MATH 3340: Abstract Algebra

Problem Set 6

Due Tuesday, March 10, 2020

You are allowed to work in groups, but the solutions you hand in should be written by you only.

Instructions on how to hand in pset:
https://gradescope-static-assets.s3-us-west-2.amazonaws.com/help/submitting_hw_guide.pdf

Cornell Libraries have free scanners, if needed:

https://olinuris.library.cornell.edu/print-scan-wifi

Prelim 1 is on March 11 in class covering chapters 2 and 3; for Section 3.5 it is up to and including Theorem 3.5.2, for Section 3.6 it is up to and including Theorem 3.6.2, for Section 3.7 it is up to and including Proposition 3.7.6 and for Section 3.8 up to and including Theorem 3.8.9.

*Please submit to gradescope solutions to **all** the problems below. Note, however, that only a subset of them will be graded, and will not necessarily be checked for details. This problem set serves mostly as a partial preparation for your prelim.*

David is holding special office hours this week on **Tuesday, March 10, 12-2 in Malott 218**.

1) Let G be a group and H a subgroup.

    a) Define the relation of G by $g \sim h$ if and only if $gh^{-1} \in H$.

        i. Prove that $\sim$ defines an equivalence relation on G.
        ii. What are the equivalence classes in S_3 for $H = \langle (1\ 2\ 3) \rangle$?
        iii. If G is a finite group, what is the relationship between $|G|$, $|H|$ and the number of equivalence classes?

b) Define $\phi : G \to G$ by $\phi(g) = g^2$.

   i. Show that ϕ is a homomorphism if and only if G is abelian.

   ii. Show that any finite group of even order contains an element of order 2.

   iii. If G is a finite abelian group, show that ϕ is one-to-one if and only if G has odd order.

2) i. Prove that every subgroup of a cyclic group is cyclic.

   ii. Prove that if C is an infinite cyclic group, then C is isomorphic to $\mathbb{Z}$.

   iii. Prove that if C is a finite cyclic group of order n, then C is isomorphic to $\mathbb{Z}_n$.

   iv. Let n be a positive integer which has the prime decomposition $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, where $p_1, \ldots, p_m$ are distinct primes. Then

$$Z_n \cong Z_{p_1^{\alpha_1}} \times \cdots \times Z_{p_m^{\alpha_m}}.$$

3) Describe all homomorphisms from Z_n to Z_k. Characterize (and prove) when such a homomorphism is onto.

4) A subgroup of the symmetric group $\mathrm{Sym}(S)$ is called a **permutation group**.

   i. For a group G and $a \in G$ define $\lambda_a : G \to G$ by $\lambda_a(g) = ag$ for $g \in G$. Show that λ_a is a bijection.

   ii. Show that $\phi : G \to \mathrm{Sym}(G)$ defined by $\phi(a) = \lambda_a$ is a group homomorphism.

   iii. Show that the kernel of ϕ has cardinality one.

   iv. Use the Fundamental Homomorphism Theorem to conclude that G is isomorphic to a subgroup of $\mathrm{Sym}(G)$.

5) a) Show that every subgroup of an abelian group is normal.

   b) Show that every factor group of an abelian group is abelian.

6) a) Let G consists of the set of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

   where $x, y, z \in \mathbb{Z}$. Show that G is a subgroup of $GL_3(\mathbb{R})$, the group of invertible 3×3 matrices over $\mathbb{R}$, with matrix multiplication.

b) Show that the subset H of G consisting of matrices as above with $x, y = 0$ and $z \in \mathbb{Z}$ is a **normal** subgroup of G.

c) Show that G/H is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.