Finite Metric Spaces of Euclidean Type[1]

Peter J. Kahn
Department of Mathematics
Cornell University
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Abstract: Finite subsets of $\mathbb{R}^n$ may be endowed with the Euclidean metric. Do all finite metric spaces arise in this way for some n? If two such finite metric spaces are abstractly isometric, must the isometry be the restriction of an isometry of the ambient Euclidean space? This note shows that the answer to the first question is no and the answer to the second is yes. These answers are reversed if the sup metric is used in place of the Euclidean metric.

1. Introduction

Metric spaces typically arise in connection with the foundations of calculus or analysis, providing the necessary general structure for discussing the concepts of convergence, continuity, compactness and the like. These spaces usually have the cardinality of the continuum. But metric spaces of finite cardinality also appear naturally in many mathematical contexts (e.g., graph theory, string metrics, coding theory) and have applications in the sciences (e.g., DNA analysis, network theory, phylogenetics).

Any finite subset X of a metric space (Y, e) inherits a metric $d = e|X \times X$, allowing us to produce many examples of finite metric spaces (X, d) when (Y, e) is known. Often (Y, e) is taken to be $(\mathbb{R}^n, d_2)$, where $\mathbb{R}^n$ is the standard Euclidean space of dimension n and d_2 is the usual Euclidean metric. In such a case we say that the induced finite metric space is of Euclidean type, and we also use this designation for any metric space isometric to it. In that case, we have an isometric embedding, say, $f : (X, d) \to (\mathbb{R}^n, d_2)$, which we call a Euclidean representation of (X, d). If $g : (X, d) \to (\mathbb{R}^n, d_2)$ is another Euclidean representation, we say that it is equivalent to f whenever there is an isometry $F : (\mathbb{R}^n, d_2) \to (\mathbb{R}^n, d_2)$ such that $g = F \circ f$.

The main result of this paper is the following theorem, which follows from more detailed results in Sections 3 and 4 (Theorem 3, Corollary 4, Theorem 5, Corollary 6):

Main Theorem. *Let (X, d) be a finite metric space. Then*
(a) Any two Euclidean representations of (X, d) in $(\mathbb{R}^n, d_2)$ are equivalent.
(b) There are finite metric spaces that admit no Euclidean representations in $(\mathbb{R}^n, d_2)$, for any n.

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This theorem implies that for finite metric spaces, equivalence classes of Euclidean representations form a proper subset of the set of all isometry classes. In a sequel to this paper, we shall study these two sets and their relationship.

Before proceeding to proofs of results (a) and (b) of the Main Theorem, we describe a slightly more general context with some features that contrast with those described in the Main Theorem.

2. The Kuratowski Embedding

In 1909 Frechet showed that every separable metric space embeds isometrically in the Banach space ℓ^∞ of bounded sequences of real numbers equipped with the sup norm [F], [H]. In 1935 Kuratowski provided an isometric embedding of a general metric space into a Banach space [K], [H] via a short, ingenious proof that we present in our more elementary and concrete setting of finite metric spaces.

We first remind the reader that $\mathbb{R}^n$ admits many well-known Banach norms other than the standard Euclidean one. In particular, for each real number $p \geq 1$, we can endow $\mathbb{R}^n$ with the L^p norm, letting d_p denote the corresponding metric. The L^p norm is given by $||x||_p = (|x_1|^p + |x_2|^p + ... + |x_n|^p)^{1/p}$. Clearly, these are extensions of the case $p = 2$, the standard Euclidean norm. It is well-known that, as $p \to \infty$, these norms converge to the so-called "sup" or L^∞ norm, $||\ \ ||_\infty$, which is given by $||x||_\infty = sup_{i=1}^n |x_i|$. We denote the corresponding sup metric by d_∞.

Now let (X, d) be a metric space of cardinality $n < \infty$, and write $X = \{x_1, x_2, ..., x_n\}$. We have n^2 non-negative real numbers a_{ij} given by $a_{ij} = d(x_i, x_j)$. These comprise an $n \times n$ symmetric matrix $A = [a_{ij}]$, the ith row of which we denote by A_i. Define a map $K : X \to \mathbb{R}^n$ by $K(x_i) = A_i \in \mathbb{R}^n$.

Theorem 1 (Kuratowski [K], [H]). *$K : (X, d) \to (\mathbb{R}^n, d_\infty)$ is an isometric embedding.*

Proof. The triangle inequalities $a_{ij} + a_{jk} \geq a_{ik}$, valid for all i, j, k in $\{1, 2, ..., n\}$, imply that $||A_i - A_j||_\infty \leq a_{ij}$, for all i, j. Moreover, since $|a_{ij} - a_{jj}| = a_{ij}$,for all i, j, the inequality is actually an equality. Therefore,

$$d_\infty(K(x_i), K(x_j)) = ||K(x_i) - K(x_j)||_\infty = d(x_i, x_j),$$

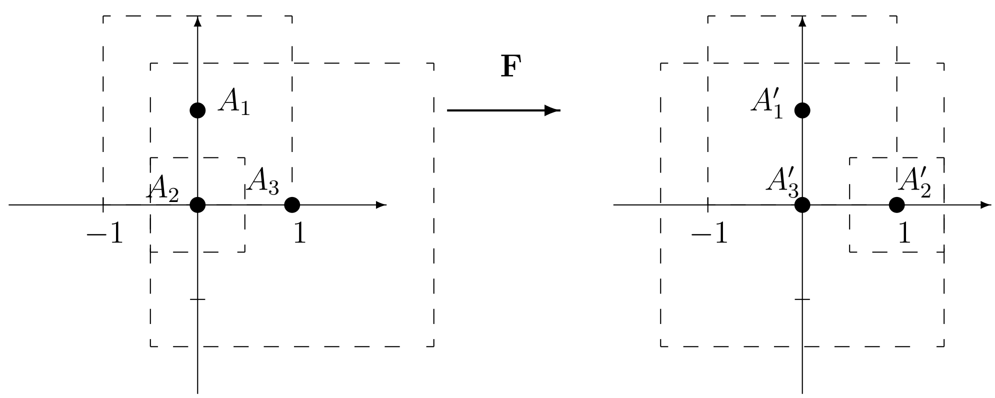
as desired. □

Theorem 1 and Main Theorem (b) exhibit the contrast between the sup metric and the Euclidean metric d_2 with respect to the existence of isometric embeddings of finite metric spaces in $\mathbb{R}^n$.

A similar contrast occurs in the case of Main Theorem (a) with respect to what we could call the uniqueness of such embeddings. The following theorem demonstrates this with respect to a simple example.

To simplify the notation, we omit the representations and directly take our finite metric spaces to be subsets of Euclidean space with the induced sup metric. In our example, there is only one subset, contained in $\mathbb{R}^2$. We assume $\mathbb{R}^2$ is included in the standard way in $\mathbb{R}^n$, for any integer $n \geq 2$. The subset X consists of the points $A_1 = (0,1), A_2 = (0,0), A_3 = (1,0)$. The induced sup metric assigns the distance 1 to each pair of these points. Thus, the permutation of $X = \{A_1, A_2, A_3\}$ that fixes A_1 and exchanges A_2 and A_3 is an isometry. We call this isometry F. In the Euclidean metric, of course, the triangle is not equilateral and F would not be an isometry. In Figure 1 below, we represent $F(A_i)$ by A'_i. The dashed boxes in the diagram represent spheres of various radii (in the sup metric). We amplify on this in the proof of the following theorem.

FIGURE 1



Theorem 2. *The self-isometry F of X does not extend to a self-isometry of $(\mathbb{R}^n, d_\infty)$.*

Proof. The proof is by contradiction. We assume that the isometry F extends to a self-isometry of $(\mathbb{R}^n, d_\infty)$, again called F. For any $P \in \mathbb{R}^n$ and non-negative real number r, define the "sphere" $S(P,r) = \{z \in \mathbb{R}^n | \; ||P - z||_\infty = r\}$. Clearly $F(S(P,r)) = S(F(P), r)$. In the case $n = 2$, Figure 1 displays the spheres $S(A_1, 1), S(A_2, 1/2)$, and $S(A_3, 3/2)$ on the left (as dashed boxes) and their respective images under F, $S(A'_1, 1), S(A'_2, 1/2)$, and $S(A'_3, 3/2)$ on the right.

Let Z denote the point $(-1/2, 0, ..., 0)$ in $\mathbb{R}^n$. A direct computation shows that

$$Z \in S(A_1, 1) \cap S(A_2, 1/2) \cap S(A_3, 3/2),$$

from which it follows that

$$F(Z) \in S(A'_1, 1) \cap S(A'_2, 1/2) \cap S(A'_3, 3/2).$$

However, we claim that

$$S(A'_1, 1) \cap S(A'_2, 1/2) \cap S(A'_3, 3/2) = \emptyset,$$

a contradiction, from which the result follows.

To verify the claim, note that if $z = (z_1, z_2, z_3, ...z_n) \in S(A'_1, 1) \cap S(A'_2, 1/2)$, then $1/2 \leq z_1 \leq 1, \; 0 \leq z_2 \leq 1/2$, and $|z_i| \leq 1/2$, for $i = 3, ..., n$. In particular, none of these coordinates z_i can be equal to $\pm 3/2$, and so such a z cannot belong to $S(A'_3, 3/2)$. This completes the proof. $\square$

Remark. Around 1926, Urysohn constructed a complete metric space in which all separable metric spaces can be isometrically embedded [U],[H]. The cited paper was published posthumously. The Urysohn construction yields a so-called universal such space U that has the property that any two isometric embeddings of finite metric spaces into U are equivalent in the sense we have been using. Indeed this space U is unique up to a strong notion of equivalence. U is quite abstract and its construction is non-trivial. Banach showed that a more familiar space, namely the Banach space of continuous, real-valued functions on the unit interval, $\mathcal{C}[0,1]$, equipped with the sup norm, can also be used to embed all separable metric spaces [B],[H]. However, it is not Urysohn universal [H]. We have been mainly interested in this paper in finite metric spaces isometrically embeded or embeddable (or not) in the the familiar Euclidean space $(\mathbb{R}^n, d_2)$ and have presented the somewhat more exotic metrics by way of providing some contrast.

Question: In light of the fact that $\lim_{p\to\infty} d_p = d_\infty$, do Theorems 1 and 2 have analogs when d_∞ is replaced by d_p, with p sufficiently large? We conjecture that the answer is yes for Theorem 2, but the case of Theorem 1 seems harder.

3. Isometric maps of finite metric spaces of Euclidean type

To fix terminology, we say that a map of metric spaces is isometric if it preserves distances. It is then injective, of course. When it is surjective as well, we call it an isometry. The main result of this section is the following:

Theorem 3. *Let $g : (X, d) \to (\mathbb{R}^n, d_2)$ and $h : (Y, d') \to (\mathbb{R}^n, d_2)$ be Euclidean representations of finite metric spaces, and let $f : X \to Y$ be an isometric map. Then, there exists an isometry $F : (\mathbb{R}^n, d_2) \to (\mathbb{R}^n, d_2)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{R}^n & \xrightarrow[F]{} & \mathbb{R}^n. \end{array}$$

Proof. The proof will proceed in several steps.

Step 1: Suppose the theorem holds when f is an isometry. Then it holds in general. For in general, we factor f into the composition

$$X \xrightarrow{f'} Y' \xrightarrow{i} Y,$$

where $Y' = f(X)$, and i is the inclusion map. Then f' is an isometry, and the theorem, applied with f' and Y' replacing f and Y respectively, yields an isometry F which works for f and Y as well.

From now on we assume that f is an isometry. In particular, we write $X = \{x_1, ...x_m\}$ and $Y = \{y_1, ..., y_m\}$, with $y_i = f(x_i), i = 1, 2, ..., m$.

Step 2: Suppose that the theorem holds when $g(x_1) = 0 = h(y_1)$. Then it holds in general.

This step is based on the fact that translation in $\mathbb{R}^n$ is an isometry. This allows us to modify the given maps by translations. In particular, define g' and h' by the formulas $g'(x_i) = g(x_i) - g(x_1)$ and $h'(y_i) = h(y_i) - h(y_1)$. Then $d(x_i, x_j) = ||g(x_i) - g(x_j)|| = ||g'(x_i) - g'(x_j)||$, for all $i, j = 1, 2, ...m$, so that g' is an isometric map with $g'(x_1) = 0$. Analogously for h'. So, by our assumption, we can apply the theorem with g' and h' replacing g and h to obtain an isometry F' satisfying $F'g' = h'f$. In particular,

$$F'(0) = F'(g'(x_1)) = h'(f(x_1)) = h'(y_1) = 0.$$

Now, define translations $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^n$ by $T_1(v) = v - g(x_1)$ and $T_2(v) = v + h(y_1)$, and let F be the isometry $T_2F'T_1$. Then, compute

$$F(g(x_i)) = T_2F'(g(x_i) - g(x_1)) = T_2F'(g'(x_i)) = T_2(h'(f(x_i)) = T_2(hf(x_i)) - h(y_1)) = h(f(x_i)),$$

as required by the theorem.

So, from now on we assume $g(x_1) = 0 = h(y_1)$

Step 3: For all $i, j = 1, 2, ..., m$, the following equalities of inner products hold:

$$< g(x_i), g(x_j) >=< h(y_i), h(y_j) > .$$

This follows from the facts that both g and h are Euclidean representations of isometric metric spaces and $g(x_1) = 0 = h(y_1)$. For then, we get

$$||g(x_i) - g(x_j)||^2 = ||h(y_i) - h(y_j||^2,$$

which we apply to the equalities

$$-2 < g(x_i), g(x_j) >= ||g(x_i) - g(x_j)||^2 - ||g(x_i) - g(x_1)||^2 - ||g(x_j) - g(x_1)||^2$$

and

$$-2 < h(y_i), h(y_j >= ||h(y_i) - h(y_j)||^2 - ||h(y_i) - h(y_1)||^2 - ||h(y_j) - h(y_1)||^2.$$

Step 4: We regard the vectors $g(x_j)$ and $h(y_j)$ in $\mathbb{R}^n$ as column vectors, $j = 1, 2, ..., m$, and we let A be the $n \times (m-1)$matrix consisting of $g(x_2), ..., g(x_m)$. Similarly, let B be the $n \times (m-1)$ matrix consisting of $h(y_2), ..., h(y_m)$.

The equalities of inner products obtained in Step 3 can be expressed as the equality of Gramian matrices

$$A^*A = B^*B.$$

According to Theorem 7.3.11, p. 452 of [HJ], there is an $n \times n$ real, orthogonal matrix V such that $B = VA$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding linear transformation. F is an isometry, and, by construction, it satisfies $F(0) = 0$ (i.e., $F(g(x_1)) = h(y_1) = h(f(x_1))$), and $F(g(x_j)) = h(y_j) = h(f(x_j)), j = 2, 3, ..., m$. That is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\big\downarrow & & \big\downarrow h \\ \mathbb{R}^n & \xrightarrow[F]{} & \mathbb{R}^n. \end{array}$$

commutes. This completes the proof. □

Recall that we say Euclidean representations $g, h : (X, d) \to \mathbb{R}^n$ are equivalent if there is an isometry $F : \mathbb{R}^n \to \mathbb{R}^n$ that makes the diagram

$$\begin{array}{ccc} X & = & X \\ g\big\downarrow & & \big\downarrow h \\ \mathbb{R}^n & \xrightarrow[F]{} & \mathbb{R}^n. \end{array}$$

commute. Then specializing Theorem 3, we get Main Theorem (a). That is,

Corollary 4. *Let (X, d) be a finite metric space of Euclidean type. Then all of its Euclidean representations in $\mathbb{R}^n$ are equivalent.* □

4. Some finite metric spaces that do not have Euclidean type

The object of this section is to present a class of simple examples of finite metric spaces that do not have Euclidean type. In this section n will always denote the cardinality of the metric spaces we are considering. It is obvious that every metric space of cardinality n has Euclidean type when $n \leq 3$. Our examples will all have cardinality 4. Of course these may be imbedded in metric spaces of any higher cardinality, and perforce these cannot have Euclidean type either.

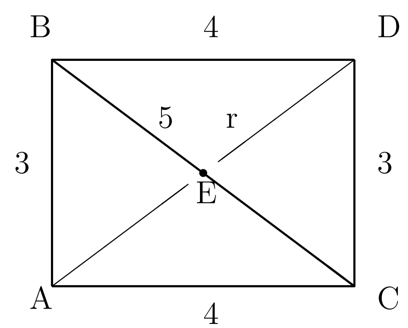
If we eliminate the triangle-inequality requirement from the definition of a metric, then we obtain what we shall call a pre-metric; a set equipped with a pre-metric will be called a pre-metric space. To construct our examples, we begin with a pre-metric space $(X, d(r))$, where X is the four-element set $\{A, B, C, D\}$, r is a positive real parameter, and $d(r)$ is the function $X \times X \to [0, \infty)$ given by the table

d(r)	A	B	C	D
A	0	3	4	r
B	3	0	5	4
C	4	5	0	3
D	r	4	3	0

Figure 1 below will aid in visualizing the example. Only the labeled vertices A, B, C, D belong to X. The connecting lines and the point E appear in he diagram to aid the visualization.

Theorem 5. (a) *$(X, d(r))$ is a metric space $\iff 1 \leq r \leq 7$.*
(b) *$(X, d(r))$ is a metric space of Euclidean type $\iff 7/5 \leq r \leq 5$.*

Figure 2



Corollary 6. $(X, d(r))$ *is a metric space that does not have Euclidean type whenever* $1 \leq r < 7/5$ *or* $5 < r \leq 7$. □

Proof of Theorem . (a): Check the triangle inequalities for the triples $\{A, B, D\}$ and $\{A, C, D\}$.

(b) $\Rightarrow$ Suppose that $(X, d(r))$ has Euclidean type. We may identify X with an isometric image in $\mathbb{R}^n$, for some n, and use the table for $d(r)$ to obtain the Euclidean distances determined by the points A, B, C, D in $\mathbb{R}^n$. Let E denote the mid-point of BC, the (common) hypotenuse of the $3-4-5$ right triangles ABC and DBC. By elementary geometry, each segment AE and DE has length $5/2$. Therefore, the piecewise-linear path $AE \cup DE$ joining A to D in $\mathbb{R}^n$ has length 5. It follows that the (straight-line) distance between A and D, i.e., r, is ≤ 5.

For the inequality $7/5 \leq r$, consider the altitude h_A (resp., h_D) in the triangle ABC (resp, DBC) from A (resp., D) to the point E_A (resp., E_D) on the hypotenuse BC (see Figure 2).

The lengths of BE_A and CE_D are each $9/5$, so that E_AE_D has length $7/5$. Note that all this is independent of where A, B, C, D are situated in $\mathbb{R}^n$, assuming the edge lengths as given. Now, let H_A (resp., H_D) be the hyperplane in $\mathbb{R}^n$ orthogonal to BC and passing through E_A (resp., E_D). These are parallel hyperplanes joined by the orthogonal segment E_AE_D. Hence, the distance between H_A and H_D is everywhere $7/5$. But $A \in H_A$ and $D \in H_D$. So $r = ||A - D|| \geq 7/5$.

(b) $\Leftarrow$ Choose any r satisfying $7/5 \leq r \leq 5$. We show that there is a Euclidean representation $(X, d(r)) \to \mathbb{R}^n$. We start with the case $r = 5$ and construct the Euclidean representation $(X, d(5)) \to \mathbb{R}^2 \hookrightarrow \mathbb{R}^n$ illustrated in Figure 3 (a). Now let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous $180°$ rotation about the axis containing the hypotenuse BC. R maps D to a point D' in $\mathbb{R}^2$ at a distance $7/5$ from A. The configuration $ABCD'$, illustrated in Figure 3 (b), gives a Euclidean representation of $(X, d(7/5))$. It remains to observe that, by the Intermediate Value Theorem, $(X, d(r))$ has a Euclidean representation obtained by using a rotation intermediate between $0°$ and $180°$.

This completes the proof of Theorem 5. □

FIGURE 3

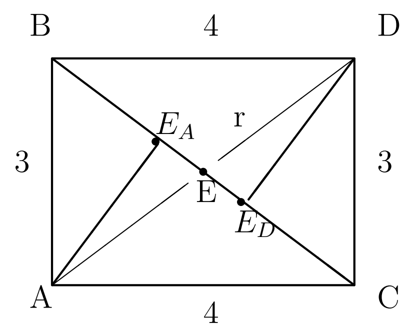
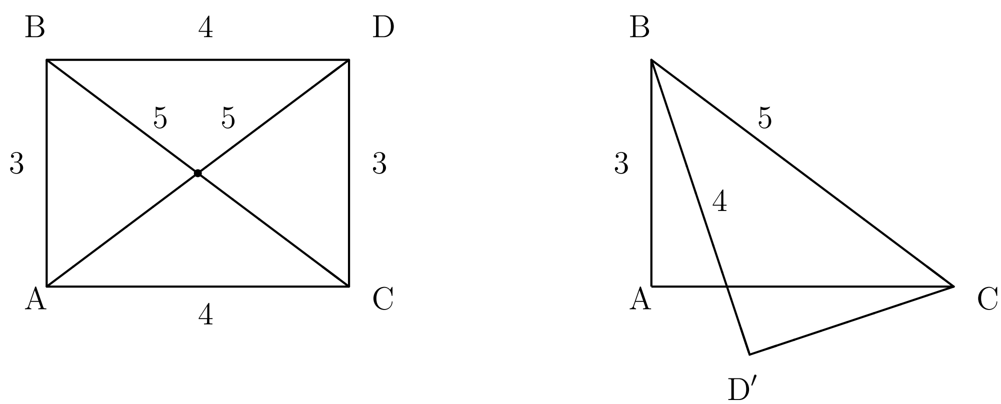


FIGURE 4



(a) (b)

Remarks: (1) Any triangle could have been used in the foregoing in place of the $3-4-5$ right triangle ABC (and its copy BCD). The results and proofs would have been essentially the same with certain numbers appearing in the statements and proofs being more awkward to express explicitly and the diagrams looking slightly different. Indeed, given any triangle with sides of lengths u, v, w, one can reproduce the foregoing arguments, resulting in a version of Theorem 5 in which the bounds in the statements (a) and (b), respectively, are replaced by explicit formulas in u, v, w. We invite the interested reader to do so. In any case, this shows that there are many examples of finite metric spaces that do not have Euclidean type.

(2) Theorem 5 (b) can be proved via analytic geometry, but we feel that our more purely geometric method is more attractive and makes the result clearer.

References

[B] Banach, S.; *Théorie des operateurs linéaires*; Chelsea Publishing Co., New York (1955).
[F] Frechet, S.; *Les dimensions d'un ensemble abstrait*, Math. Ann. 68 (1909-10), 145-168.
[H] Heinonen, Juha; *Geometric embeddings of metric spaces*; Lectures in the Finnish Graduate School of Mathematics, University of Jyväskylä, (January, 2003).
[HJ] Horn, R.A., Johnson, C.R.; *Matrix Analysis*; Cambridge University Press (2013)
[K] Kuratowski, C.; *Quelques problèmes concernant les espaces métriques non-séparables*; Fund.Math. 25 (1935), 534-545.
[U] Urysohn, P.; *Sur un espace métrique universel*; Bull. Sc. Math. 2^e serie 51 (1927), 43-64.