapsto \mathcal{I}_x \) is Borel on Borel. This will be taken up in the next lecture.
Lecture 22

We return to the task of proving that $x \mapsto \mathcal{M}(P_x)$ is Borel on Borel, which we have seen is sufficient to prove the Luzin-Novikov Uniformization Theorem. Let $E \subseteq P \subseteq X \times Y^\omega$ consist of those $(x, z)$ such that $z$ is one-to-one. Observe that if $x \in X$ and $e \in E_x$, then $f \mapsto e \circ f$ defines a homeomorphism from $\omega^\omega$ to $P_x$. Thus the following are equivalent for $A \subseteq P_x$:

- $A$ is meager in $P_x$;
- there is an $e \in E_x$ such that $\{ f \in \omega^\omega \mid e \circ f \in A \}$ is meager in $\omega^\omega$;
- for all $e \in E_x$, $\{ f \in \omega^\omega \mid e \circ f \in A \}$ is meager in $\omega^\omega$.

We will need the following theorem.

**Theorem 52** (Montgomery, Novikov). *Suppose that $X$ is a set equipped with a $\sigma$-algebra $\mathfrak{A}$ and $Y$ is a Polish space. If $A \subseteq X \times Y$ is in the $\sigma$-algebra generated by products of sets in $\mathfrak{A}$ with Borel sets, then for every open $U \subseteq Y$

$$\{ x \in X \mid A_x \text{ is meager in } U \}$$

is in $\mathfrak{A}$. 

*Proof*. We only sketch the proof. Let $\mathfrak{B}$ consist of all sets $A \subseteq X \times Y$ for which the conclusion of the theorem is true. One shows that $\mathfrak{B}$ contains all products elements of $\mathfrak{A}$ with open sets, and is closed under taking complements and countable unions. \qed

Now let $W$ be Polish and $A \subseteq W \times X \times Y^\omega$ be Borel. Define

$$B := \{ (w, x, e, f) \in W \times X \times Y^\omega \times \omega^\omega \mid e \in E_x \text{ and } (w, x, e \circ f) \in A \}.$$ 

Clearly $B$ is Borel. By Theorem 52,

$$C = \{ (w, x, e) \in W \times X \times Y^\omega \mid B_{w,x,e} \text{ is meager} \}$$

is Borel. By the above observation, the following are equivalent:

- $A_{w,x}$ is meager;
- there is an $e \in E_x$ such that $B_{w,x,e}$ is meager;
- for all $e \in E_x$, $B_{w,x,e}$ is meager.

Thus

$$\{ (w, x) \in W \times X \mid A_{w,x} \text{ is meager} \}$$

coincides with both

$$\{ (w, x) \in W \times X \mid \exists e \in E((e \in E_x) \text{ and } (w, x, e) \in C)) \}$$

which is analytic and

$$\{ (w, x) \in W \times X \mid \forall e \in E((e \in E_x) \text{ implies } (w, x, e) \in C)) \}$$
which is coanalytic. It follows that this set is Borel and hence the map \( x \mapsto \mathcal{M}(P_x) \) is Borel on Borel. This finishes the proof of the Luzin-Novikov Uniformization Theorem.

If \( \Gamma \) is a countable group and \( \Gamma \) acts on a Polish space by Borel permutations, then

\[
E^\Gamma_X := \{(x, y) \in X^2 \mid \exists \gamma \in \Gamma (\gamma \cdot x = y)\}
\]

is a Borel equivalence relation in which each equivalence class is countable. This is called the \textit{orbit equivalence relation} associated to the action. These are often known as \textit{countable Borel equivalence relations (CBERs)}, although this is obviously an abuse of the language. In fact this observation has a converse, which explains the important role that group theory often plays in the study of countable Borel equivalence relations.

**Corollary 15** (Feldman-Moore). If \( E \) is a countable Borel equivalence relation on a Polish space \( X \), then \( E \) is the orbit equivalence relation of countable group \( \Gamma \) acting on \( X \) by Borel permutations. Moreover, such a \( \Gamma \) can be found which is generated by elements of order 2.

**Proof.** By the Luzin-Novikov Uniformization Theorem, there is a countable collection of Borel functions \( \mathcal{F} \) such that \( E = \bigcup \mathcal{F} \). Let \( \mathcal{G} \) be the collection of all intersections of the form \( f_0 \cap f_1^{-1} \cap U \times V \) such that \( f_0, f_1 \in \mathcal{F} \) and \( U \) and \( V \) are disjoint basic open sets. Observe that each element \( g \) of \( \mathcal{G} \) is a Borel bijection between two disjoint Borel sets \( A \) and \( B \). Furthermore, \( \mathcal{G} \) is countable and \( \bigcup \mathcal{G} = E \setminus \Delta \). Extend each \( g \) in \( \mathcal{G} \) to a Borel involution \( \gamma \) as follows:

\[
\gamma(x) := \begin{cases} 
  g(x) & \text{if } x \in A \\
  g^{-1}(x) & \text{if } x \in B \\
  x & \text{if } x \not\in A \cup B 
\end{cases}
\]

Define \( \Gamma \) to be group generated by all such extensions, observing that \( \Gamma \) is as desired. \( \square \)
We will now turn our discussion to Borel graphs and their Borel chromatic numbers. A **Borel graph** is exactly what it sounds like: a set of vertices $X$ which is a Polish space and a symmetric irreflexive relation $G$ which is regarded as the adjacency relation coming from the edge set. Our main interest will be in the **Borel chromatic number** of a Borel graph. Recall that a set $I$ of vertices is independent if $I^2 \cap G = \emptyset$ — i.e. no two elements of $I$ are connected by an edge. A **Borel coloring** of a Borel graph is a partition of the vertex set into Borel independent sets. The Borel chromatic number of the graph is the minimum number of pieces needed in a Borel coloring.

The Borel chromatic number is always at least as large as the conventional chromatic number, in which the independent sets are not required to be Borel. In some cases, however, it may be larger as the next examples show.

First, suppose that $\alpha \in \mathbb{C}$ is such that $|\alpha| = 1$ and yet there is no $n \in \mathbb{Z}$ such that $\alpha^n = 1$. Let $T$ denote the unit modulus complex numbers and define the Borel graph $G$ on $T$ by $(x, y) \in G$ iff $x = \alpha y$ or $y = \alpha x$. The conventional chromatic number of this graph is 2: if $X \subseteq T$ is such that for every $y \in T$ there is a unique $n \in \mathbb{Z}$ with $\alpha^n y \in X$, then we define $A$ to consist of all those $y$ such that this unique $n$ is even and set $B = T \setminus A$. This is easily seen to define a coloring of the graph.

There are no Borel 2 colorings of this graph, however. If $A, B \subseteq T$ is any 2 coloring of $G$, then $\alpha A = B$ and $\alpha B = A$. In particular, both $A$ and $B$ are invariant under the map $x \mapsto \alpha^2 x$. Since $H := \{\alpha^{2n} \mid n \in \mathbb{Z}\}$ is a countable dense subgroup of $T$, it follows that if $A$ has the Baire property (for instance if it is Borel), it must be either meager or comeager. But this contradicts the fact that $\alpha A$ and $A$ are a partition of $T$ with $x \mapsto \alpha x$ a homeomorphism of $T$. This shows that in fact $T$ has no 2 coloring in which the color sets are Baire measurable. Similarly, there is no Haar measurable 2 coloring of this graph.

There is, however, a Borel three coloring. Let $r \in [0, 1)$ be such that $\alpha = \exp ((2\pi i) r)$. For simplicity, suppose that $r < 1/2$; the other case is a routine modification. Let $n$ be such that $nr < 1 < (n+1)r$. Define

$$A := \{\exp ((2\pi is) \mid \exists k \in \mathbb{Z}(0 \leq 2kr \leq s < 2k + 1r \leq nr)\}$$

$$B := \{\exp ((2\pi is) \mid \exists k \in \mathbb{Z}(0 \leq (2k+1)r \leq s < (2k+2)r \leq nr)\}$$

$$C := \{\exp ((2\pi is) \mid \exists k \in \mathbb{Z}(rn \leq s < 1)\}.$$

The sets $A, B, C$ are each $F_\sigma$ and are pairwise disjoint. It is easily checked that this defines a 3 coloring of the graph.
Next we turn to a Borel graph (actually a class of them) which plays an important role in characterizing when the Borel chromatic number of a graph is countable. Fix a sequence $\langle t_n \mid n \in \omega \rangle$ of finite binary sequences such that the length of $t_n$ is $n$ and for every $s \in 2^{<\omega}$, there is an $n$ such that $s$ is an initial part of $t_n$. Define $G_0$ to be the set

$$\{(t_n\upharpoonright i)^-x, t_n\upharpoonright (1-i)^-x) \mid (n \in \omega) \land (i \in \{0,1\}) \land (x \in 2^\omega)\}$$

It can be checked that this graph is acyclic and hence has chromatic number 2. The next claim, however, implies that its Borel chromatic number is uncountable.

**Proposition 8.** If $E \subseteq 2^\omega$ is independent and Baire measurable, then $E$ is meager.

*Proof.* It suffices to show that if $E$ has the Baire property and it is nonmeager, then $E$ is not independent. Let $s \in 2^{<\omega}$ be such that $[s] \setminus E$ is meager. Let $n$ be such that $t_n$ extends $s$. It follows that $[t_n\upharpoonright 0] \setminus E$ and $[t_n\upharpoonright 1] \setminus E$ are both meager. Consequently there is a comeager set of $x \in 2^\omega$ such that $t_n\upharpoonright 0^-x$ and $t_n\upharpoonright 1^-x$ are both in $E$. Thus $E$ fails to be independent. □

The next result, which characterizes when analytic graphs have countable Borel chromatic number, has a number of applications which we will investigate in future lectures. It is often referred to as the $G_0$-dichotomy.

**Theorem 53** (Kechris-Solecki-Todorcevic). *Suppose that $G$ is an analytic graph on $X$. Exactly one of the following holds:

1. $G$ has countable Borel chromatic number;
2. there is a continuous function $\phi : 2^\omega \to X$ such that if $(x,y) \in G_0$, then $(\phi(x),\phi(y)) \in G$.*
We will now give a proof of the $G_0$-dichotomy stated at the end of the last lecture. The proof is not the original and is essentially due to Ben Miller (I am working from lecture notes of Clinton Conley).

We observed last lecture that $G_0$ does not admit a countable coloring with Baire measurable color sets and, in particular, $2^\omega$ can’t be covered by countably many analytic $G_0$-independent sets. This is already a consequence of the following proposition of independent interest.

**Proposition 9.** Every analytic independent set in an analytic graph is contained in a Borel independent set.

**Proof.** Let $G$ be an analytic graph on a Polish space $X$. If $Y \subseteq X$, set

$$Y_G := \{ x \in X \mid \exists y \in Y((x, y) \in G) \}$$

Observe that $Y$ is independent if and only if $Y \cap Y_G = \emptyset$. Also, if $Y$ is analytic, then so is $Y_G$.

Now let $A$ be an analytic independent set. Since $A_G$ is disjoint from $A$, the separation theorem implies that there is a Borel set $B$ which contains $A$ and is disjoint from $A_G$. Notice that $B_G$ is disjoint from $A$ and thus, by another application of the separation theorem, there is a Borel set $C$ which contains $A$ and is disjoint from $B_G$. It follows that $B \cap C$ is a Borel independent set which contains $A$. \hfill \Box

**Proof of the $G_0$-dichotomy.** Let $G$ be an analytic graph on a Polish space $X$ and suppose that the Borel chromatic number of $G$ is uncountable. By increasing the topology on $X$, we may and will assume that $X$ is a closed subspace of $\omega^\omega$. Fix a continuous function $\gamma : \omega^\omega \to X \times X$ such that $G$ is the range of $\gamma$. Let $I$ be the $\sigma$-ideal generated by the Borel independent sets, noting that every analytic independent set is in $I$ by Proposition 9. Observe that by shrinking $X$ if necessary, we may assume that if $U$ is open and $U \cap X$ is nonempty, then it is not in $I$.

For each $n \in \omega$, let $G_{0,n}$ to be the graph on $2^n$ defined by

$$G_{0,n} := \{(t_k \triangledown \langle i \rangle \triangledown x, t_k \triangledown \langle 1-i \rangle \triangledown x) \mid (k < n) \land (i \in \{0, 1\}) \land (x \in 2^{n-k-1})\}.$$

Set $G_{0,\langle \omega \rangle} := \bigcup_{n=0}^{\infty} G_{0,n}$. If $e, f \in G_{0,\langle \omega \rangle}$, we say that $e$ is extended by $f$ if this is true coordinatewise. If $e = (s, t) \in G_{0,n}$, we will let $e \triangledown \langle i \rangle$ denote $(s \triangledown \langle i \rangle, t \triangledown \langle i \rangle)$.

Recursively construct $\langle u_s \mid s \in 2^{<\omega} \rangle$ and $\langle q_e \mid e \in G_{0,\langle \omega \rangle} \rangle$ such that:

1. if $s, t \in 2^{<\omega}$ and $s$ is an initial part of $t$, then $u_s$ is an initial part of $u_t$;
(2) if \( e, f \in G_{0,<\omega} \), then \( e \) is extended by \( f \) if and only if \( q_e \) is an initial part of \( q_f \).

(3) if \( e = (s, t) \in G_{0, n} \), then the \( \gamma \)-image of \([q_e]\) is contained in \([u_s] \times [u_t]\).

(4) for each \( n \in \omega \) and each \( I \in \mathcal{I} \), there exist \( \langle x_s \mid s \in 2^n \rangle \) and \( \langle z_e \mid e \in G_{0, n} \rangle \) such that:
   - for each \( s \in 2^n \), \( x_s \in [u_s] \setminus I \);
   - for each \( e = (s, t) \in G_{0, n} \), \( \gamma(z_e) = (x_s, x_t) \).

We will say that the pair of sequences \( \langle x_s \mid s \in 2^n \rangle \) and \( \langle z_e \mid e \in G_{0, n} \rangle \) are a realization of \( \langle u_s \mid s \in 2^n \rangle \) and \( \langle q_e \mid e \in G_{0, n} \rangle \) which avoids \( I \).

To see that the construction can be carried out, start by defining \( u_\varepsilon = \varepsilon \). All of the requirements are vacuously satisfied except for (4), which is just a restatement of our assumption that \( X \notin \mathcal{I} \).

Now suppose that we have carried out the construction for all \( s \in 2^n \) and \( e \in G_{0, n} \). Observe that since \( \mathcal{I} \) is countably directed, if we partition \( \mathcal{I} \) into countably many pieces, one of them is cofinal in \( (\mathcal{I}, \subseteq) \). It follows that it is sufficient to show that for each Borel \( I \in \mathcal{I} \), we can construct the extensions \( \langle u_s \mid s \in 2^{n+1} \rangle \) and \( \langle q_e \mid e \in G_{0, n+1} \rangle \) which satisfy (1)–(3) and the instance of (4) for this fixed \( I \): it must be that for cofinally many \( I \in \mathcal{I} \), the same extension works.

Let \( I \in \mathcal{I} \) be a Borel set. Define \( A \) to be the set of all \( x_{t_n} \) where \( \langle x_s \mid s \in 2^n \rangle \) and \( \langle z_e \mid e \in G_{0, n} \rangle \) are a realization of \( \langle u_s \mid s \in 2^n \rangle \) and \( \langle q_e \mid e \in G_{0, n} \rangle \) avoiding \( I \). Observe that \( A \) is analytic and not in \( \mathcal{I} \). In particular, it is not independent. Let \( \langle x_{s^-(i)} \mid s \in 2^n \rangle \) and \( \langle z_{e^-(i)} \mid e \in G_{0, n} \rangle \) for \( i \in \{0, 1\} \) be two realizations of \( \langle u_s \mid s \in 2^n \rangle \) and \( \langle q_e \mid e \in G_{0, n} \rangle \) which avoid \( I \). Pick \( G_{(t_n^-(i), t_n^-(1-i))} \in \omega^{\omega^2} \) such that
\[
\gamma(G_{(t_n^-(i), t_n^-(1-i))}) = (x_{t_n^-(i)}, x_{t_n^-(1-i)}).
\]
Fix an \( m \) such that if \( x_s \neq x_t \) for \( s, t \in 2^{n+1} \), then \( x_s \upharpoonright m \neq x_t \upharpoonright m \); set \( u_s := x_s \upharpoonright m \) for each \( s \in 2^{n+1} \). By continuity of \( \gamma \), there is a \( \bar{m} \) such that if we set \( q_e = z_e \upharpoonright \bar{m} \), then (3) is satisfied. Observe that we have now constructed \( \langle u_s \mid s \in 2^{n+1} \rangle \) and \( \langle q_e \mid e \in G_{0, n+1} \rangle \) such that (3)–(3) are satisfied. Moreover, \( \langle x_s \mid s \in 2^{n+1} \rangle \) and \( \langle z_e \mid e \in G_{0, n+1} \rangle \) are a realization of \( \langle u_s \mid s \in 2^{n+1} \rangle \) and \( \langle q_e \mid e \in G_{0, n+1} \rangle \) which avoids \( I \). As noted above, this ensures the construction can be completed.

Given the construction of \( \langle u_s \mid s \in 2^{<\omega} \rangle \) and \( \langle q_e \mid e \in G_{0, <\omega} \rangle \), define \( \phi(a) = x \) if for every \( n \), \( u_{a|n} \) is an initial part of \( x \). By (1), this defines a continuous function from \( 2^\omega \) into \( X \). Now suppose that \( (a, b) \in G_0 \). For some \( n_0 \), \( a \upharpoonright n_0 \neq b \upharpoonright n_0 \). Observe that for all \( n \geq n_0 \), \( e_n := (a \upharpoonright n, b \upharpoonright n) \) is in \( G_{0, n} \). By (2), there is a \( z \in \omega^\omega \) such that for all \( n \), \( q_{e_n} \) is an initial part of \( z \). It now follows from (3) that \( \gamma(z) = (\phi(a), \phi(b)) \). In particular, \( (\phi(a), \phi(b)) \in G \), as desired. \( \square \)
If \((X, E)\) and \((Y, F)\) are two Polish spaces equipped with equivalence relations, we say that \(E\) is \textit{Borel reducible} to \(F\), written \(E \leq_B F\) if there is a Borel function \(\phi : X \to Y\) such that \((x_0, x_1) \in E\) if and only if \((\phi(x_0), \phi(x_1)) \in F\); the map \(\phi\) is called a reduction. A reduction which is an injection is an \textit{embedding}. This provides a measure of relative complexity of a equivalence relations on Polish spaces. Usually one considers this ordering in the context of equivalence relations which themselves have regularity properties – typically equivalence relations of interest are at least analytic and often Borel.

The simplest Borel equivalence relations are just the equality relation \(\Delta_X\) on a Polish space \(X\). A Borel equivalence relation is \textit{smooth} if it is reducible to \(\Delta_X\) for some Polish space \(X\). A more complicated equivalence relation is \(E_0 \subseteq 2^\omega \times 2^\omega\), defined by \((a, b) \in E_0\) iff there is an \(m\) such that for all \(n > m\), \(a(n) = b(n)\). Observe that if \(h : \omega \to \omega\) has the property that \(h^{-1}(n)\) is infinite for each \(n\), then the map \(a \mapsto a \circ h\) defines a continuous reduction from \(\Delta_{2^\omega}\) to \(E_0\). The next proposition shows that in fact \(\Delta_{2^\omega} <_B E_0\).

**Proposition 10.** If \(\phi : 2^\omega \to X\) is a Borel map into a Polish space and \(\phi(a) = \phi(b)\) whenever \((a, b) \in E_0\), then there is a comeager set \(G \subseteq 2^\omega\) such that \(\phi\) is constant on \(G\). In particular, \(E_0\) is not smooth.

**Proof.** Observe that if \(B \subseteq X\) is a Borel set, then \(\phi^{-1}(B)\) is a Borel set which is invariant under finite modifications. It follows that \(\phi^{-1}(B)\) is meager or comeager for each Borel \(B \subseteq X\). Define \(\mathcal{F}\) to be the collection of all Borel subsets \(B\) of \(X\) such that \(\phi^{-1}(B)\) is comeager. It follows that \(\mathcal{F}\) is a countably complete ultrafilter on the Borel subsets of \(X\). This implies that \(\bigcap \mathcal{F} = \{x\}\) for some \(x \in X\). It follows that \(G := \phi^{-1}(x)\) is the desired comeager set. \(\square\)

One of the main motivations for studying Borel reducibility is that it gives a way of measuring the complexity of classification problems which arise in mathematics. Often the members of a class of mathematical structures — e.g. countable groups, separable Banach spaces, compact metric spaces, finitely generated groups — can be naturally viewed as points in some fixed Polish space. For instance, since every compact metric space is homeomorphic to a compact subspace of \([0, 1]^\omega\), the elements of this class could be viewed as points in the Polish space \(K([0, 1]^\omega)\). The class of all finitely generated groups can be coded as those \((G, \ast) \in \mathcal{P}(\omega) \times \mathcal{P}(\omega^3)\) such that \((G, \ast)\) is a finitely generated group. This is a Borel subset of \(\mathcal{P}(\omega) \times \mathcal{P}(\omega^3)\) and thus a Polish space once the topology is strengthened. Typically one is interested in some
natural equivalence such as isometry, isomorphism, or homeomorphism on the members of this class. This corresponds to an equivalence relation on the associated Polish space; this equivalence relation is almost always analytic and often is Borel. Notice, for instance, that any isomorphism between finitely generated groups is determined by where the isomorphism sends the generators. Thus the isomorphism equivalence relations on the space of finitely generated groups is Borel and has countable equivalence classes.

The notion of Borel reducibility allows one to compare these classification problems both to each other and to benchmarks such as $\Delta_{2\omega}$ and $E_0$. Notice that this is of course not the only way to measure the complexity of a classification problem and in some cases its is completely ill equipped to reflect this complexity. For instance if the class of objects under consideration is countably infinite, Borel reducibility offers no stratification of the complexity. Needless to say, few would argue that determining whether two finite groups are isomorphic is much harder than determining whether two natural numbers are equal. In cases where one classification problem is not Borel reducible to another, however, one does obtain a powerful statement about the relative difficulty of these classification problems.

The next two theorems, which we will derive from the $G_0$-dichotomy, explain the special role that $\Delta_{2\omega}$ and $E_0$ play in the context of Borel equivalence relations.

**Theorem 54** (Silver). **If $E$ is a coanalytic equivalence relation on a Polish space $X$, then either $E \leq_B \Delta_{\omega}$ or else $\Delta_{2\omega} \leq_B E$ and this is witnessed by a continuous embedding.**

*Proof. Since the complement of $E$ is an analytic graph on $X$, it is subject to the $G_0$-dichotomy. First observe that any independent set for $X^2 \setminus E$ is contained in an $E$-equivalence class. In particular, if $X^2 \setminus E$ has a countable Borel chromatic number, then $E$ has countably many equivalence classes, each of which is Borel. It follows that in this case, $E \leq \Delta_\omega$. Now suppose that $\phi: 2^\omega \to X$ is such that if $(a, b) \in G_0$, then $(\phi(a), \phi(b)) \not\in E$. Define $F = (\phi \times \phi)^{-1}(E)$, noting that $F$ is a coanalytic equivalence relation. If $a \in 2^\omega$, then $[a]_F = \{b \in 2^\omega \mid (a, b) \in F\}$ is a $G_0$-independent set which has the Baire property and hence is meager. By the Kuratowski-Ulam theorem, $F$ is meager and by Mycielski’s theorem, there is a continuous injection $\psi: 2^\omega \to 2^\omega$ such that if $a \neq b$, then $(\psi(a), \psi(b)) \not\in F$ and consequently $(\phi(\psi(a)), \phi(\psi(b))) \not\in E$. This $\phi \circ \psi$ is a continuous embedding of $\Delta_{2\omega}$ into $E$. □
Theorem 55 (Harrington-Kechris-Louveau). If $E$ is a Borel equivalence relation, then either $E \leq_B \Delta_{2\omega}$ or else there is a continuous embedding to $E_0$ into $E$.

We will prove only the special case of this theorem, in which $E$ is a countable Borel equivalence relation. In this case, the alternative $E \leq_B \Delta_{2\omega}$ takes a special form. If $E$ is an equivalence relation a (partial) transversal for $E$ is a set which intersects each $E$-class in (at most) one point.

Proposition 11. If $E$ is a countable Borel equivalence relation on a Polish space $X$, the following are equivalent:

1. $E$ is smooth;
2. $E$ admits a Borel transversal;
3. $X$ can be covered by countably many partial Borel transversals.

Proof. If $E$ is smooth, let $\phi : X \to Y$ be a Borel function such that $(x_0, x_1) \in E$ if and only if $\phi(x_0) = \phi(x_1)$. In particular, $f$ is one-to-one and hence has a Borel range. By replacing $Y$ if necessary, we may assume that $\phi$ is surjective. An application of the Luzin-Novikov uniformization theorem to $F := \{(y, x) \in Y \times X \mid \phi(x) = y\}$ yields both that $F$ contains the graph of a Borel function $g : Y \to X$ and that it can be partitioned into countably many partial Borel functions $\{h_n \mid n \in \omega\}$. It follows that the range of $g$ is a Borel transversal and that the ranges of the $h_n$’s are partial Borel transversals which cover $X$. This establishes that (1) implies both (2) and (3). To see that (2) implies (1), observe that if $T \subseteq X$ is a Borel transversal, then $(X \times T) \cap E$ is a Borel function from $X$ to $X$ which defines a reduction of $E$ to $\Delta_X$. Similarly if $\{T_n \mid n \in \omega\}$ is a cover of $X$ by partial Borel transversals, then set $h_n := (X \times T_n) \cap E$ and set $\phi(x) = h_n(x)$ if $n$ is minimal such that $x \in \text{dom}(h_n)$. This is a Borel function since each $h_n$ is a Borel function and hence has a Borel domain. It is easily seen that $\phi$ witnesses $E \leq_B \Delta_X$. \hfill \Box

We now will now complete the proof of Theorem 55 when $E$ has countable classes. Define $G := E \setminus \Delta_X$ and apply the $G_0$-dichotomy to $G$. Observe that the $G$-independent sets are exactly the transversals of $E$. Thus by Proposition 11 $G$ has countable Borel chromatic number precisely when $E$ is smooth. Suppose now that there is a continuous $\phi : 2^{\omega} \to X$ such that if $(a, b) \in G_0$, then $(\phi(a), \phi(b)) \in E$. If we define $F := (\phi \times \phi)^{-1}(E)$, then $F$ is an equivalence relation which contains $G_0$ and hence $E_0$. Since each $F$-equivalence class is Borel and invariant under finite modifications, each must be meager or comeager. Moreover, the later is clearly impossible: since each $E$-class is countable,
each class is $F_\sigma$ and thus contains a dense open set if it is comeager, which contradicts the fact that each $E_0$-class — and hence each $F$-class — is dense. It follows from the Kuratowski-Ulam theorem that $F$ is meager.

**Claim 2.** If $Z \subseteq 2^\omega \times 2^\omega$ is meager, then there exist $u_k^i \in 2^{<\omega}$ for each $k \in \omega$ and $i \in \{0, 1\}$ such that:

- for all $k$, $u_k^0 \neq u_k^1$ have the same length;
- if $a, b \in 2^\omega$ are not $E_0$-related, then

$$(u_0^{a_0} \cdots u_1^{a_1} \cdots , u_0^{b_0} \cdots u_1^{b_1} \cdots )$$

is not in $Z$.

**Proof.** Let $\langle Z_k \mid k \in \omega \rangle$ be an increasing sequence of nowhere dense sets which covers $Z$. Given $\langle u_k \mid k < n \rangle$, pick $u_n^0$ and $u_n^1$ in $2^{<\omega}$ of equal length such that if $s_0 \neq s_1$ are any binary sequences of length $s = \sum_{k<n} \ell h(u_k)$, then

$$(|s_0 \cap u_n^0| \times |s_1 \cap u_n^1|) \cup (|s_1 \cap u_n^1| \times |s_0 \cap u_n^0|) \cap Z_n = \emptyset.$$  

Suppose now that $a, b \in 2^\omega$ are not $E_0$-related and let $k \in \omega$ be arbitrary. Fix an $n > k$ such that $a(n) \neq b(n)$ and observe that now

$$[u_0^{a_0} \cdots u_n^{a_n}, u_0^{b_0} \cdots u_n^{b_n}] \cap Z_n = \emptyset.$$  

Since $k$ was arbitrary, $(u_0^{a_0} \cdots u_1^{a_1} \cdots , u_0^{b_0} \cdots u_1^{b_1} \cdots )$ is not in $Z$.  

By applying the claim to $F$, there is a continuous $\psi : 2^\omega \to 2^\omega$ which witnesses $E_0 \leq_B F$. In particular, $\phi \circ \psi$ witnesses that $E_0 \leq_B E$.

We will now turn to showing that for some continuous $\theta : 2^\omega \to 2^\omega$, $\phi \circ \psi \circ \theta$ is a continuous embedding of $E_0$ into $E$. By applying the Luzin-Novikov theorem to $\{(x, a) \in 2^\omega \times X \mid \phi(\psi(a)) = x\}$ it follows that $2^\omega$ is a union of countably many Borel sets on which $\phi \circ \psi$ is injective. Thus by the Baire Category Theorem, there is a nonmeager Borel set $B \subseteq 2^\omega$ such that $\phi \circ \psi \upharpoonright B$ is one-to-one and hence and embedding of $E_0 \upharpoonright B$ into $E$. We are now finished by the following claim.

**Claim 3.** If $B \subseteq 2^\omega$ is a nonmeager Borel set, then there exist $u_k \in 2^{<\omega}$ for each $k \in \omega$ such that if $a \in 2^\omega$, then

$$\theta(a) := u_0 ◦ (a_0) ◦ u_1 ◦ (a_1) ◦ u_2 ◦ (a_2) \cdots \in B.$$  

In particular, $\theta$ is a continuous embedding of $E_0$ into $E_0 \upharpoonright B$.

**Proof.** Start by letting $u_0$ be such that $[u_0] \setminus B$ is a countable union of nowhere dense sets $\langle Z_k \mid k \in \omega \rangle$. Given $\langle u_k \mid k < n \rangle$, select $u_n$ such that if $s$ is any binary sequence of length $n + \sum_{k<n} \ell h(u_k)$, then $\langle s \cap u_n \rangle$ is disjoint from $Z_k$ for all $k < n$. It follows that the sequence has the desired property.
We will now turn to a discussion of infinite dimensional Ramsey theory. The classical Ramsey theorem can be stated as follows.

**Theorem 56 (Ramsey).** If $d, m \in \omega$ and $[\omega]^d = \bigcup_{i<m} K_i$, then there exists an infinite $H \subseteq \omega$ and an $i < m$ such that $[H]^d \subseteq K_i$.

It is reasonable to ask to what extent this theorem remains true if we allow $d = \omega$: Under what hypotheses is it true that whenever $[\omega]^\omega = \bigcup_{i<m} K_i$, there is an infinite $H \subseteq \omega$ and an $i < m$ such that every infinite subset of $H$ is in $K_i$?

Without further assumptions, this strengthening is false. To see this, define $\sigma : [\omega]^\omega \to [\omega]^\omega$ by $\sigma(A) = A \setminus \{\min(A)\}$ and let $\sim$ be the minimum equivalence relation such that for all $A, B \sim \sigma(A)$. Observe that the graph $G$ in which $A$ and $\sigma(A)$ are adjacent is acyclic. Let $d$ be the path metric given by this graph. Notice in particular if $A \sim B$, then $d(A, B)$ is even if and only if $d(\sigma(A), B)$ is odd. Let $X \subseteq [\omega]^\omega$ meet each $\sim$-class in exactly one element. Define $A \in K_0$ if $d(A, X)$ is even, where $X \in X$ is the unique element such that $A \sim X$. Set $K_1 = [\omega]^\omega \setminus K_0$. It follows from the above observation that if $H \subseteq \omega$ is infinite, then $H$ is in $K_0$ if and only if $\sigma(H) \subseteq H$ is in $K_1$.

Int this construction, however, the set $X$, which is chosen using the Axiom of Choice, is at least a priori not regular. We will see that in fact such an $X$ can not be Souslin measurable. It will be fruitful at this point to pause to introduce a definition. The *Ellentuck topology* on $[\omega]^\omega$ is generated by the following basic open sets:

$$[a, A] := \{B \in [a \cup A]^\omega \mid a \text{ is an initial part of } B\}$$

where $a$ is a finite subset of $\omega$ and $A \subseteq \omega$ is infinite. It is readily checked that in this topology these sets are both open and closed and that they separate points. The Ellentuck topology is stronger than the usual metric topology on $[\omega]^\omega$, which is generated by $[a, \omega]$ where $a \subseteq \omega$ is finite. In particular, the sets which are Baire measurable in the Ellentuck topology form a $\sigma$-algebra which contains the metric open sets and which is closed under the $\mathcal{A}$-operation. In particular, every Souslin measurable subset of $[\omega]^\omega$ has the Baire property in the Ellentuck topology.

**Theorem 57 (Ellentuck).** Suppose that $[\omega]^\omega = \bigcup_{i<m} K_i$ where each $K_i$ has the Baire property with respect to the Ellentuck topology. There exists an infinite $H \subseteq [\omega]^\omega$ such that $[H]^\omega \subseteq K_i$ for some $i < m$.

This theorem was first proved when the $K_i$ are Borel by Galvin and Prikry and when the $K_i$ are analytic by Silver. Todorcevic showed
that Ellentuck’s proof can be abstracted so as to provide a general means of proving infinite dimensional partition theorems starting from pigeonhole principles. For instance, the basic pigeonhole principle underlying Ramsey’s theorem asserts that *if an infinite set is partitioned into finitely many pieces, at least one is infinite.* Pigeonhole principles such as Hindman’s Theorem, the Hales-Jewett Theorem, and Gowers’ FIN\(_k\)-Theorem each give rise to much more powerful infinite dimensional Ramsey Theorems. It is these infinite dimensional Ramsey theorems which are often most useful in practice.

I will finish this lecture by stating a strengthening of the Galvin-Prikry Theorem which is based on a result of Milliken. Let \( T \) be a tree with no maximal elements. A *strong subtree* of \( T \) is a subset \( S \subseteq T \) which is a tree such that for some subset \( I(S) \subseteq \omega \) the following is true:

- if \( s \in S \) and \( lh(s) \notin I \), then \( s \) has a unique immediate successor in \( S \);
- if \( s \in S \) and \( lh(s) \in I \), then every immediate successor \( s \) in \( T \) is in \( S \).

If \( d \leq \omega \), define \( S^d(T) \) to be the collection of all strong subtrees of \( T \) such that \( I(S) \) has cardinality \( d \). Observe that for all \( m \), \( S^\omega(m^{<\omega}) \) is a \( G_\delta \)-subset of \( \mathcal{P}(m^{<\omega}) \) and hence a Polish space. The next result is sometimes referred to as Milliken’s Theorem.

**Theorem 58.** If \( S^\omega(m^{<\omega}) \) is partitioned into finitely many Souslin measurable pieces, then there is an \( T \in S^\omega(m^{<\omega}) \) such that \( S^\omega(T) \) is contained in one piece of the partition.

Notice that the map \( T \mapsto I(T) \) is continuous on \( S^\omega(m^{<\omega}) \) and that colorings which depend only on \( I(T) \) correspond to colorings of \( [\omega]^{<\omega} \). Thus the Galvin-Prikry Theorem for Souslin measurable partitions is a special case of Milliken’s Theorem.
We will prove Ellentuck’s Theorem through a series of lemmas. A subset \( \mathcal{X} \) of \([\omega]^{\omega}\) is **completely Ramsey** if whenever \([a, A]\) is a basic open set in the Ellentuck topology, there is an infinite \( B \subseteq A \) such that \([a, B]\) is either contained in or disjoint from \( \mathcal{X} \). We will show that every set with the Baire property in the Ellentuck topology is completely Ramsey. This is sufficient since if \([\omega]^{\omega} = \bigcup_{i<m} K_i \) and each \( K_i \) has the Baire property, then there is an infinite \( A \subseteq \omega \) such that \([0, A] = [A]^{\omega} \) is either contained in or disjoint from each \( K_i \). Since the \( K_i \)'s cover \([\omega]^{\omega}\), there must be an \( i < m \) such that \([A]^{\omega} \subseteq K_i \).

**Lemma 7.** Every Ellentuck open set is completely Ramsey.

**Proof.** Fix an Ellentuck open set \( \mathcal{X} \subseteq [\omega]^{\omega} \). It will be useful to introduce some terminology. If \( A \in [\omega]^{\omega} \) and \( a \in [\omega]^{<\omega} \), we say that \( A \) **accepts** \( a \) if \([a, A] \subseteq \mathcal{X}\). We say that \( A \) **rejects** \( a \) if there is no infinite subset of \( A \) which accepts \( a \); we say that \( A \) **decides** \( a \) if \( A \) either accepts or rejects \( a \). The following are obvious from the definitions:

1. if \( A \) accepts (rejects) \( a \), then so does every infinite subset of \( A \);
2. there is an infinite \( B \subseteq A \) such that \( B \) decides \( a \);
3. if \( A \) rejects \( a \) and \( B \) is a finite modification of \( A \), then \( B \) rejects \( a \).
4. if \( A \) rejects \( a \), then \( \{ n \in A \mid A \) accepts \( a \cup \{ n \}\} \) is finite.

Item 4) follows from the observation that if \( B := \{ n \in A \mid (\max(a) < n) \land (A \) accepts \( a \cup \{ n \}\}\} \) then \([a, B] = \bigcup_{n \in B} [a \cup \{ n \}, B]\).

Toward showing that \( \mathcal{X} \) is completely Ramsey, let \( a \in [\omega]^{<\omega} \) and \( A \in [\omega]^{\omega} \) be given. Without loss of generality, we may assume that \( \max(a) < \min(A) \).

**Claim 4.** There is an \( B \in [A]^{\omega} \) which decides \( a \cup b \) whenever \( b \in [B]^{<\omega} \).

**Proof.** Construct \( \langle n_k \mid k \in \omega \rangle \) and \( \langle A_k \mid k \in \omega \rangle \) by recursion. Let \( A_0 \subseteq \omega \) be an infinite set which either accepts or rejects \( a \). Given \( A_k \), let \( n_k = \min(A_k) \) and let \( A_{k+1} \subseteq A_k \setminus \{ n_k \} \) be such that \( A_{k+1} \) decides \( a \cup b \) for each subset \( b \) of \( \{ n_i \mid i \leq k \} \). Set \( B := \{ n_k \mid k \in \omega \} \), noting that \( B \) decides \( a \cup b \) whenever \( b \in [B]^{<\omega} \).

**Claim 5.** Either \( B \) accepts \( a \) or else there is a \( C \in [B]^{\omega} \) which rejects \( a \cup b \) whenever \( b \in [B]^{<\omega} \).

**Proof.** Define \( \langle n_k \mid k \in \omega \rangle \) recursively as follows: \( n_k \) is the least element \( n \) of \( B \) which is greater than \( n_i \) for \( i < k \) and all \( p \) such that \( B \) accepts \( a \cup b \) whenever \( b \in [B]^{<\omega} \).
a \cup b \cup \{p\}$ for some $b \subseteq \{n_i \mid i < k\}$. It follows that $C := \{n_k \mid k \in \omega\}$ is as desired. □

If $C$ accepts $a$, then $[a, C] \subseteq \mathcal{X}$ by definition and we are done. Suppose now that $C$ rejects $a \cup b$ whenever $b \in [C]^{<\omega}$. We claim that $[a, C] \cap \mathcal{X} = \emptyset$. Suppose for contradiction that there is a $D \in [a, C] \cap \mathcal{X}$. Since $\mathcal{X}$ is open, there is an initial part $d$ of $D$ such that $[d, D] \subseteq \mathcal{X}$. Note, however, that $a$ is an initial part of $d$ and that $b := d \setminus a$ is a finite subset of $C$. Since $D$ is an infinite subset of $C$ which accepts $a \cup b$, $C$ did not reject $a \cup b$, contrary to our assumption. Thus $[a, C] \cap \mathcal{X} = \emptyset$. □

Observe that Lemma 7 implies that nowhere dense sets are completely Ramsey as well.

We pause to note the following consequence of this lemma.

**Theorem 59** (Galvin, Nash-Williams). If $\mathcal{F} \subseteq \text{FIN}$, then there is an infinite $H \subseteq \omega$ such that either $[H]^{<\omega} \cap \mathcal{F} = \emptyset$ or else every infinite subset of $H$ has an initial segment in $\mathcal{F}$.

**Proof.** Define $\mathcal{X} = \bigcup_{a \in \mathcal{F}} [a, \omega]$. By Lemma 3, there is an $H \in [\omega]^{\omega}$ such that $[\emptyset, H] = [H]^{\omega}$ is either contained in or disjoint from $\mathcal{X}$. It is easily checked that this $H$ satisfies the conclusion of the theorem. □

Since finite Boolean combinations of completely Ramsey sets are completely Ramsey, the following Lemma is sufficient in order to complete the proof of Ellentuck’s Theorem.

**Lemma 8.** Every meager set in the Ellentuck topology is nowhere dense.

**Proof.** Let $\mathcal{X}$ be an Ellentuck meager set and let $\langle \mathcal{X}_k \mid k \in \omega \rangle$ be a sequence of nowhere dense sets which covers $\mathcal{X}$. It suffices to show that if $[a, A]$ is an Ellentuck open set, there is a $B \in [A]^{\omega}$ such that $[a, B] \cap \mathcal{X} = \emptyset$. Construct $\langle A_k \mid k \in \omega \rangle$ such that, setting $n_k = \min(A_k)$:

- for all $k \in \omega$, $\max(a) < n_k < n_{k+1}$;
- if $b \subseteq \{n_i \mid i < k\}$, then $[a \cup b, A_k] \cap \mathcal{X}_k = \emptyset$.

Set $B := \{n_k \mid k \in \omega\}$ and suppose that $k \in \omega$. Observe that

$[a, B] \subseteq \bigcup\{[a \cup b, A_k] \mid b \subseteq \{n_i \mid i < k\}\}$

which is disjoint from $\mathcal{X}_k$ by construction. Thus $[a, B] \cap \mathcal{X} = \emptyset$. □
Proposition 12. Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$. If $B \subseteq X$ is a Borel set, then so is:

$$[B]_E := \bigcup \{[x]_E \mid x \in B\}.$$  

Proof. By the Luzin-Novikov theorem $E = \bigcup_{n=0}^{\infty} f_n$ where each $f_n$ is a partial Borel injection from $X$ to $X$. Since $f_n(E)$ is Borel for each $n$, it follows that $[B]_E = \bigcup_{n=0}^{\infty} f_n(E)$ is Borel. \[\square\]

Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$. A marker for $E$ is a Borel set $A$ which intersects each $E$-equivalence class. A sequence of markers $\langle A_k \mid k \in \omega \rangle$ is vanishing if it is decreasing and has empty intersection. Observe that in order for a vanishing sequence of markers to exist, each equivalence class of $E$ must be infinite. Such an $E$ is said to be aperiodic.


Proof. Since this is trivially true if the underlying Polish space is countable and since all uncountable Polish spaces are Borel isomorphic, we may assume that $E$ is an equivalence relation on $2^\omega$. For each $n$, define $s_n : 2^\omega \to 2^n$ by $s_n(a) \in 2^n$ the lexicographically least $s$ such that $[s]$ intersects the equivalence class of $a$. Each $s_n$ is a Borel function and therefore the set $A_n$ consisting of all $a \in 2^\omega$ such that $a \upharpoonright n = s_n(a)$ is a Borel set. Clearly each $A_n$ intersects each equivalence class and also $A_{n+1} \subseteq A_n$. Define $T := \bigcap_{n=0}^{\infty} A_n$ and observe that $T$ intersects each equivalence class in at most one element. It follows that $\langle A_n \setminus T \mid n \in \omega \rangle$ is a vanishing sequence of markers. \[\square\]

Next we note the following characterization of which countable Borel equivalence relations are smooth. If $X$ is a Polish space and $E$ is a Borel set $A$ which intersects each $E$-equivalence class, then the Borel structure on $X$ induces a $\sigma$-algebra on $X/E$: it consists of all $\{[x]_E \mid x \in B\}$ such that $B$ is Borel.

Theorem 60. Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$. The following are equivalent:

1. $E \leq_B \Delta_{2^\omega}$ (i.e. $E$ is smooth);
2. there is a Borel transversal for $E$;
3. $X$ is a countable union of partial Borel transversals;
4. the quotient $X/E$ is a standard Borel space.
Proof. We have already shown the equivalence of the first three items in previous lecture notes. Suppose now that $\phi : X \rightarrow 2^\omega$ is a Borel function such that $(x, y) \in E$ if and only if $\phi(x) = \phi(y)$. If follows that if $\{[x]_E \mid x \in B\}$ is in the $\sigma$-algebra associated to $X/E$, then $B$ is Borel. The map $\psi : [x]_E \mapsto \phi(x)$ induced by $\phi$ maps this set to $\phi(B)$, which is Borel. Thus $\psi$ defines an isomorphism between $X/E$ and a standard Borel space. In the converse direction, the map $x \mapsto [x]_E$ defines a Borel reduction from $E$ to $\Delta_{X/E}$. If $X/E$ is standard Borel, this witnesses that $E$ is smooth. \(\square\)

**Theorem 61** (Slaman-Steel, Weiss). Suppose that $E$ is a Borel equivalence relation. The following are equivalent:

1. $E$ is hyperfinite;
2. $E$ is the orbit equivalence relation of a Borel action of $\mathbb{Z}$.

Proof. Suppose first that $E$ is hyperfinite and fix an increasing sequence $\langle F_n \mid n \in \omega \rangle$ of Borel equivalence relations with finite classes such that $\bigcup_{n=0}^\infty F_n = E$ and such that $F_0 = \Delta_X$. Fix a Borel linear ordering $\leq$ on $X$ and define a relation $\prec$ on $X$ by $x \prec y$ if $x \neq y$, $(x, y) \in E$ and

$$\min[x]_{F_n} < \min[y]_{F_n}$$

where $n$ is maximal such that $(x, y) \notin F_n$. Observe that, by the homework exercises, $\prec$ is a Borel relation and that the restriction to each equivalence class is a linear order. Moreover, an easy inductive argument shows that each $F_n$-class is an interval in this order. It follows that each $E$-class is isomorphic to a suborder of $\mathbb{Z}$. For $n \leq \omega$, define $B_n$ to be the set of all $x \in X$ such that $[x]_E$ has cardinality $n$ and has a $\prec$-maximum or $\prec$-minimum element.

Define a Borel automorphism $f : X \rightarrow X$ as follows. On $X \setminus \bigcup_{n<\omega} B_n$, $f(x)$ is the $\prec$-least element of $[x]_E$ which is greater than $x$. That this is a Borel function follows from the Luzin-Novikov Theorem. To see this, let $\{g_k \mid k \in \omega\}$ be partial Borel functions which partition $E$. For each $k$,

$$Y_k := \{x \in X \mid \forall m((g_n(x) = g_m(x)) \vee (g_n(x) \prec g_m(x)))\}$$

is Borel and $f \upharpoonright Y_k = g_k \upharpoonright Y_k$.

If $n$ is finite, define $f \upharpoonright B_n \rightarrow B_n$ by permuting the cyclic order which $\prec$ induces on each finite class: $f(x) = y$ if $(x, y) \in E$ and $y$ is the $\prec$-successor of $x$ or, if $x = \max[x]_E$, then $y = \min[x]_E$. As in the previous case, this defines a Borel automorphism.

Finally, let $\langle T_n \mid n \in \mathbb{Z} \rangle$ be a partition of $B_\omega$ into Borel transversals for $E \upharpoonright B_\omega$. Define $f \upharpoonright B_\omega$ by $f(x) = y$ if $(x, y) \in E$ and for some $k$, $x \in T_k$ and $y \in T_{k+1}$. This finishes the proof of the forward implication. \(\square\)
Proof of Slaman-Steel, reverse implication. Next suppose that \( f : X \to X \) is a Borel automorphism which generates \( E \). Since the set of \( x \) whose equivalence class is finite forms a Borel set and is \( E \)-invariant, it suffices to prove the theorem when \( E \) has only infinite classes. Let \( \langle A_k \mid k \in \omega \rangle \) be a vanishing sequence of markers and let \( Y \) be the set of all \( y \in X \) for that for all \( k \)

\[
Z_k(x) := \{ n \in \mathbb{Z} \mid f^n(y) \in A_k \}
\]

is both cofinal and coinitial in \( \mathbb{Z} \). By the homework exercises, the restriction of \( E \) to \( X \setminus Y \) is smooth and therefore hyperfinite. Define \( F_k \) on \( Y \) by \( (x, y) \in F_k \) if either \( x = y \in A_k \) or else there is a (necessarily unique) sequence \( \langle z_i \mid i \leq m \rangle \) in \( Y \setminus A_k \) such that \( \{ z_0, z_m \} = \{ x, y \} \) and \( z_{i+1} = f(z_i) \) for all \( i < m \). Since the sequence connecting \( x \) to \( y \) is unique if it exists, \( F_k \) is Borel. By our definition of \( Y \) each \( F_k \)-class is finite and clearly \( F_k \subseteq F_{k+1} \) for all \( k \). Since \( \langle A_k \mid k \in \omega \rangle \) is vanishing, \( \bigcup_{k=0}^{\infty} F_k = E \).

Our next goal will be to prove that tail equivalence on \( \omega^\omega \) is hyperfinite. This will be a consequence of a more general result. A Borel equivalence relation \( E \) is hypersmooth if it is a countable increasing union of smooth equivalence relations. If \( X \) is Polish, then eventual equality on \( X^\omega \) is a typical example of a hypersmooth equivalence relation.

Since every equivalence relation with only finite equivalence classes is smooth, it follows that every hyperfinite equivalence relation is hypersmooth. The next theorem clarifies the relationship between these two notions.

**Theorem 62** (Dougherty-Jackson-Kechris). A Borel equivalence relation is hyperfinite if and only if it is both hypersmooth and all of its equivalence classes are countable.

**Proof.** It suffices to show that hypersmooth countable Borel equivalence relations are hyperfinite. To this end, let \( X \) be a Polish space and suppose that \( \langle \phi_k \mid k \in \omega \rangle \) be a sequence of countable-to-one Borel functions from \( X \) to \( 2^\omega \) such that \( \phi_k(x) = \phi_k(y) \) implies \( \phi_{k+1}(x) = \phi_{k+1}(y) \). Define \( E \) by \( (x, y) \in E \) if and only if there is a \( k \) such that \( \phi_k(x) = \phi_k(y) \); our task is to prove that \( E \) is hyperfinite.

Observe that our assumptions on the \( \phi_k \)'s imply that \( \phi_k(x) \mapsto \phi_{k+1}(x) \) is a well defined countable-to-one Borel function. By the Luzin-Novikov Theorem, there are Borel functions \( i_k : 2^\omega \to \omega \) such that if \( \phi_k(x) \neq \phi_k(x) \), then either \( \phi_{k+1}(x) \neq \phi_{k+1}(y) \) or else \( i_k(\phi_k(x)) \neq i_k(\phi_k(y)) \). For
each $x$, set $p_k(x) := i_k(\phi_k(x))$ and observe that if $x \neq y$, then
\[
\langle p_i(x) \mid i < k \rangle \neq \langle p_i(y) \mid i < k \rangle \quad \text{or} \quad \phi_k(x) \neq \phi_k(y)
\]
Furthermore, if $\phi_k(x) = \phi_k(y)$, then $p_m(x) = p_m(y)$ for all $m \geq k$. Define $(x, y) \in F_k$ if and only if:

- $\phi_k(x) = \phi_k(y)$ and
- for all $i < k$ there exists a $q$ such that
  \[
  2^n q \leq p_i(x), p_i(y) < 2^n (q + 1).
  \]

\(\square\)

**Theorem 63** (Dougherty-Jackson-Kechris). If $X$ is Polish and $f : X \to X$ is a Borel function, then the equivalence relation generated by $f$ is hypersmooth. If in addition $f$ is at most countable-to-one, then it generates a hyperfinite equivalence relation.

**Proof.** Without loss of generality, $X = 2^\omega$. For each $x \in X$, define $s_n(x) \in 2^n$ to be the lexicographically least element $s$ such that
\[
I_n(x) := \{i \in \omega \mid f^i(x) \upharpoonright n = s\}
\]
is infinite. Observe that $s_n(x)$ is an initial part of $s_{n+1}(x)$ for all $n$ and $x$. Define $\psi(x) = \bigcup_n s_n(x)$ and set $S := \{x \in X \mid (x, \psi(x)) \in E\}$. Observe that $\psi$ witnesses that $E \upharpoonright S$ is smooth and therefore it suffices to show that $E \upharpoonright (X \setminus S)$ is hypersmooth.

Notice that if $x \in X \setminus S$, $\psi(x) \notin \{f^i(x) \mid i \in \omega\}$ and thus if $i_n(x) := \min I_n(x)$, then $\lim_{n \to \infty} i_n(x) = \infty$. Furthermore, if $x, y \in X$ and $f^k(x) = f^l(y)$ for some $k, \ell \in \omega$, then whenever $k \leq i_n(x)$ and $\ell \leq i_n(y)$,
\[
i_n(x) - k = i_n(y) - \ell \quad \text{and} \quad f^{i_n(x)}(x) = f^{i_n(y)}(y).
\]
In particular, if $x, y \in X \setminus S$, then $(x, y) \in E$ if and only if for some $n$, $f^{i_n(x)}(x) = f^{i_n(y)}(y)$. Define $\phi_n(x) = f^{i_n(x)}(x)$, noting that if $\phi_n(x) = \phi_n(y)$, then $\phi_{n+1}(x) = \phi_{n+1}(y)$. Thus $\langle \phi_n \mid n \in \omega \rangle$ witnesses that $E$ is hypersmooth. \(\square\)

Proofs of hyperfinite can be notoriously difficult. An example of this is the following deep result of Gao and Jackson.

**Theorem 64** (Gao-Jackson). Any Borel action of an abelian group on a Polish space generates a hyperfinite orbit equivalence relation.

The following are notorious open problems.

**Problem 1.** If $G$ is a countable amenable group, must every Borel action of $G$ generate a hyperfinite orbit equivalence relation.

The answer is known even for the solvable group $\langle a, b \mid a^{-1}ba = b^2 \rangle$.

**Problem 2.** Is a countable increasing union of hyperfinite equivalence relations hyperfinite?
Lecture 30

There are few tools which can be used to directly show that a countable Borel equivalence relation is complex. Often one uses techniques which employ measure or category in order to prove that complexity arises. One example of this was our proof that $E_0$ is not smooth: we showed that if $f: 2^\omega \to X$ is any Borel function into a Polish space which is $E_0$-invariant, then $f$ is constant on both a conull set and a comeager set. It turns out that measure is a more powerful tool in drawing out the complexity of a countable Borel equivalence relation.

The next result shows that category is not useful in proving that equivalence relations are not hyperfinite.

**Theorem 65** (Hjorth-Kechris, Sullivan-Weiss-Wright, Woodin). Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$. Then there is a comeager set $G$ such that $E \upharpoonright G$ is hyperfinite.

**Proof.** Fix a Borel linear order $\leq$ on $X$. Applying the Feldman-Moore theorem, let $\{g_k \mid k \in \omega\}$ be Borel involutions which generate $E$. Define $f_k(x) := \min(x, g_k(x))$. If $s \in \omega^{<\omega}$ define $f_s(x) = f_{s(n-1)}(\cdots f_{s(0)}(x))$ and let $E_s$ be the equivalence relation on $X$ defined by $(x,y) \in E_s$ if and only if $f_s(x) = f_x(y)$. Notice that $E_s$ is a Borel equivalence relation contained in $E$ whose equivalence classes have cardinality at most $2^{lh(s)}$. Furthermore, if $s$ is an initial part of $t$, then $E_s \subseteq E_t$. If $a \in \omega^\omega$, define $E_a = \bigcup_{n=0}^{\infty} E_{a\upharpoonright n}$.

Observe that it is sufficient to show that for some $a \in \omega^\omega$, there is a comeager set of $x$ such that $[x]_{E_a} = [x]_E$. By the Kuratowski-Ulam Theorem, it is sufficient to show that for every $x \in X$, the set of $a \in A$ such that $[x]_{E_a} = [x]_E$ is comeager. Furthermore, since each $E$-equivalence class is countable, it is sufficient to show that if $(x,y) \in E$, then the set of $a$ such that $(x,y) \in E_a$ is dense and open. Clearly it is open since if $(x,y) \in E_a$, then for some $n$, $(x,y) \in E_{a\upharpoonright n}$ and thus $(x,y) \in E_b$ for all $b \in [a \upharpoonright n]$. To see that the set is dense, let $s \in \omega^{<\omega}$ be arbitrary and fix a $k$ such that $g_k(f_s(x)) = f_s(y)$. It follows that $f_k(f_s(x)) = f_k(f_s(y))$ and hence that $(x,y) \in E_{s\upharpoonright \langle k \rangle}$. □

A countable Borel equivalence relation $E$ on a Borel measure space $(X, \mu)$ is $\mu$-hyperfinite if $E$ is hyperfinite when restructed to a $\mu$-conull set. The notion of amenability plays an important role in the study of $\mu$-hyperfiniteness. A countable discrete group $G$ is said to be *amenable* if it supports a finitely additive, translation invariant probability measure. Since the space of finitely additive probability measures is compact as a subspace of $[0,1]^{\mathcal{P}(G)}$, it can be shown that $G$ is amenable if for every $\varepsilon > 0$ and every finite $F \subseteq G$, there is a finite set $A \subseteq G$ such
that
\[ \frac{|A \triangle F \cdot A|}{|A|} < \varepsilon. \]

In fact the existence of \((\varepsilon, F)\)-Følner sets for each \(\varepsilon\) and \(F\) is equivalent to amenability. It should be clear that every finite group is amenable and it is not hard to show that every abelian group is amenable. Furthermore, it is not difficult to show that the class of amenable groups is closed under taking extensions, directed unions, quotients, and subgroups. In particular, every solvable group is amenable.

Any nonabelian free group, on the other hand, is not amenable. For instance for the group freely generated by \(a\) and \(b\), there is no finitely additive probability measure which invariantly measures
\[ A := \{ w \in \mathbb{F}_2 \mid w \text{ begins with } a \}. \]

For instance if \(\mu\) is an invariant finitely additive measure on \(\mathbb{F}_2\) with \(\mu(\mathbb{F}_2) < \infty\), then since \(\{ b^n A \mid n \in \mathbb{Z} \}\) is a pairwise disjoint family, \(\mu(A) = 0\). On the other hand, \(\mathbb{F}_2 \setminus A \cap a \cdot (\mathbb{F}_2 \setminus A) = \emptyset\), which implies \(\mu(\mathbb{F}_2 \setminus A) = 0\) and hence \(\mu(\mathbb{F}_2) = 0\).

We will eventually prove the following results.

**Theorem 66.** Suppose that \(G\) is a countable discrete group acting ergodically by measure preserving transformations on a Borel probability measure space \((X, \mu)\). If \(E^G_X\) is \(\mu\)-hyperfinite, then \(G\) is amenable.

**Theorem 67.** Suppose that \(G\) is a nondiscrete nonabelian free group on two generators. If \(G\) acts continuously on a locally finite measure space \((X, \mu)\), then the orbit equivalence relation is not \(\mu\)-hyperfinite.

**Problem 3.** Suppose that \(E\) is a countable Borel equivalence relation on a \(\sigma\)-finite measure space \((X, \mu)\). If \(E\) is \(\mu\)-hyperfinite, is \(E\) hyperfinite?

**Problem 4.** Is a countable increasing union of hyperfinite equivalence relations hyperfinite?

**Problem 5.** Is the orbit equivalence relation of a Borel action of a countable amenable group hyperfinite?

The last two questions are known to have a positive answer if hyperfinite is replaced by \(\mu\)-hyperfinite for any Borel probability measure \(\mu\). Also, a positive answer to the second problem can be used to reduce the last problem to the finitely generated case. Note, however, that there are finitely generated solvable groups such as the Baumslag-Solitar group \(\langle a, b \mid a^{-1}ba = b^2 \rangle\) for which the last problem remains open.
The goal of the next two lectures is to prove the following result.

**Theorem 68** (Connes-Feldman-Weiss). The orbit equivalence relation of a countable discrete amenable group acting on a Borel probability space $(X, \mu)$ is $\mu$-hyperfinite.

Note that there is no requirement that the group action interacts with the measure $\mu$ (in particular, we are not assuming that the action preserves measure or even preserves being measure 0). This theorem complements the following theorem of Dye and the Slaman-Steel theorem to show that, up to a null set, any two orbit equivalence relations of amenable group actions are isomorphic.

**Theorem 69** (Dye). Suppose that $\mathbb{Z}$ acts ergodically on atomless Borel probability spaces $(X, \mu)$ and $(Y, \nu)$. Then there is a measure preserving isomorphism between $(X, E^\mathbb{Z}_X)$ and $(Y, E^\mathbb{Z}_Y)$.

The Connes-Feldman-Weiss theorem will be proven in two stages: first by reducing the general case to the finitely generated case and then proving it in the finitely generated case. The reduction is an immediate consequence of the following theorem.

**Theorem 70** (Dye, Krieger). Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$ equipped with a Borel probability measure $\mu$. If $E$ is a countable increasing union of $\mu$-hyperfinite equivalence relations, then $E$ is $\mu$-hyperfinite.

We will prove this by establishing the following characterization of $\mu$-hyperfiniteness.

**Proposition 13.** Suppose that $E$ is a countable Borel equivalence relation on a Polish space $X$ equipped with a Borel probability measure $\mu$. The following are equivalent:

1. $E$ is $\mu$-hyperfinite;
2. whenever $\varepsilon > 0$ and $R \subseteq E$ is a relation such that $R_x = \{y \in X \mid (x, y) \in R\}$ is finite for all $x \in X$, then there is a Borel subequivalence relation $F \subseteq E$ with finite classes such that

$$1 - \varepsilon \leq \mu(\{x \in X \mid R_x \subseteq [x]_F\}).$$

**Proof of Proposition 13.** The forward implication follows from the following obvious fact.

**Fact 2.** If $E$ is an increasing union of a sequence of equivalence relations $\langle E_n \mid n \in \omega \rangle$ and $R \subseteq E$ is a relation such that $R_x$ is finite for all
x, then for all \( \varepsilon > 0 \) there is an \( n \) such that \( \mu(\{ x \in X \mid R_x \not\subseteq [x]_{E_n}\}) < \varepsilon \).

To see the reverse implication, let \( E = \bigcup_{n=0}^{\infty} f_n \) where each \( f_n \) is a function. Construct an increasing sequence of finite equivalence relations \( F_n \) such that for all \( n \), the set \( X_n \) of \( x \in X \) such that for some \( i < n \), \( f_i(x) \not\in [x]_{F_n} \) has measure less than \( 1/n \). If \( x, y \in X \setminus \bigcap_{n=0}^{\infty} X_n \), then for some \( i \), \( f_i(x) = y \). Let \( n \) be such that \( i < n \) and \( x, y \not\in X_n \). Then \( y = f_i(x) \in [x]_{F_n} \).

**Proof of Theorem 70.** Let \( E, X \) and \( \mu \) be given as in the statement of the theorem. First observe that by discarding an \( E \)-invariant null set if necessary, we may assume that \( E = \bigcup_{n=0}^{\infty} E_n \) where each \( E_n \) is hyperfinite. Let \( \varepsilon > 0 \) and \( R \subseteq E \) be a relation such that for all \( x \), \( R_x \) is finite. By Fact 2, there is an \( n \) such that \( \mu(\{ x \in X \mid R_x \not\subseteq [x]_{E_n}\}) < \varepsilon/2 \).

By Proposition 13, there is a Borel equivalence relation \( F \subseteq E_n \) with finite classes such that \( \mu(\{ x \in X \mid R_x \cap E_n \not\subseteq [x]_F\}) < \varepsilon/2 \). It follows that \( F \) is as desired. \( \square \)

Before proceeding, we need to define some terminology. Fix for a moment a Polish space \( X \) and a countable Borel equivalence relation \( E \). A partial finite subequivalence relation or \( f.s.r. \) is a Borel set \( F \subseteq E \) which is an equivalence relation on some \( Y \subseteq Y \) such that each \( F \)-class is finite. A subset of \( X \) is \( E \)-related if it is contained in some \( E \)-class. If \( \Phi \) is a Borel family of finite \( E \)-related sets, then a \( f.s.r. \) \( F \subseteq E \) is \( \Phi \)-maximal if each \( F \)-class is in \( \Phi \) and every element of \( \Phi \) intersects an \( F \)-class.

**Proposition 14.** Suppose \( X \) is Polish and \( E \) is a countable Borel equivalence relation on \( X \). If \( \Phi \) is a Borel family of \( E \)-related finite subsets of \( X \), then there is a \( \Phi \)-maximal \( f.s.r. \) \( F \subseteq E \).

**Proof.** Fix a Borel linear ordering \( \leq \) of \( X \) and fix a countable basis \( \langle W_k \mid k \in \omega \rangle \) for \( X \). Using the Feldman-Moore theorem, fix Borel involutions \( \langle g_k \mid k \in \omega \rangle \) which generate \( E \). If \( a \) is a finite subset of \( X \), let \( \langle a_i \mid i < n \rangle \) be the \( \leq \)-increasing enumeration of \( a \).

**Claim 6.** There is a Borel function \( c \) which maps the finite \( E \)-related subsets of \( X \) into \( \omega \) such that if \( a \neq b \) and \( c(a) = c(b) \), then \( a \cap b = \emptyset \).

**Proof.** Clearly it suffices to define a function \( c \) which has a countable range instead. If \( a \) is \( E \)-related, define \( c(a) = (\langle k_i \mid i < n \rangle, \langle m_{i,j} \mid i < j < n \rangle) \) where \( n = |a| \) and:
\[ \langle k_i \mid i < n \rangle \text{ is the lexicographically least sequence such that } a_i \in U_{k_i} \text{ and } W_{k_i} \cap W_{k_j} = \emptyset \text{ if } i \neq j < n. \]

\[ \langle m_{i,j} \mid i < j < n \rangle \text{ is the lexicographically least sequence such that } a_j = g_{m_{i,j}}(a_i) \text{ if } i < j < n. \] We will also set \( m_{j,i} := m_{i,j} \text{ if } j > i. \)

Observe that if 
\[
c(a) = c(b) =: (\langle k_i \mid i < n \rangle, \langle m_{i,j} \mid i < j < n \rangle)
\]
and \( a_i = b_i' \) for some \( i, i' < n \), then the pairwise disjointness of \( \{W_{k_j} \mid j < n\} \) implies that \( i = i' \). Furthermore, if \( j \neq i \), then
\[
a_j = g_{m_{i,j}}(a_i) = g_{m_{i,j}}(b_i) = b_j
\]
and hence \( a = b. \)

Define Borel sets \( \langle X_k \mid k \in \omega \rangle \) by
\[
X_k = \bigcup \{ a \in \Phi \mid c(a) = k \text{ and } a \cap \bigcup_{i<k} X_i = \emptyset \}.
\]

Let \( F_k \) be the Borel equivalence relation on \( X_k \) whose equivalence classes are the sets \( a \in \Phi \) such that \( c(a) = k \). Finally, set \( F = \bigcup_{k=0}^{\infty} F_k \).

Clearly \( F \subseteq E \) is an f.s.r. and every \( F \)-class is in \( \Phi \). Furthermore, each \( F \)-class is an \( F_k \)-class for some \( k \). To see that \( F \) is \( \Phi \)-maximal, suppose that \( a \in \Phi \). If \( c(a) = k \), then either \( a \cap X_i \neq \emptyset \) for some \( i < k \) or else \( a \) is an \( F_k \)-class. If \( a \cap X_i \neq \emptyset \), then there is a \( b \in \Phi \) such that \( c(b) = i \) and \( b \cap \bigcup_{j<i} X_j = \emptyset \). But this means that \( b \) is an \( F_i \)-class and hence an \( F \)-class. \( \square \)
Lecture 32

A countable Borel equivalence relation $E$ on a Polish space $X$ is said to be amenable if there are Borel relations $R_n \subseteq E$ with finite vertical sections such that for each $(x, y) \in E$,

$$\lim_{n \to \infty} \frac{|(R_n)_x \Delta (R_n)_y|}{|(R_n)_x \cup (R_n)_y|} = 0.$$ 

Just as in the case of hyperfiniteness, we say that $E$ is $\mu$-amenable if $E \upharpoonright Z$ is amenable for some conull $Z \subseteq X$.

Observe that any hyperfinite Borel equivalence relation is amenable: if $E = \bigcup_{n=0}^{\infty} F_n$ witnesses that $E$ is hyperfinite, then whenever $(x, y) \in F_n$,

$$(F_n)_x \Delta (F_n)_y = [x]_{F_n} \Delta [y]_{F_n} = \emptyset.$$ 

Also, if $G$ is an countable discrete amenable group acting in a Borel way on a Polish space $X$, then $E^G_X$ is amenable. To see this fix a sequence $\langle A_n \mid n \in \omega \rangle$ of finite subsets of $G$ such that if $S \subseteq G$ is finite and $\varepsilon > 0$, then there is an $n$ such that $|S \cdot A_n \Delta A_n| < \varepsilon |A_n|$. It follows that $F_n := \{(x, y) \in E \mid y \in A_n \cdot x\}$ defines a witness to $E$ being amenable. Thus the Connes-Feldman-Weiss theorem can be rephrased as saying that the $\mu$-amenable equivalence relations are exactly the $\mu$-hyperfinite equivalence relations.

Before we proceed, we need to develop some language and machinery. Recall that if $\mu$ and $\nu$ are Borel probability measures on a Polish space $X$, then $\nu$ is absolutely continuous with respect to $\mu$ (denoted $\nu << \mu$) if every $\mu$-null set is $\nu$-null. We will need the following well known theorem from measure theory.

**Theorem 71.** If $\nu << \mu$ are Borel probability measures on a Polish space $X$, then there is a Borel function $\frac{d\nu}{d\mu} : X \to [0, \infty)$ such that $\nu(A) = \int_A \frac{d\nu}{d\mu} d\mu$. Moreover $\frac{d\nu}{d\mu}$ is unique up to $\nu$-a.e. equivalence.

If $E$ is a countable Borel equivalence relation on $X$ and $\mu$ is a Borel probability measure on $X$, then we say $\mu$ is $E$-quasi-invariant if whenever $f$ is a Borel bijection whose graph is contained in $E$, $\mu$ and $f_*\mu$ are mutually absolutely continuous (here $f_*(\mu)(A) = \mu(f(A))$). Given such an $E$, fix a sequence of Borel involutions $\langle g_k \mid k \in \omega \rangle$ which generates $E$. By modifying the involutions, we may additionally assume that $g_i(x) \neq g_j(x)$ unless $i = j$ or $g_i(x) = x = g_j(x)$. In the context of $\mu$-hyperfiniteness and $\mu$-amenability, the assumption that $\mu$ is $E$-quasi invariant is a minor one as the next proposition shows.
Proposition 15. Suppose that $E$ is a countable Borel equivalence relation. If $\mu$ is a Borel probability measure, then there is a Borel probability measure $\nu$ such that $\mu << \nu$ and $\nu$ is $E$-quasi invariant.

Proof. Define $\nu(A) = \sum_{n=0}^{\infty} 2^{-n-1} \mu(g_n A)$. For any $n$,

$$\nu(g_n A) \geq 2^{-n-1} \mu(g_n g_n A) = 2^{-n-1} \mu(A).$$

Now suppose that $\mu$ is $E$-quasi invariant. Define $D : E \to (0, \infty)$ by

$$D(x, y) := \frac{dg_k \mu}{d\mu}.$$ 

Thus if $f \subseteq E$ is a Borel automorphism, then $\mu(f(A)) = \int_A D(x, f(x))d\mu$. Observe that on a conull set, if $x, y, z$ are $E$-related, then $D(x, y)D(y, z)$. 

A Borel graph $G \subseteq E$ is said to be bounded (with respect to $D$) if there exists an $n$ such that for all $x$, if $(x, y) \in G$, then there is an $i < n$ with $y = f_i(x)$. Observe that this implies that $G$ has bounded degree and that for some $0 < M < \infty$, $1/M \leq D(x, y) \leq M$ whenever $(x, y) \in G$. The proof from the previous lecture proves the following characterization.

Proposition 16. Suppose that $E$ is a countable Borel equivalence relation on a Borel probability space $(X, \mu)$ and that $\mu$ is $E$-quasi invariant. $E$ is $\mu$-hyperfinite provided that whenever $G \subseteq E$ is a $\mu$-bounded Borel graph and $\varepsilon > 0$, there is a Borel equivalence relation $F \subseteq E$ with finite classes such that

$$\mu(\{x \in X \mid G_x \not\subseteq [x]_F\}) < \varepsilon.$$

We will also need the following propositions which we state without proof.

Proposition 17. If $E$ is an amenable equivalence relation on a Borel probability space $(X, \mu)$ and $Y \subseteq X$ is a Borel set with positive measure, then $E \upharpoonright Y$ is $\mu$-amenable.

Proposition 18. Suppose that $(X, \mu)$ is a Borel probability space and $E$ is a $\mu$-amenable countable Borel equivalence relation on $X$. If $\mu$ is $E$-quasi invariant and $G \subseteq E$ is a bounded Borel graph, then for each $\varepsilon > 0$ and a conull set of $x \in X$, there exists a finite $A \subseteq [x]_E$ such that $|\partial G(A)|_x < \varepsilon|A|_x$. 

We are now ready to give a proof of the Connes-Feldman-Weiss theorem. Fix a bounded Borel graph $G \subseteq E$ and let $\varepsilon > 0$ be given. Let $M$
be sufficiently large such that for any \( x \), \( \sum_{(x,y) \in G} D(x,y) \leq M \). Recursively construct sequences \( \langle X_n \mid n \in \omega \rangle \), \( \langle \Phi_n \mid n \in \omega \rangle \), and \( \langle F_n \mid n \in \omega \rangle \) such that:

- \( X_n = X \setminus \bigcup_{m<n} \text{dom}(F_m) \).
- \( \Phi_n \) is the collection of all finite \( E \)-related \( A \subseteq X_n \) such that for any/all \( x \in A \) \( \partial G_n(A) \cap x \leq \varepsilon M^3 |A| \), where \( G_n = G \restriction X_n \).
- \( F_n \) is a \( \Phi_n \)-maximal f.s.r.

Claim 7. The \( \mu \)-measure on \( \bigcap_{n=0}^{\infty} X_n \) is 0.

Proof. Set \( X_0 := \bigcap_{n=0}^{\infty} X_n \) and \( G_\omega = G \restriction X_\omega \). If \( \mu(X_\omega) > 0 \), then \( E \restriction X_\omega \) is \( \mu \)-amenable by Proposition 17. By Proposition 18, there is an \( E \)-related \( A \subseteq X_\omega \) such that \( \partial G_\omega(A) \cap x < \varepsilon M^3 |A| \). Since \( A \) is finite and \( G \) is bounded, there is an \( n \) such that \( \partial G_n(A) = \partial G_\omega(A) \). But now \( A \) witnesses that \( F_n \) is not \( \Phi_n \)-maximal, which is a contradiction. \( \square \)

Define \( F = \bigcup_{n=0}^{\infty} F_n \). We have that \( F \) is a Borel f.s.r. of \( E \) and that the domain of \( F \) is conull. We will need the following computations (taken from [Kechris-Miller]), which are left to the reader.

Claim 8. If \( B \subseteq X \) is Borel, then \( \mu(\bigcup_{x \in B} \{x\} \cup G_x) \leq M^3 \mu(B) \).

Claim 9. If \( T \) is a Borel transversal for \( F_n \), then for any \( A \subseteq \text{dom}(F_n) \), \( \mu(A) = \int_T |[x]_{F_n} \cap A| x d\mu(x) \).

Putting this all together, set \( B_n := \{x \in \text{dom}(F_n) \mid G_x \not\subseteq [x]_{F_n} \} \) and let \( T_n \) be a transversal for \( F_n \). Then by Claim 9

\[
\mu(B_n) = \int_{T_n} |[x]_{F_n} \cap B_n| x d\mu(x) \leq \frac{\varepsilon}{M^3} \int_{T_n} |[x]_{F_n}| x d\mu(x) = \frac{\varepsilon}{M^3} \mu(\text{dom}(F_n)).
\]

By Claim 8,

\[
\mu(\bigcup_{x \in B_n} \{x\} \cup G_x) \leq M^3 \mu(B_n) \leq \varepsilon \mu(\text{dom}(F_n)).
\]

Summing over \( n \),

\[
\mu(\{x \in X \mid G_x \not\subseteq [x]_{F}\}) \leq \sum_{n=0}^{\infty} \varepsilon \mu(\text{dom}(F_n)) = \varepsilon.
\]

This completes the proof.
Lecture 33

In this lecture we will see an example of a non amenable equivalence relation. Recall that if $A$ is a subring of $\mathbb{R}$, then $\text{PSL}_2(A)$ is the family of all homeomorphisms of the real projective line $P^1(\mathbb{R}) := \mathbb{R} \cup \{\infty\}$ which have the form

$$
t \mapsto \frac{at + b}{ct + d}
$$

where $a, b, c, d \in A$ and $ad - bc = 1$. The group $\text{PSL}_2(\mathbb{Z})$ has a finite index free group on 2 generators, namely

$$
\langle t \mapsto t + 2, t \mapsto \frac{t}{1 - 2t} \rangle.
$$

On the other hand, its action on $P^1(\mathbb{R})$ gives a hyperfinite equivalence relation. In fact the map $\phi : 2^\omega \to P^1(\mathbb{R})$ defined implicitly by

$$
\phi(\langle 0 \rangle \triangle a) = \frac{1}{1 + \frac{a}{\alpha}} \quad \phi(\langle 1 \rangle \triangle a) = 1 + \phi(a)
$$

$$
\phi(\langle 0 \rangle \triangle a) = -\phi(\tilde{a}) \quad \phi(1 \triangle a) = \phi(a)
$$

has the property that $a$ is tail equivalent to $b$ if and only if $\phi(a)$ and $\phi(b)$ are in the same $\text{PSL}_2(\mathbb{Z})$ orbit.

Any proper enlargement of the ring $A$ yields different behavior, however.

Theorem 72 (Ghys-Carrier). If $A$ is a dense unital subring of $\mathbb{R}$, then $\text{PSL}_2(A)$ contains a dense free group on two generators.

Theorem 73 (Ghys-Carrier). If $\Gamma$ is a non-discrete rank 2 free subgroup of $\text{PSL}_2(\mathbb{R})$, then the action of $\Gamma$ on $\mathbb{R}$ is not $\lambda$-amenable.

As a consequence of this result and the result of Connes-Feldman-Weiss, we obtain:

Corollary 16. The group $G$ of $C^1$ piecewise $\text{PSL}_2(\mathbb{Z}[1/2])$ homeomorphisms of $P^1(\mathbb{R})$ which fix $\infty$ is not amenable.

Proof of Corollary 16. With some effort, one can the orbit equivalence relations of $G$ and of $\text{PSL}_2(\mathbb{Z}[1/2])$ are the same when restricted to $\mathbb{R}$. By the results of Ghys-Carrier, this orbit equivalence relation is not $\lambda$-amenable. By Connes-Feldman-Weiss, $G$ can not be amenable. □

This result is remarkable because the group $G$ in the corollary does not, by arguments of Brin-Squier and Monod, contain a nonabelian free group. It is also a close relative of Thompson’s group $F$: if we change $\mathbb{Z}[1/2]$ to $\mathbb{Z}$ in the definition of $G$, then by an observation of Thurston, we have a copy of $F$. 
We will now turn to proving a generalization of Theorem 73. Before we start, we will introduce some terminology. A measured Polish space is a Polish space $X$ equipped with a Borel measure $\mu$. A measured Polish space is locally finite if every point has a neighborhood with finite measure.

We will need a few facts about measured Polish spaces. The proofs are left as exercises.

**Fact 3.** If $B$ is a measurable set in a measured Polish space, then
$$\mu(B) = \sup\{\mu(E) \mid E \subseteq B \text{ and } E \text{ is closed}\}.$$  

**Fact 4.** If $(X, \mu)$ is a locally finite measured Polish space and $E \subseteq X$ is a measurable set of positive measure, then for every $\varepsilon > 0$ there is an open $U$ such that $(1 - \varepsilon)\mu(U) < \mu(E \cap U) < \infty$.

If $(X, \mu)$ is a measured Polish space and $G$ is a topological group, we say that $G$ acts continuously on $(X, \mu)$ if the action of $G$ on $X$ is continuous in the usual sense (i.e. $(g, x) \mapsto g \cdot x$ is continuous) and if $g \mapsto \mu(g \cdot E)$ is continuous for every measurable $E$.

We are now ready to state the abstract form of the Ghys-Carrier Theorem. Recall that a group $G$ acts freely on a set $S$ if whenever $g \cdots s = s$, $g$ is the identity.

**Theorem 74.** Suppose that $G$ is a non-discrete free group of rank 2 and that $G$ acts freely and continuously on a locally finite measured Polish space $(X, \mu)$. The resulting orbit equivalence relation is not $\mu$-amenable.

We will take up the proof of this result in the next lecture. For now we will prove the following proposition which will play an important role in the main proof.

**Proposition 19.** Suppose that $G$ is a separable metric group acting continuously on a locally finite measured Polish space $(X, \mu)$. If $E \subseteq X$ has positive measure, then there is an open set $V \subseteq G$ containing the identity and an $\varepsilon > 0$ such that if $g \in V$, then $\mu(gE \cap E) \geq \varepsilon$.

**Proof.** Let $E$ be given and fix an open $U \subseteq X$ such that
$$0 < \frac{3}{4} \mu(U) < \mu(U \cap E) < \mu(U) < \infty$$
and set $\varepsilon = \mu(U)/4$. By continuity of the action, for each $x \in U$, there is a $\delta_x > 0$ and an open set $W_x \subseteq U$ such that if $d(g, \text{id}) < \delta_x$, then $gW_x \subseteq U$. Let $\delta > 0$ be such that
$$\mu(\{x \in U \mid \delta_x \geq \delta\}) > \frac{3}{4} \mu(U).$$
Define $W := \bigcup \{W_x \mid \delta_x \geq \delta\} \subseteq U$ and observe that $gW \subseteq U$ whenever $d(g,\text{id}) < \delta$. Also, since $x \in W_x$, it follows that $\mu(W) > \frac{3}{4}\mu(U)$ and hence that $\mu(E \cap W) > \frac{1}{4}\mu(U)$. Fix an open $V \subseteq G$ containing the identity such that whenever $g \in V$, $d(g,\text{id}) < \delta$ and $\mu(g(E \cap W)) > \frac{1}{2}\mu(U)$.

It suffices to show that if $g \in V$, then $\mu(U \cap gE \cap E) \geq \varepsilon$. On one hand $\mu(U \cap E) > \frac{3}{4}\mu(U)$ by choice of $U$. On the other hand, since $g(U \cap W) \subseteq (gE) \cap U$, it follows that

$$\mu(U \cap gE \cap E) = \mu((U \cap gE) \cap (U \cap E)) > \frac{1}{4}\mu(U) = \varepsilon.$$

□
Before proceeding further, let us see an example of a nondiscrete free subgroup of $\text{PSL}_2(\mathbb{Z}[1/2])$. Define

$$\alpha = \begin{pmatrix} 1/2 & -4 \\ 1/4 & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1/2 & -1/4 \\ 4 & 0 \end{pmatrix}.$$ 

These matrices correspond to the fractional linear transformations $t \mapsto 2 - 16/t$ and $t \mapsto 1/8 - 1/(16t)$. In order to see that $\langle \alpha, \beta \rangle$ is a nondiscrete group, first observe that the eigenvalues of $\alpha$ are $\lambda_0 = (1 + \sqrt{15}i)/4$ and $\lambda_1 = (1 - \sqrt{15}i)/4$. Both are unit magnitude complex numbers which are not roots of 1. Pick $\gamma \in \text{PSL}_2(\mathbb{C})$ such that $\gamma \alpha \gamma^{-1}$ is diagonal with entries $\lambda_0$ and $\lambda_1$. Then $(\gamma \alpha \gamma^{-1})^n = \gamma \alpha^n \gamma^{-1}$ is diagonal with entries $\lambda_0^n$ and $\lambda_1^n$. Since $\{\gamma \alpha^n \gamma^{-1} \mid n \in \omega\}$ accumulates to the identity, so does $\{\alpha^n \mid n \in \omega\}$. In particular, the group generated by $\alpha$ is not discrete.

It now suffices to show that the above matrices generate a free group. Define $X$ to be the set of all rational numbers in $\mathbb{P}_1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ which can be represented by a fraction with an odd denominator and let $Y$ denote the remaining rational numbers in $\mathbb{P}_1(\mathbb{R})$. By the Ping-Pong Lemma, it suffices to show that if $n \neq 0$ is an integer, then $\alpha^n Y \subseteq X$ and $\beta^n X \subseteq Y$. Let $X_0$ consist of those elements of $X$ which can be represented by a fraction of the form $(4p + 2)/q$ where $q$ is odd. Notice that $\alpha(X_0 \cup Y) \subseteq X_0$ and that $X_0$ and $Y$ are disjoint. It follows that $\alpha^n Y \subseteq X$ whenever $n$ is a nonzero integer. Similarly, $\beta^n X \subseteq Y$.

We now turn to a proof of the generalized Ghys-Carrier Theorem.

**Theorem 75.** Suppose that $(X, \mu)$ is a locally finite measured Polish space. If $G = \langle a, b \rangle$ is a free nondiscrete metrizable group which is acting freely and continuously on $(X, \mu)$, then the orbit equivalence relation is not $\mu$-amenable.

**Proof.** Suppose for contradiction that $E^G_X$ is $\mu$-amenable. By appealing to a reformulation of an equivalence relation being $\mu$-amenable, we can fix an assignment $x \mapsto \nu_x$ such that:

- for each $x$, $\nu_x$ is a finitely additive probability measure supported on the orbit of $x$;
- if $x$ and $y$ are in the same orbit, then $\nu_x = \nu_y$.

If $A \subseteq X \times X$ is $\mu$-measurable, then $x \mapsto \nu_x(A_x)$ is $\mu$-measurable.

For $u \in \{a, b\}$, define $\Gamma_u$ to be all those elements of $G$ which are representable by a reduced word beginning with $u$ and ending with $u^{-1}$.

**Claim 10.** Both $\Gamma_a$ and $\Gamma_b$ accumulate to the identity.
Positive measure, one obtains a contradiction by finding a \(g\) in \(\Gamma_a\). Thus there is an \(h \in \{e, a, ab, ab^{-1}\}\) and a \(\Gamma' \subseteq G\) which accumulates to the identity such that \(h\Gamma'h^{-1} \subseteq \Gamma_a\). Since conjugation is continuous, it follows that \(\Gamma_a\) also accumulates to the identity. Furthermore, \(b\Gamma_a b^{-1} \subset \Gamma_b\) and thus \(\Gamma_b\) accumulates to the identity as well. \(\square\)

Claim 11. \(Y := \{x \in X : \phi(x) \neq 1/2\}\) has positive measure with respect to \(\mu\).

Proof. Suppose not. Using our assumption that \(\Gamma\) acts continuously on \((X, \mu)\), find an open neighborhood \(V\) of the identity such that if \(g\) is in \(V\), then \(Y, bg^{-1} \cdot Y, b^2g^{-2} \cdot Y\) have total measure less than that of \(X\). Now if \(x\) is outside these sets and \(g \in V \cap \Gamma_a\), we have that \(\phi(x), \phi(bg^{-1} \cdot x), \phi(b^2g^{-2} \cdot x)\) are each 1/2, contradicting that there sum is at most 1. It must therefore be that \(Y\) has positive measure. \(\square\)

Let \(Y_a = \{y \in Y : \phi(y) > 1/2\}\) and \(Y_b = \{y \in Y : \phi(y) < 1/2\}\). By Claim 11, \(Y\) has positive measure, either \(Y_a\) or \(Y_b\) have positive measure. If \(Y_a\) has positive measure, then by Claim 10 Proposition 19 there is a \(g\) in \(\Gamma_b\) such that \((g \cdot Y_a) \cap Y_a\) has positive measure and in particular is nonempty. This contradicts our observation that if \(\phi(y) > 1/2\) and \(g\) is in \(\Gamma_b\), then \(\phi(g \cdot y) < 1/2\). Similarly, if \(Y_b\) has positive measure, one obtains a contradiction by finding a \(g\) in \(\Gamma_a\) such that \(g \cdot Y_b\) intersects \(Y_b\). It must be, therefore, that the orbit equivalence relation is nonamenable. \(\square\)
Lecture 35

We will now turn our attention to Gale-Stewart games and their determinacy. Fix a set $S$ and a subset $\Gamma$ of $S^\omega$. In what follows, $S$ will be given the discrete topology and $S^\omega$ will be given the product topology. The Gale-Stewart Game associated to $\Gamma$ is described as follows. Players I and II alternately play a sequence of elements $\langle s_k \mid k \in \omega \rangle$ of $S$: Player I starts by playing $s_0$, Player II responds with $s_1$, Player I responds with $s_2$, and so on. If the resulting sequence $\langle s_k \mid k \in \omega \rangle$ is in $\Gamma$, then we declare Player I the winner; otherwise Player II wins.

At each stage of the game, both players have complete information as to what has been played thus far, including the number of plays which have already been made. Also, it is important to note that there is no notion of a “draw” in which neither player wins or loses. The game of chess could be coded as a Gale-Stewart game, provided we declared black the winner whenever the game ended in a stalemate. Typically the set $S$ is countable, although it is often convenient to work with some countable set other than $\omega$. If $\Gamma \subseteq S^\omega$ we will say that $\Gamma$ is a game played on $S$.

Typically we are interested in restriction on $\Gamma$ which imply that it is determined: i.e. one of the two players has a strategy which allows them to always beat the other player, no matter how they play. Formally, a strategy $\sigma$ is a function from $S^{<\omega}$ into $S$. Player I is said to follow $\sigma$ if $s_{2k} = \sigma(\langle s_i \mid i < 2k \rangle)$ and similarly Player II is said to follow $\sigma$ if $s_{2k+1} = \sigma(\langle s_i \mid i < 2k + 1 \rangle)$. Player I has a winning strategy if there is a $\sigma$ such that in every play in which Player I follows $\sigma$, Player I wins (similarly one defines a winning strategy for Player II). We say that $\Gamma$ is determined if one of the two players has a winning strategy.

Notice that if $S = \{0, 1\}$ and $\Gamma \subseteq 2^\omega$ is such that neither $\Gamma$ nor $2^\omega \setminus \Gamma$ contain a perfect subset, then the game associated to $\Gamma$ is not determined. Such a $\Gamma$ can readily be constructed using the axiom of choice. On the other hand, sets which are simply topologically, tend to yield determined games.

Theorem 76 (Zermelo). If $\Gamma$ is a closed (or open) game on $S$, then $\Gamma$ is determined. In particular, the game of chess is determined.

Proof. Fix $\Gamma$. A sequence $\langle s_i \mid i < k \rangle$ is a losing position for Player I if there is a strategy $\sigma$ such that Player II wins every play of the game which starts with $\langle s_i \mid i < k \rangle$ in which they follow $\sigma$. Observe that if $\langle s_i \mid i < 2k \rangle$ is not a losing position for Player I, then there is an $s_{2k}$ such that $\langle s_i \mid i < 2k + 1 \rangle$ is not a losing position for Player I. Also, if $\langle s_i \mid i < 2k + 1 \rangle$ is not a losing position for Player I, then for any
$s_{2k+1} \in S, \langle s_i \mid i < 2k+2 \rangle$ is not a losing position for Player I. Define $\sigma$ to be any strategy which plays such a $s_{2k}$ in response to $\langle s_i \mid i < 2k \rangle$.

Now suppose that Player II does not have a winning strategy. We claim that $\sigma$ is a winning strategy for Player I. To see this, first observe that Player II not having a winning strategy is the same as $\varepsilon$ not being a losing position for Player I. Thus in any play $\langle s_k \mid k \in \omega \rangle$ of the game in which Player I follows $\sigma$, none of the finite plays of the game are losing positions for Player I. If $\langle s_k \mid k \in \omega \rangle$ were not in $\Gamma$, then there would be an $n$ such that any infinite sequence extending $\langle s_k \mid k < n \rangle$ is not in $\Gamma$. But then $\langle s_k \mid k < n \rangle$ is a losing position for Player I, contrary to our assumption.

In spite of the simplicity of this proof, it serves as the basis for all known proofs of determinacy of more complicated games. Typically given a payoff set $\Gamma \subseteq \omega^\omega$, one tries to find a set $S$ and a closed $\Gamma^* \subseteq S^\omega$ such that winning strategies in $\Gamma^*$ can be translated into winning strategies for $\Gamma$. We will say that $\Gamma^*$ is equivalent to $\Gamma$. Typically, however, the set $S$ is much larger than $\omega$, with its size correlated to the complexity of $\Gamma$.

**Theorem 77** (Martin). If $\Gamma$ is a Borel game on $S$, then there is a cardinal $\theta < \beth_\omega(S)$ such that $\Gamma$ is equivalent to a closed game on $\theta$. In particular, Borel games are determined.

In order to prove the determinacy of analytic games — which seem just slightly more complicated than Borel games — one must strengthen the axioms of ZFC. A cardinal $\theta$ is the $\omega_1^{st}$ Erdős cardinal if it is the least cardinal such that whenever $f : [\theta]^{<\omega} \to \omega$, there is an uncountable $H \subseteq \theta$ such that $f \upharpoonright [H]^n$ is constant for each $n$. This cardinal is necessarily inaccessible (in fact necessarily weakly compact) and thus $(V_{\theta+}, \in)$ is a model of ZFC. In particular, ZFC does not prove the existence of the $\omega_1^{st}$ Erdős cardinal. On the other hand, if $I$ is a set equipped with a countably complete nonprinciple ultrafilter, then the $\omega_1^{st}$ Erdős cardinal exists and is less than $|I|$.

**Theorem 78** (Martin). If $\Gamma$ is analytic game on $\omega$ and $\theta$ is the $\omega_1^{st}$ Erdős cardinal, then $\Gamma$ is equivalent to a closed game played on $\theta$. In particular, if the $\omega_1^{st}$ Erdős cardinal exists, then every analytic game is determined.

Remarkably this result is more or less sharp. A slightly weaker assertion — that $x^\sharp$ exists for every $x \subseteq \omega$ — is sufficient to carry out Martin’s argument (and similarly is not provable in ZFC) and is in fact equivalent to the determinacy of analytic games by a result of
Harrington. With a stronger large cardinal assumption, one can prove more. We leave the large cardinal notions undefined but note that every Woodin cardinal is a limit of measurable cardinals.

**Theorem 79** (Martin-Steel). *If $\Gamma$ is a $\Pi^1_{n+1}$ game on $\omega$ and $\theta$ is a measurable cardinal which is greater than $n$ Woodin cardinals, then $\Gamma$ is equivalent to a closed game played on $\theta$. In particular, if there are infinitely many Woodin cardinals, then all projective games are determined.*

This result is also sharp: if all projective games are determined, then for each $n$, there is an inner model of set theory in which there is a measurable cardinal above $n$ Woodin cardinals.