INTRODUCTION TO POLYGONS

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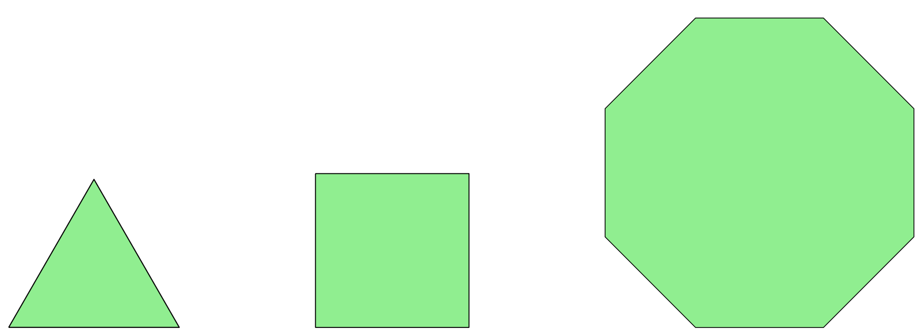
Contents

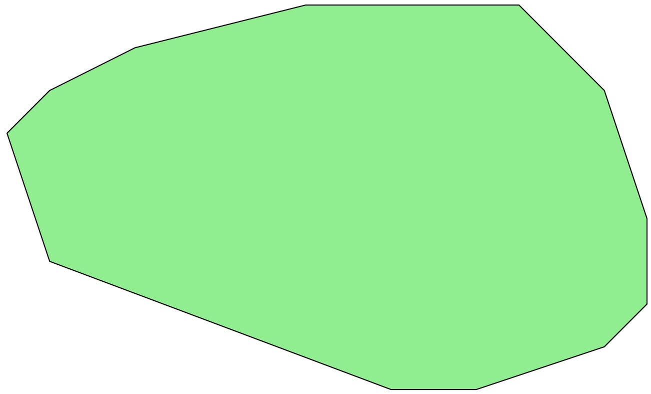
1. Introduction

This document aims to guide the reader into the world of polytopes, focusing on the familiar setting of polygons, the two-dimensional polytopes. We will assume the reader is comfortable with the Cartesian plane and ordered pairs of numbers. Let's get right into it!

Intuitively, polygons are certain 2-dimensional shapes. You may think of things like:



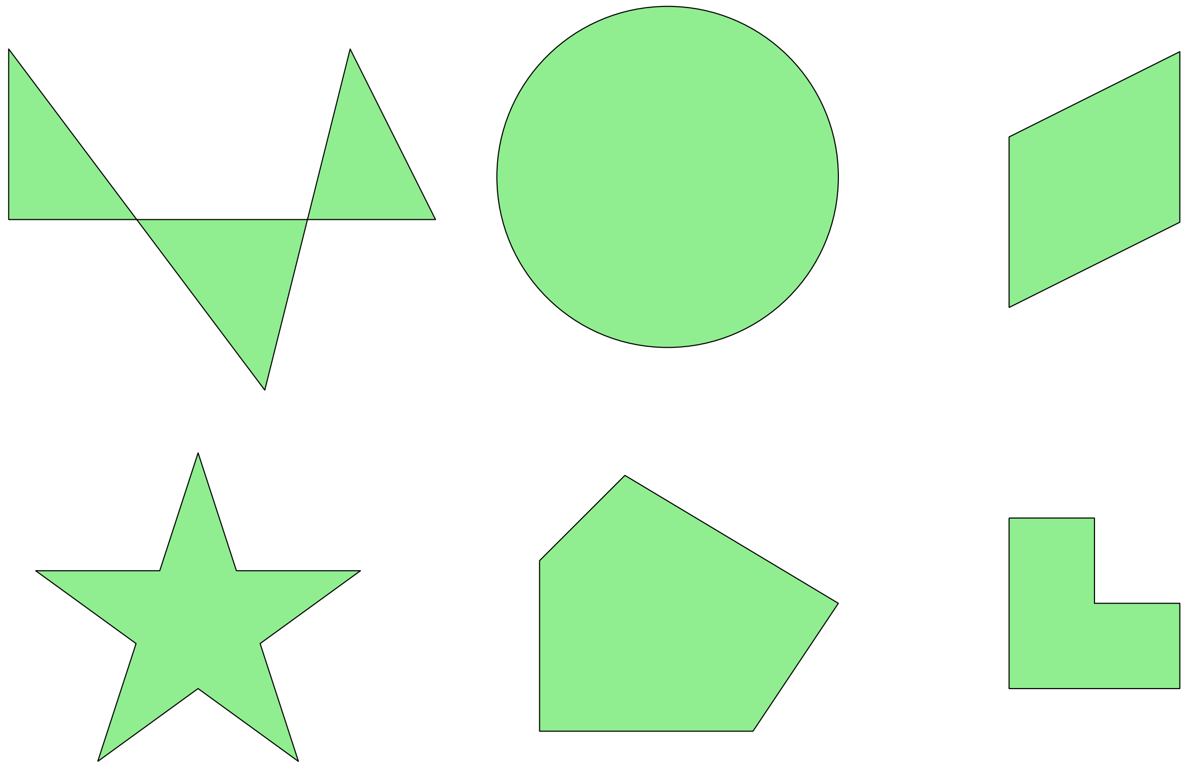
However, general polygons can look less uniform. The following is a more exotic polygon:



Before, we move, on try formulating your own definition of the word polygon based on your previous experience in math classes (and use the above confirmed polygons as inspiration!).

Definition 1.1. A polygon is ...

Exercise 1.2. Using your definition, try to guess which of the following should be considered polygons:



It turns out there are different answers to this question that lead to completely different definitions the word polygon. All but one of the above regions may be considered examples of different kinds of polygons! In this document, we will focus only on one particularly elegant definition: convex polygons. For this, we turn to the notion of convex sets.

2. Convex Sets

Consider a collection of points of the Cartesian coordinate plane, that is a set of tuples of the form (a, b) where a and b are real numbers. We will call such a collection a *subset* of $\mathbb{R}^2$, where $\mathbb{R}$ is a fancy shorthand for the set of all real numbers, and the 2 indicates we are dealing with ordered pairs. You can think of $\mathbb{R}$ as the usual number line, and $\mathbb{R}^2$ as a flat plane with the usual x and y coordinate axes. We will work with subsets of these two sets.

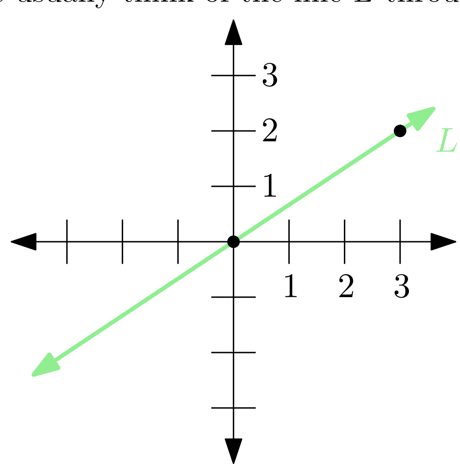
For example, we may think of the line L connecting the points $(0, 0)$ and $(3, 2)$. To view L as a collection of ordered pairs (that is a subset of the plane $\mathbb{R}^2$), we note that L is the subset of all pairs (x, y) such that $y = 2/3x$. In mathematical set notation, we would write

$$L = \{(x, y) \in \mathbb{R}^2 \mid y = \frac{2}{3}x\}$$

In case the reader is not familiar, let's break that notation down:

$$\underbrace{L = \{}_{L\text{ is the set}}\overbrace{(x, y) \in \mathbb{R}^2}^{\text{of all ordered pairs }(x,y)\text{ of real numbers}}\underbrace{\mid y = \tfrac{2}{3}x\}}_{\text{such that } y = \frac{2}{3}x.}$$

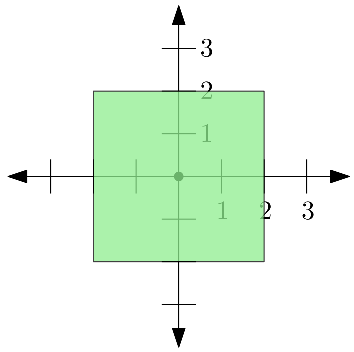
In the Cartesian plane, we usually think of the line L through its graph:



As another example, we could think of the set

$$S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2\}$$

together with its graph



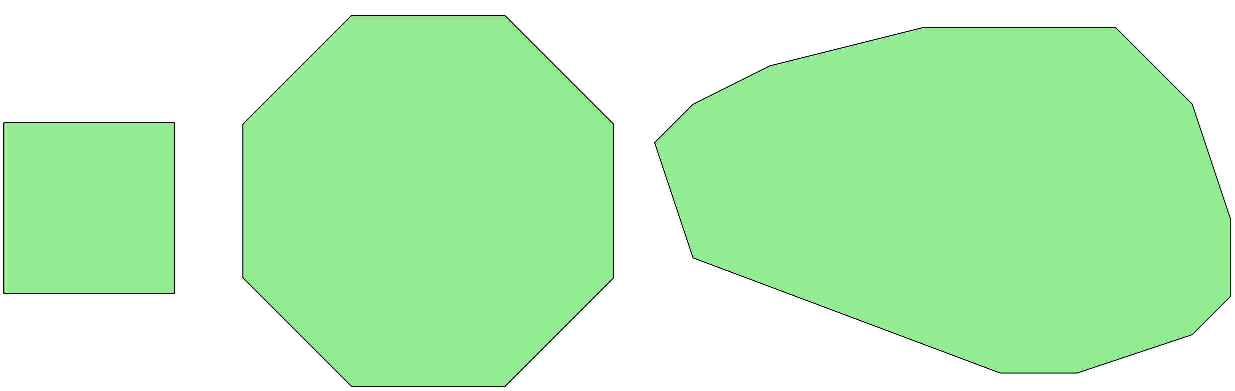
Exercise 2.1. Use set notation to describe the following sets:

a) The set S of all points lying on the graph of the function x^2.

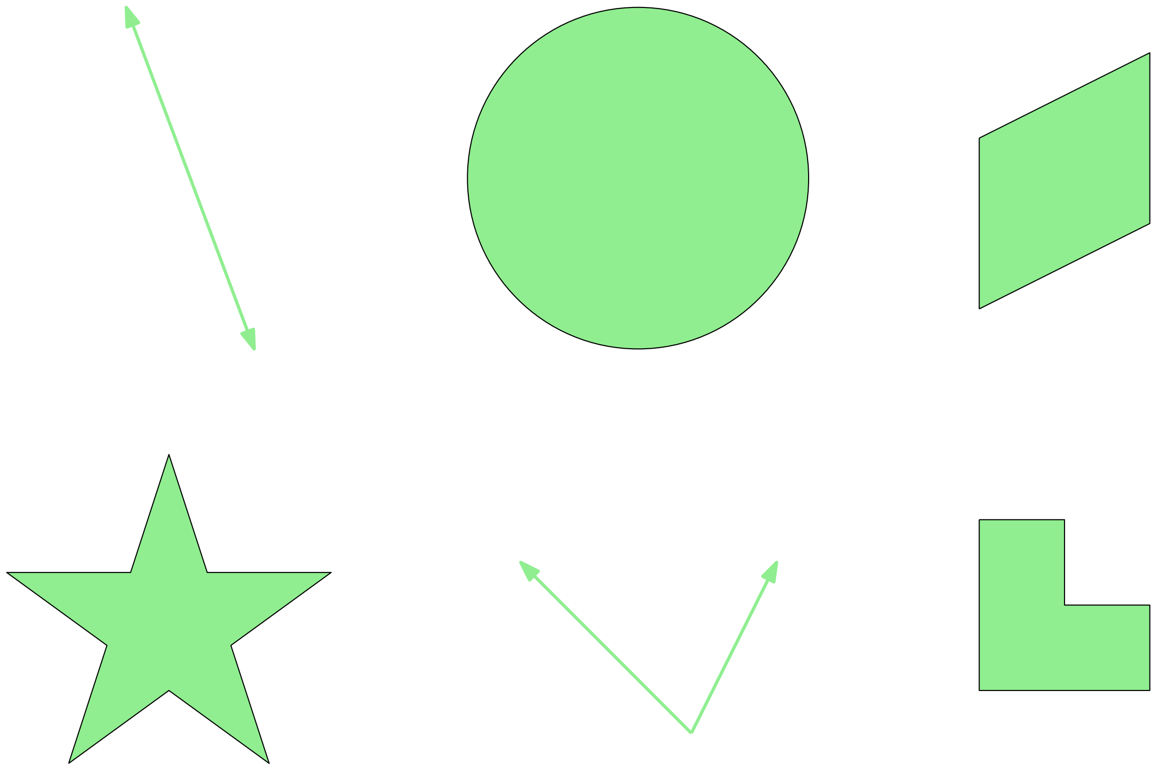
b) The set T of all points with positive x component.

Definition 2.2. We say a subset P of $\mathbb{R}^2$ is *convex* if for any points $p, q \in P$, (read p and q in P), the line segment between p and q is contained entirely in P.

For example, several of the sets we have drawn so far are convex. Try drawing a few line segments between points inside each of the sets below. Being convex means that your line segments should never have to leave the shape.

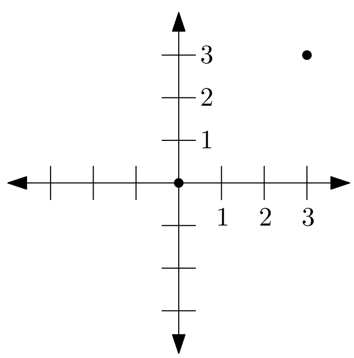


Exercise 2.3. Which of the following sets are convex and which are not?



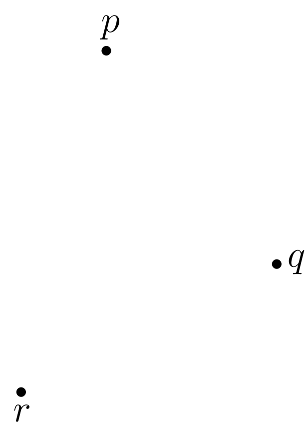
3. Polygons

Now that we understand what is and is not a convex set, let's try to create convex sets from points. As a warmup, what is the smallest convex hull containing the points $(0, 0)$ and $(3, 3)$? Draw it below!



When we only have two points, the smallest convex set containing them is just the line segment connecting them. This is the minimum that any convex set containing the points would have to contain, and it is convex.

Example 3.1. What happens if we have three points not on a line? Let's call them p, q, and r.



Denote by S the smallest convex set containing p, q, and r. At a glance, it looks like S should be the triangle with corners p, q, and r. Let's try to see why!

Since S is convex, $p, q \in S$ means the line segment $\overline{pq}$ should lie entirely in S. Draw this! Since $r \in S$ also, the line segment between r and any point on $\overline{pq}$ should also lie entirely in S. Draw some of these segments! Based on your drawing, conclude that S must contain the triangle with corners p, q, and r. Since the triangle is convex, it must be the smallest convex set containing p, q, and r.

Definition 3.2. Given points $p_1, p_2, \dots p_n \in \mathbb{R}^2$ (think of each p_i as a pair (x_i, y_i)) their *convex hull* $\mathrm{Conv}(\{p_1, \dots, p_n\})$ is the smallest convex set containing $p_1, p_2, \dots, p_n$.

Based on Example 3.1, $\mathrm{Conv}(\{p, q, r\})$ is the triangle with corners p, q, r.

We can now give a formal definition of polygons.

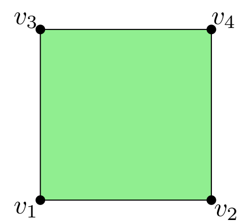
Definition 3.3. A (convex) *polygon* is a subset P of $\mathbb{R}^2$ of the form $P = \mathrm{Conv}(\{p_1, \dots, p_n\})$ for some points $p_1, \dots, p_n$ not lying on a line.[1].

4. Vertices and Edges

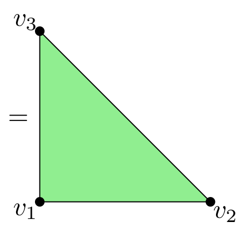
Based on the definition, what do polygons look like? They look roughly like flattened circles. They have a number of corner points connected by straight lines lying on the boundary of the polygon. The corner points are called *vertices*, and the boundary line segments are called *edges*. Let's think about this with an example.

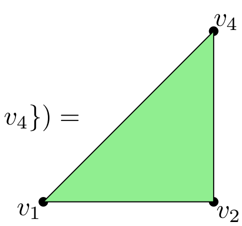
Example 4.1. Consider the points $v_1 = (0, 0)$, $v_2 = (1, 0)$, $v_3 = (0, 1)$, and $v_4 = (1, 1)$. The convex hull $P = \mathrm{Conv}(\{v_1, v_2, v_3, v_4\})$ is the unit square.

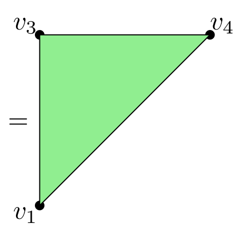
[1]What we are calling polygons are more generally known as *convex polygons*. This distinguishes them from other definitions of polygon, such as *concave polygons* or *self-intersecting polygons*, that we will not cover in this document. The intrigued reader may enjoy the demonstration on the website `https://www.mathopenref.com/polygonconcave.html`

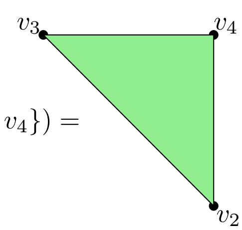


The vertices of P are v_1, v_2, v_3, v_4. The edges of P are the boundary of the square, the line segments $\overline{v_1v_2}$, $\overline{v_1v_3}$, $\overline{v_2v_4}$, and $\overline{v_3v_4}$. Note that $\overline{v_2v_3}$ is not an edge. One property that characterizes vertices of a polygon is that leaving one out from a convex hull produces a smaller convex set. For the square,

$\text{Conv}(\{v_1, v_2, v_3\}) =$ 

$\text{Conv}(\{v_1, v_2, v_4\}) =$ 

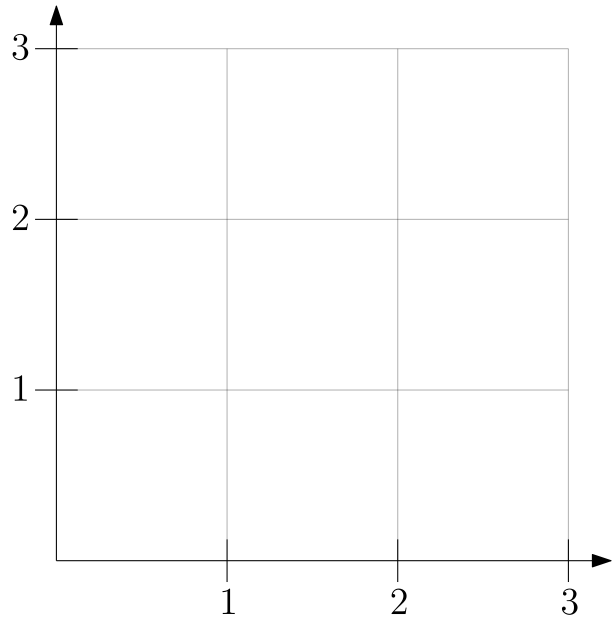
$\text{Conv}(\{v_1, v_3, v_4\}) =$ 

$\text{Conv}(\{v_2, v_3, v_4\}) =$ 

which are all strictly smaller than $\text{Conv}(\{v_1, v_2, v_3, v_4\})$. The vertices are the smallest set of points defining a given polygon. For example,

$$P = \text{Conv}(\{v_1, v_2, v_3, v_4\}) = \text{Conv}(\{v_1, v_2, v_3, v_4, (1/2, 1/2)\}).$$

Exercise 4.2. Consider the polygon $P = \text{Conv}(\{(0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1)\})$. Draw P below. What are the vertices and edges of P?



Vertices of P:

Edges of P:

5. Integer Points

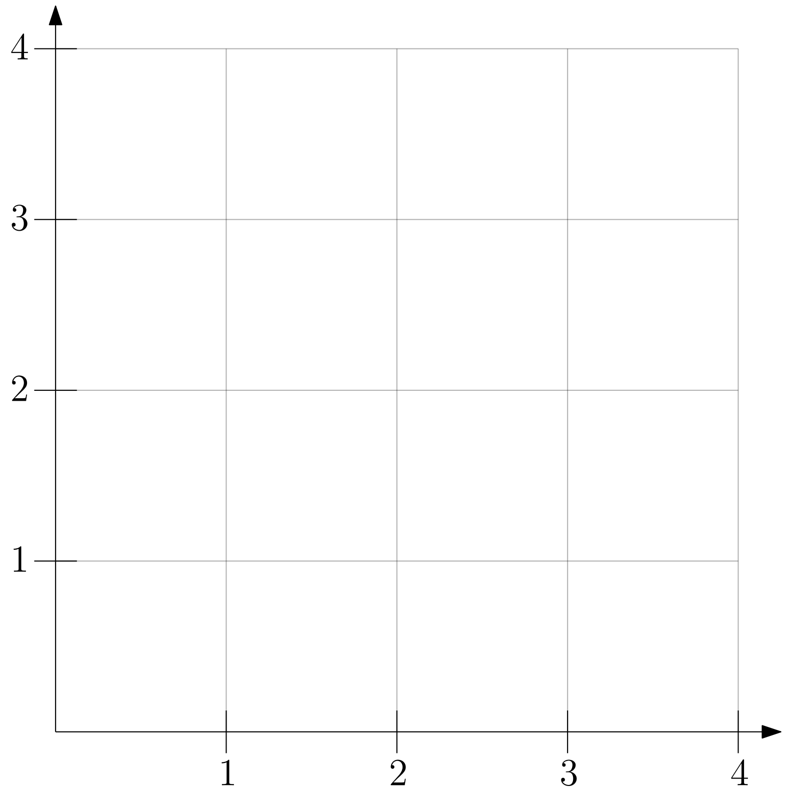
Given a polygon P, one can look at the collection of points lying in P that have integer coordinates, called the *integral* or *integer* points of P. The set notation for the integers is $\mathbb{Z}$, and the set notation for "and" is $\cap$. Consider the set of integer points in P. This is the set of ordered pairs (x, y) such that x and y are both integers AND (x, y) lies in P. In set notation, this set is denoted

$$P \cap \mathbb{Z}^2.$$

We will be interested in the number of integer points, denoted $|P \cap \mathbb{Z}^2|$, and will connect this number to the area of P later on.

This is of particular interest when the vertices of P are themselves integer points. In this case, the polygon P is called an *integral polygon*.

Exercise 5.1. Let $P = \mathrm{Conv}(\{(0,0),(4,0),(0,4)\})$. Draw P below and identify its integer points. What is $|P \cap \mathbb{Z}^2|$?



Ehrhart theory is a field of discrete math that asks about how $|P \cap \mathbb{Z}^2|$ changes as P gets bigger. To make getting bigger precise, consider the following definition.

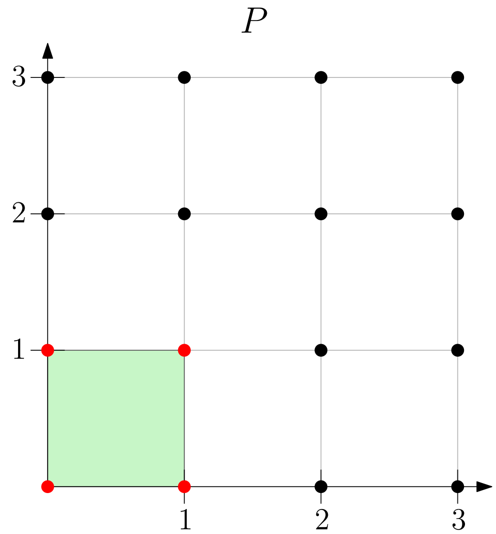
Definition 5.2. For any integer $t > 0$ and any polygon P, define the tth *dilate* of P to be the polygon tP defined by

$$tP = \{(tx, ty) : (x, y) \in P\}.$$

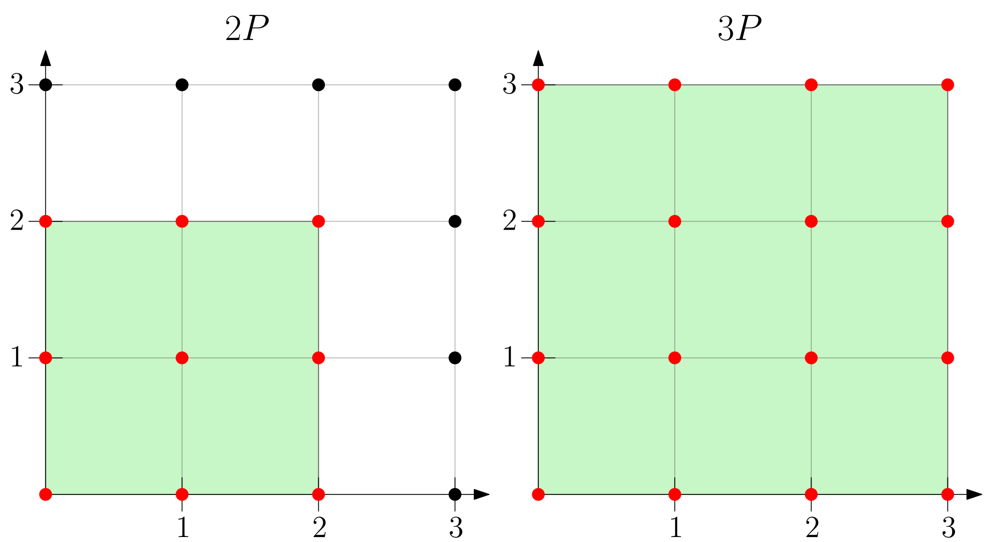
We will be interested in the numbers $|tP \cap \mathbb{Z}^2|$ as t increases. Since these numbers depend on t, we will give them the following function style notation.

Definition 5.3. For a polygon P, denote by $L_P(t)$ the function of t given by $L_P(t) = |tP \cap \mathbb{Z}^2|$ for all integers $t \geq 0$.

Example 5.4. Let's start with our good friend the square $P = \text{Conv}(\{(0,0), (1,0), (0,1), (1,1)\})$. The only integer points in P are the vertices, so $L_P(1) = 4$.



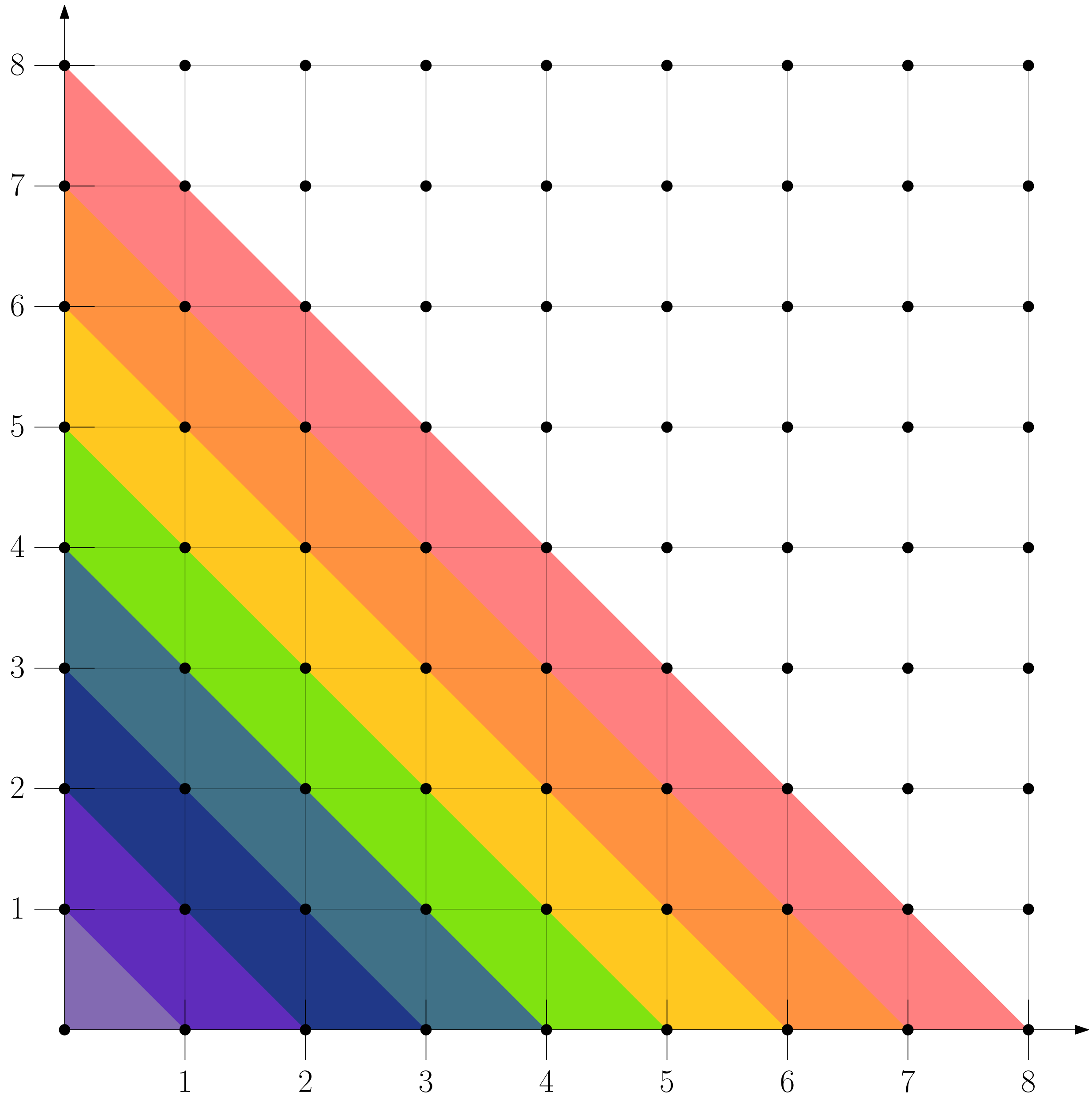
We similarly observe that $L_P(2) = 9$ and $L_P(3) = 16$:



Exercise 5.5. For the square above, show that $L_P(t) = (t+1)^2$ for all integers $t \geq 1$.

Example 5.6. Let's try something harder. Consider the triangle $P = \mathrm{Conv}(\{(0,0),(1,0),(0,1)\})$. What happens to the number of integer points as we grow the triangle to $\mathrm{Conv}(\{(0,0),(8,0),(0,8)\})$? That is, what are the values $L_P(1), L_P(2), \ldots, L_P(8)$ and how do they differ?

Below is shown the overlay of all the polygons tP as t goes from 1 to 8. Fill in the chart with the integer point counts of each.



t	$\lvert tP \cap \mathbb{Z}^2 \rvert$
1	
2	
3	
4	

t	$\lvert tP \cap \mathbb{Z}^2 \rvert$
5	
6	
7	
8	

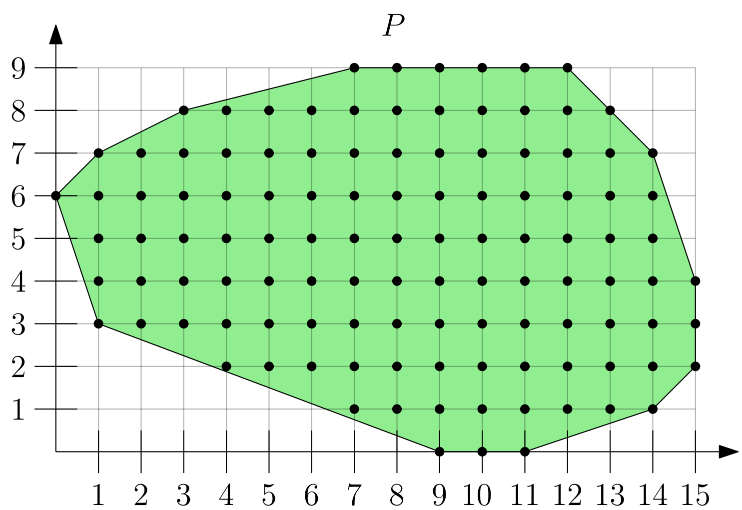
Exercise 5.7. Can you give an explicit formula for $|P \cap \mathbb{Z}^2|$ as a function of t? Even harder, can you express it as a polynomial in t? In either case, try to argue (prove) why your answer is correct.

The previous two examples are both special cases of a general theorem about polytopes.

Theorem 5.8 (Ehrhart's Theorem (for polygons))**.** *If P is an integral polygon, then $L_P(t)$ is a quadratic polynomial in t, called the Ehrhart polynomial of P.*

This theorem is valid for any polytope, not just the squares and triangles.

Example 5.9. For instance, in the case of the "exotic" polytope from earlier:



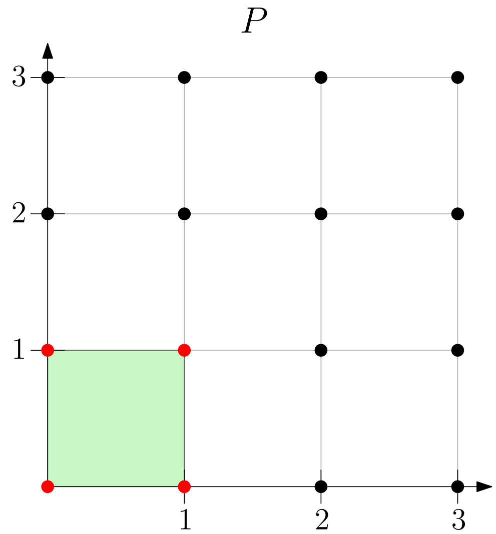
While much less obvious than a square or triangle, P has Ehrhart polynomial

$$L_P(t) = \frac{203}{2}x^2 + \frac{19}{2}x + 1,$$

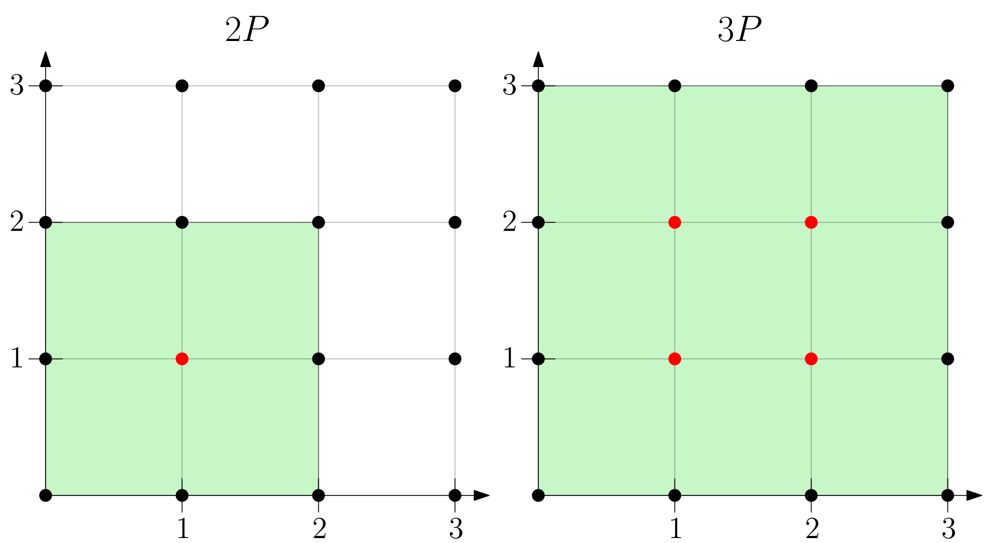
a quadratic polynomial as expected.

We now turn to a similar question. Given a polygon P, call the *interior* P^o all the points of P that do not lie on an edge or vertex. Instead of counting integer points in P, we want to count integer points in P^o.

Example 5.10. Let's return our best pal, the square $P = \mathrm{Conv}(\{(0,0),(1,0),(0,1),(1,1)\})$. The only integer points in P are the vertices, so $L_{P^o}(1) = 0$, since we do not want to count vertices.



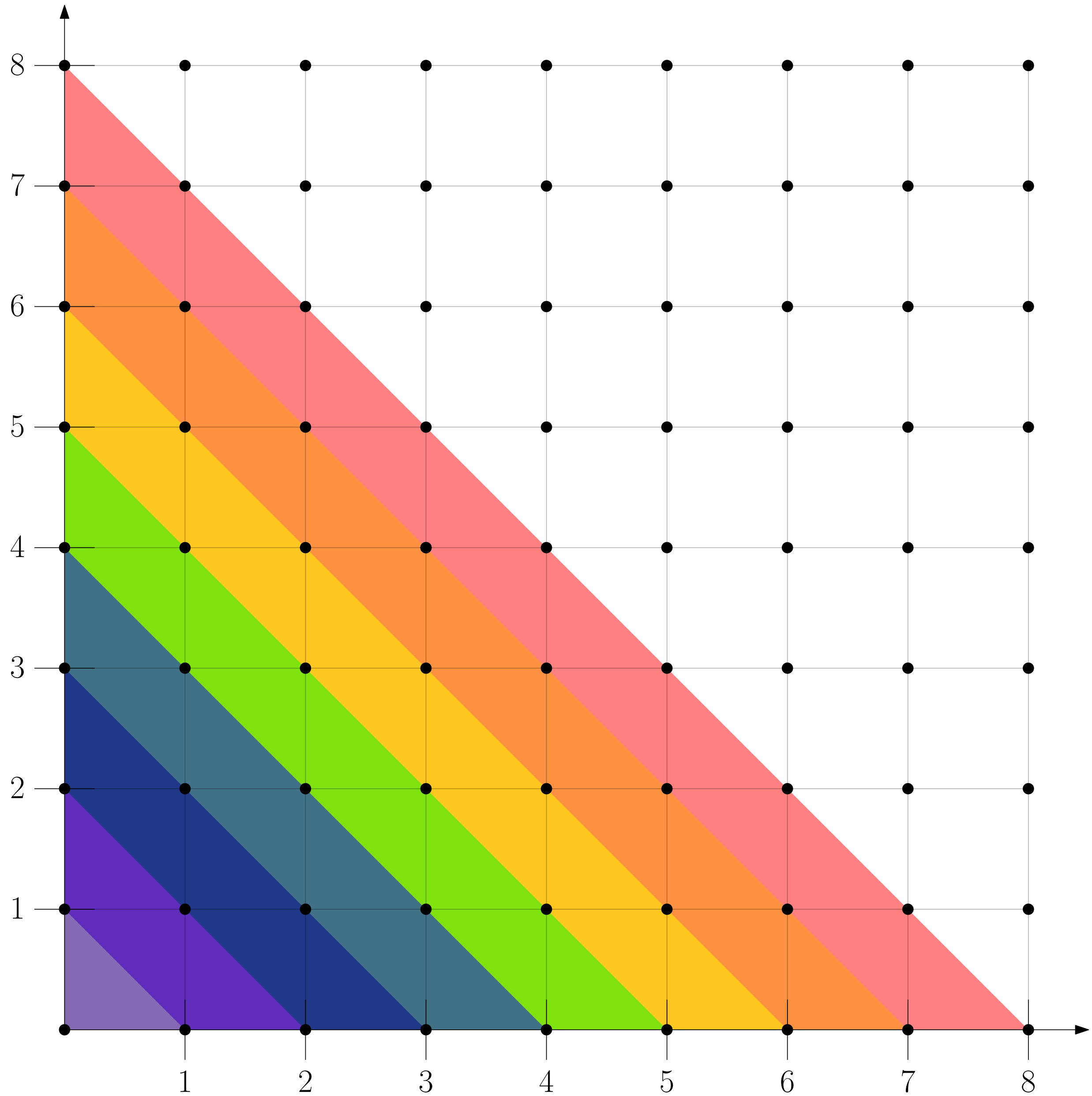
We similarly observe that $L_{P^o}(2) = 1$ and $L_{P^o}(3) = 4$:



Exercise 5.11. For the square above, show that $L_{P^o}(t) = (t-1)^2$ for all integers $t \geq 1$.

Example 5.12. Can we answer the same question for a triangle? Recall the triangle $P = \mathrm{Conv}(\{(0,0),(1,0),(0,1)\})$. What happens to the number of integer points in the interior as we grow the triangle to $\mathrm{Conv}(\{(0,0),(8,0),(0,8)\})$? That is, what are the values $L_{P^o}(1), L_{P^o}(2), \ldots, L_{P^o}(8)$ and how do they differ?

Below is shown the overlay of all the polygons tP as t goes from 1 to 8. Fill in the chart with the interior integer point counts of each.



t	$\lvert tP^o \cap \mathbb{Z}^2 \rvert$
1	
2	
3	
4	

t	$\lvert tP^o \cap \mathbb{Z}^2 \rvert$
5	
6	
7	
8	

Exercise 5.13. Can you give an explicit polynomial formula for $|P^o \cap \mathbb{Z}^2|$ as a function of t? Try to prove your answer is correct.

Compare your answers for the square and triangle interiors with your answers for the square and triangle. Do you see any relation between $L_P(t)$ and $L_{P^o}(t)$ when P is the square or triangle?

It turns out that $L_P(t)$ and L_{P^o} are very closely related for any polygon! The proof of the following theorem is involved, so we refer the interested reader to [1] for the details.

Theorem 5.14 (Ehrhart's Reciprocity Theorem (for polygons))**.** *If P is an integral polygon, then $L_{P^o}(t) = L_P(-t)$ for all integers $t \geq 0$. In particular, $L_{P^o}(t)$ is a quadratic polynomial in t.*

Example 5.15. Let P be the unit square $P = \text{Conv}(\{(0,0),(1,0),(0,1),(1,1)\})$. In Example 5.4, we found $L_P(t) = (t+1)^2$. In Example 5.10, we observed $L_{P^o}(t) = (t-1)^2$. We can easily check that reciprocity holds:

$$L_P(-t) = (-t+1)^2 = (-(-t+1))^2 = (t-1)^2 = L_{P^o}(t).$$

6. Area

For any polygon P, one can ask how much space is covered by P. This quantity is called the *area* of P. For some special polygons, you should already know how to find the area.

Example 6.1. Let P, Q, and R be the polygons

$$\begin{aligned} P &= \text{Conv}(\{(0,3),(2,0),(0,0)\}) \\ Q &= \text{Conv}(\{(-1,0),(3,0),(-1,1),(3,1)\}) \\ R &= \text{Conv}(\{(-2,0),(2,0),(0,-2),(0,2)\}) \end{aligned}$$

Find the areas of P, Q, and R using your prior knowledge of geometry.

In general though, how on earth would you do this? Not all polygons have nice geometric formulas in some variables for their areas. In this section, we discuss two ways of computing the area of any polygon using integer point counts.

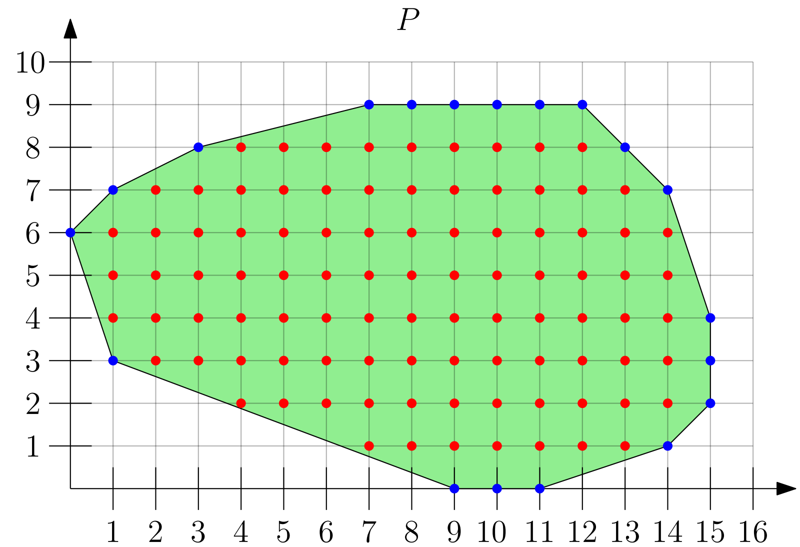
6.1. **Pick's Theorem.**

We first give a method for calculating areas by counting boundary and interior integer points of a polygon. Given a polygon P, let $i(P)$ denote the number of integer points of P lying in the interior of P, and let $b(P)$ denote the number of integer points of P lying on the boundary of P.

Theorem 6.2 (Pick's Theorem)**.** *If P is an integral polygon, then*

$$\text{Area}(P) = i(P) + \frac{b(P)}{2} - 1.$$

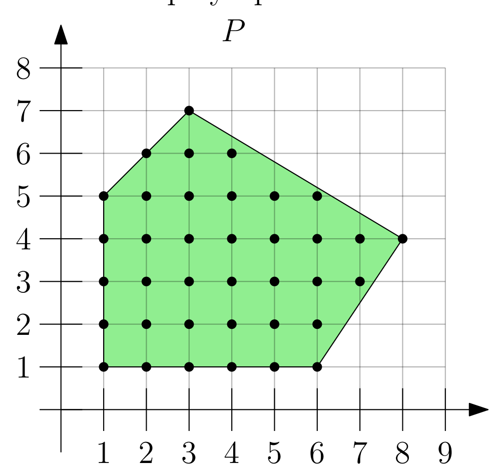
Example 6.3. In the polygon P below, we have $i(P) = 93$ (the red points) and $b(P) = 19$ (the blue points).



By Pick's Theorem,

$$\text{Area}(P) = 93 + \frac{19}{2} - 1 = \frac{203}{2}.$$

Exercise 6.4. What is the area of the polytope P below?



6.2. **Ehrhart Polynomials.** Let's return to Example 6.1. Let P, Q, and R be the polygons

$$\begin{aligned} P &= \text{Conv}(\{(0,3),(2,0),(0,0)\}) \\ Q &= \text{Conv}(\{(-1,0),(3,0),(-1,1),(3,1)\}) \\ R &= \text{Conv}(\{(-2,0),(2,0),(0,-2),(0,2)\}) \end{aligned}$$

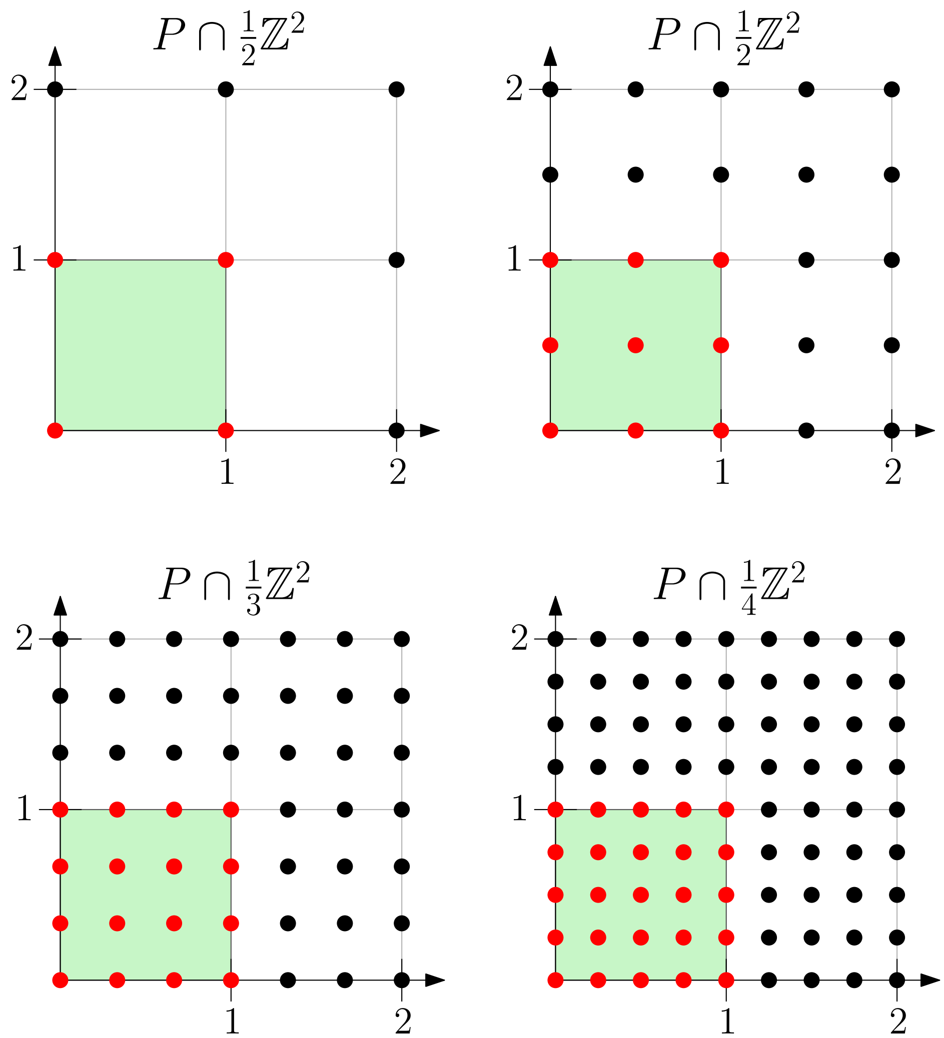
Let's look at their Ehrhart polynomials:

$$\begin{aligned} L_P(t) &= 3t^2 + 3t + 1 \\ L_Q(t) &= 4t^2 + 5t + 1 \\ L_R(t) &= 8t^2 + 4t + 1. \end{aligned}$$

What do you notice about these polynomials versus the areas that you calculated in Example 6.1? Using your observation, complete the following theorem statement:

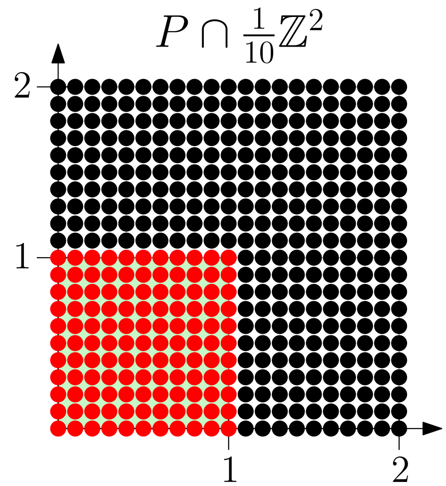
Theorem 6.5. *If P is an integral polygon, then the* ______________________________ *of $L_P(t)$ equals the* ______________________________ *of P.*

Proof. Let's talk a little about the reason why this theorem would be true. If you haven't taken calculus, you may wish to skip this explanation. What we're doing when we compute $L_P(t)$ is growing P by a factor of t and counting the integer points tP contains. What if instead, we did the opposite. Let's shrink the all the integer points by a factor of t, and count how many land inside P (without changing P). Consider for example P the unit square:

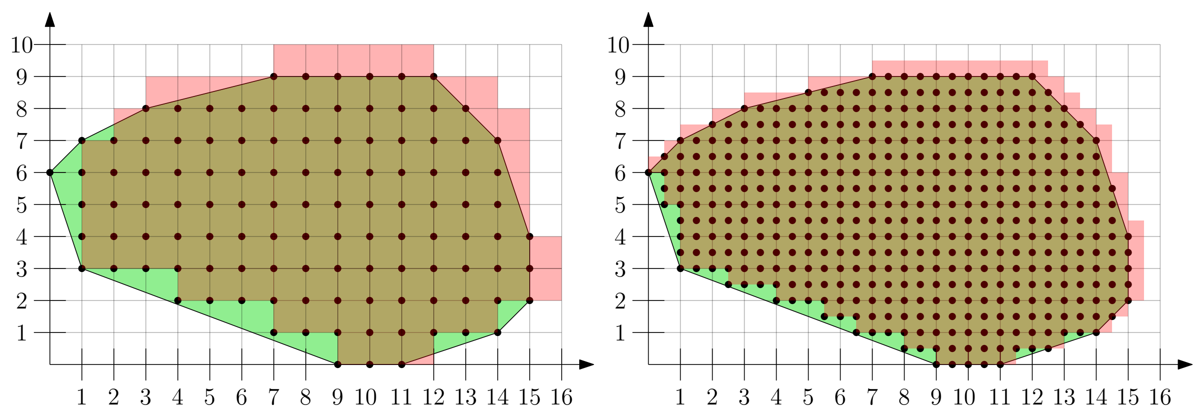


- How do the numbers of contracted integer points inside of P compare to the number of integer points in the dilates of P in Example 5.10?
- What does the picture look like as we contract the $\mathbb{Z}^2$ more and more?

For example, here is the tenth dilation:



Imagine we associate to each dot a tiny $\frac{1}{t} \times \frac{1}{t}$ box with bottom left corner at the dot. How does the area of all these boxes compare to the area of P? We demonstrate on the polygon from earlier:



As we contract $\mathbb{Z}^2$ more, the $\frac{1}{t} \times \frac{1}{t}$ boxes get smaller and we cover P with more of them. For large enough t, the area of P should be roughly the same as the number of tiny boxes times the area of each box.

- The number of tiny boxes is $L_P(t)$, one for each dot in $P \cap \frac{1}{t}\mathbb{Z}^2$.
- The area of each tiny box is $\frac{1}{t^2}$

Say $L_P(t) = at^2 + bt + c$. Then for large t,

$$\text{Area}(P) \approx \frac{1}{t^2} L_P(t) = a + \frac{b}{t} + \frac{c}{t^2}.$$

But for large enough t, both b/t and c/t^2 are very small, so $\text{Area}(P) \approx a$. If you know limits, then you should recognize that we are taking one here as $t \to \infty$ and can conclude that $\text{Area}(P) = a$. $\square$

Example 6.6. Compare the area found in Example 6.3 to the Ehrhart polynomial found in Example 5.9. Both methods for finding area give the same answer!

The following exercise guides the reader through a proof of Pick's Theorem. We recall Pick's Theorem below for convenience.

Theorem 6.7 (Pick's Theorem)**.** *If P is an integral polygon then*

$$\text{Area}(P) = i(P) + \frac{b(P)}{2} - 1,$$

where $i(P)$ is the number of integer points in the interior of P and $b(P)$ is the number of integer points on the boundary of P.

Exercise 6.8. Follow the steps below to give a proof of Pick's Theorem.

By Theorem 5.8, we can we write

$$L_P(t) = a_1t^2 + a_2t + a_3$$

for some numbers a_1, a_2, a_3.

(i) What is a_3?
(ii) By Theorem 6.5, what is a_1?
(iii) Use Theorem 5.14 to write a formula for $i(P)$.
(iv) Derive a formula for $b(P)$ your formula for $i(P)$.
(v) Deduce Pick's Theorem.

7. Triangulations

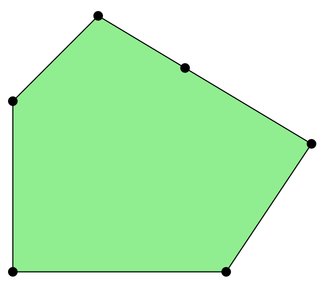
In this section, we discuss triangulations, subdivisions of a polygon into triangles. The intuition of a triangulation of a polygon P is very simple: it is a collection of triangles that exactly cover P and do not overlap. We now give a formal definition.

Definition 7.1. A collection $\mathcal{T}$ of triangles is a *triangulation* of a polygon P if

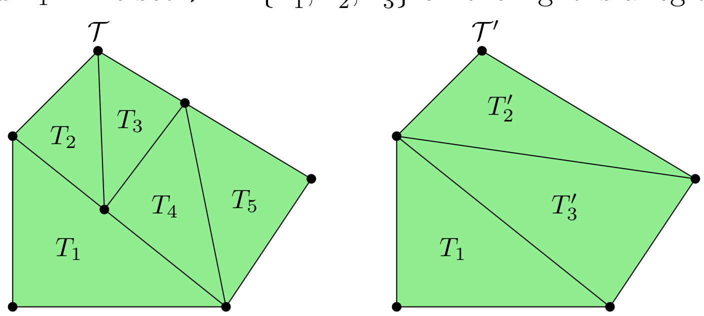
- Every point of P is in at least one triangle,
- Each triangle $T \in \mathcal{T}$ is contained within P,
- Any two triangles intersect in exactly a vertex of both or an edge of both.

Example 7.2. Consider the polytope P shown below.

$P =$



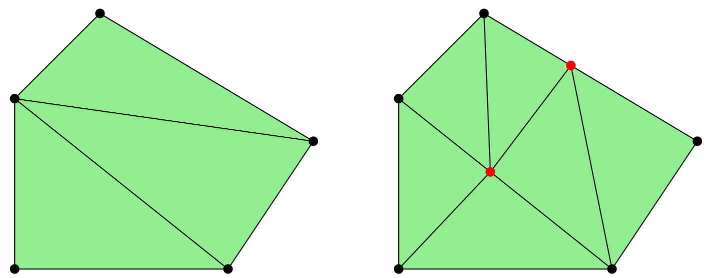
The image below shows two arrangements of triangles of P. The set $\mathcal{T} = \{T_1, T_2, T_3, T_4, T_5\}$ is not a triangulation. For example, T_1 and T_4 do not intersect in exactly a vertex or edge of both. Nor do T_3 and T_1. The set $\mathcal{T}' = \{T_1', T_2', T_3'\}$ on the right is a legitimate triangulation.



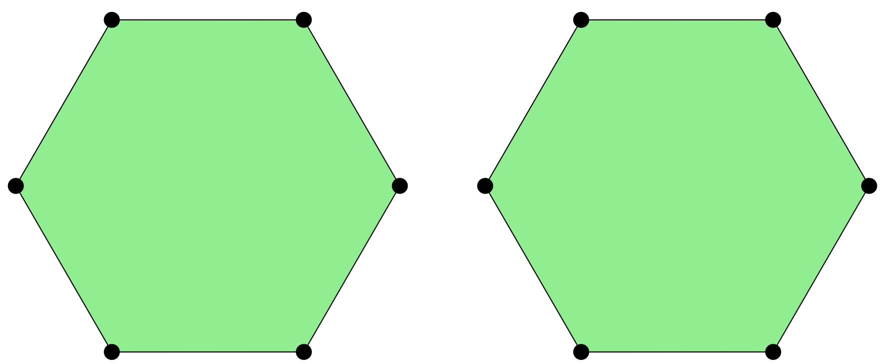
Triangulations are further differentiated by the property of having no new vertices.

Definition 7.3. A triangulation $\mathcal{T}$ of a polygon P has *no new vertices* if the vertices of each $T \in \mathcal{T}$ are contained in the vertices of P.

Example 7.4. Both sets of triangles below are triangulations of the polytope P from Example 7.2. The one on the left uses no new vertices, while the one on the right has two new vertices, indicated in red.



Example 7.5. For the polygon shown below, draw two different triangulations with no new vertices.

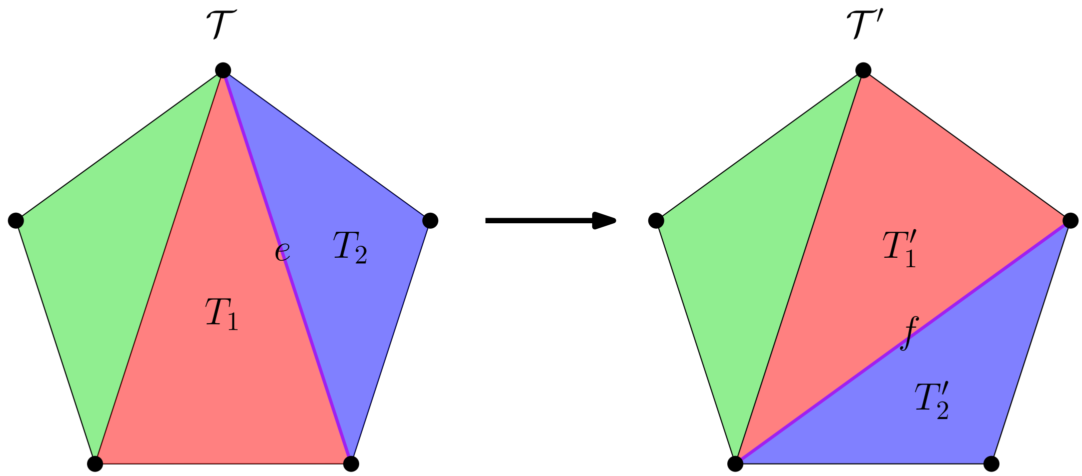


Triangulations with no new vertices have an interesting relation to each other: any two are connected by a sequence of moves!

Definition 7.6. Let $\mathcal{T}$ be a triangulation of a polygon P (with no new vertices). A *bistellar flip* is preformed on $\mathcal{T}$ to produce a new triangulation (also no new vertices) as follows:

- Pick $T_1, T_2 \in \mathcal{T}$ that share an edge, let's call it e.
- The other four edges of T_1 and T_2 form a 4-sided polygon inside P, say Q
- The edge e is a diagonal of Q. Call the other f.
- Form a new triangulation from $\mathcal{T}$ by replacing the edge e by the edge f and taking the resulting two new triangles T_1' and T_2' instead of T_1 and T_2.

Example 7.7. The following is an example of a bistellar flip between $\mathcal{T}$ and $\mathcal{T}'$ along the edge e indicated.

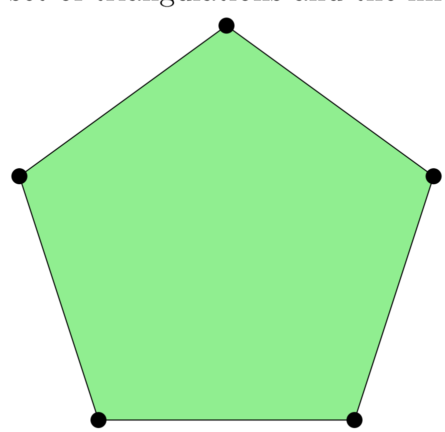


Exercise 7.8. If $\mathcal{T}$ is a triangulation of P with no new vertices, and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by a bistellar flip, show that $\mathcal{T}'$ is also a triangulation of P with no new vertices.

A cool fact is that all triangulations (with no new vertices) of a polygon are connected by bistellar flips!

Theorem 7.9 ([2])**.** *If $\mathcal{T}$ and $\mathcal{T}'$ are triangulations of a polygon P with no new vertices, then there is a sequence of bistellar flips transforming $\mathcal{T}$ into $\mathcal{T}'$.*

Exercise 7.10. On the following page, draw all the triangulations of the polytope shown below with no new vertices. Draw a line between any two triangulations when they differ by a bistellar flip. How does the set of triangulations and the lines connecting them look?



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[3] G. Ziegler. *Lectures on polytopes*, volume 152 of *Undergraduate texts in mathematics*. Springer-Verlag New York, 1995.