Dual Spaces

This section uses language and notation similar to the approach taken in the text by Hoffman and Kunze, but gives a bit more detail and shows why we are mainly interested in the case of finite dimensional vector spaces. There are many important results on dual spaces that are in the exercises of this section. It is important to try them all (as always, but especially true here), since we will use many of the concepts in class or in future homework assignments.

Let V be a vector space over the field F. We define the *dual space of* V to be $V^* = \mathrm{Hom}_F(V, F)$. The linear transformations in V^* are usually called *linear functionals*. We begin with a few simple examples.

Example 1. Let F be any field and let $V = F^{n\times 1}$ for a positive integer n. Given any set of n elements of F, $\{a_1, \ldots, a_n\}$ define $f: V \longrightarrow F$ by $f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_nx_n$. Note that the isomorphism $\mathcal{L}: F^{1\times n} \longrightarrow \mathrm{Hom}_F(F^{n\times 1}, F) = \left(F^{n\times 1}\right)^*$ of the first part of the section on the "Matrix of a Linear Transformation" asserts that the examples $f = L_A$, $A = (a_1, \ldots, a_n)$ give all such examples.

Example 2. Let F be a field and n a positive integer. Define the *trace* $\mathrm{Tr}: F^{n\times n} \longrightarrow F$ for a matrix A with entries A_{ij} by $\mathrm{Tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$.

Example 3. Let F be an arbitrary field and let $a \in F$. Define $E_a: F[x] \longrightarrow F$ on the basis $\left\{1, x, x^2, \ldots, x^i, \ldots\right\}$ by $E_a(x^i) = a^i$. This linear functional is called *evaluation at* a and is written $E_a(f) = f(a)$.

Example 4. Let $X = [0,1] \subseteq \mathbb{R}$ be the closed interval. Let $\mathcal{C}(X)$ denote the vector space of all continuous functions on X with values in $\mathbb{R}$. Define $\mathrm{Int}(f) = \int_0^1 f(t)dt$. Int is a linear functional by the standard properties of definite integrals.

Let V be a vector space over the field F. Let $\mathcal{B}$ be a basis for V. We define the *dual set* $B^* = \{\, b^* \mid b \in \mathcal{B}\,\}$ where $b^*: V \longrightarrow F$ is defined for $a \in \mathcal{B}$ by

$$b^*(a) = \begin{cases} 0 & a \neq b \\ 1 & a = b\,. \end{cases}$$

That is, we have

$$b^*(a) \quad = \quad \delta_{a,b} \tag{1}$$

where $\delta_{a,b}$ is the *Kronecker delta function*: 1 when $a = b$, 0 otherwise.

It will be useful to briefly study such collections of vectors and linear functionals independently of how they might be constructed. Our applications below will be to the pair $\mathcal{B}$ and $\mathcal{B}^*$ which we constructed above for $\mathcal{B}$ a basis of V.

Definition 5. Let V be a vector space over a field F. Two subsets $\mathcal{B} = \{ b_i \mid i \in I \} \subseteq V$ and $\mathcal{B}' = \{ f_i \mid i \in I \} \subseteq V^*$ are said to form a *dual pair* if the *duality equations* hold:

$$f_i(b_j) \;=\; \delta_{i,j}$$

for all $i, j \in I$.

Lemma 6. *Let V be a vector space over F. Let $\mathcal{B} \subseteq V$ and $\mathcal{B}' \subseteq V^*$ form a dual pair.*

a) $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent over F. Consequently $|I| = |\mathcal{B}| = |\mathcal{B}'|$.

b) If V is finite dimensional, then $\mathcal{B}$ is a basis of V if and only if $\mathcal{B}'$ is a basis of V^*.

c) If $\mathcal{B}$ is a basis of V, then $\mathcal{B}'$ is a basis of V^* if and only if $\dim V$ is finite.

Proof. If $\sum_{i\in I} \alpha_i b_i = 0$ in V, then applying f_j to both sides of the equation yields $\alpha_j = \alpha_j f_j(b_j) = 0$ as all other terms on the left are 0 and $f_j(b_j) = 1$ by the duality equations. Similarly, if $\sum_{i\in I} \beta_i f_i = 0$ in V^*, applying the linear functional to b_j yields $\beta_j = 0$. Now the function $I \longrightarrow \mathcal{B}$ given by $i \mapsto b_i$ is onto by definition and is one-to-one since the b_i must be distinct as they are linearly independent, and a similar argument holds for $\mathcal{B}'$.

Now $\mathcal{B}$ is a basis for V if and only if $\mathcal{B}$ spans V. If $\dim V$ is finite, then by our earlier computation, $\dim \mathrm{Hom}_F(V, F) = \dim V \cdot \dim F$, that is $\dim V^* = \dim V$. Since we found $\dim V$ linearly independent elements in V^*, they must also span and hence are a basis. Conversely, if $\dim V^* = |\mathcal{B}'|$, then the linearly independent set $\mathcal{B}$ has the correct number of elements to be a basis of V as well.

For the last assertion, given a basis $\mathcal{B}$ we can define a linear functional $\theta : V \longrightarrow F$ by $\theta(b_i) = 1$ for all $i \in I$. We now show $\theta \in \mathrm{Span}_F(\mathcal{B}')$ if and only if V is finite dimensional. If $\theta = \sum_{i\in I} \alpha_i f_i$, then by applying both sides to $b_j \in \mathcal{B}$, and arguing as before, we obtain $\theta(b_j) = 1 = \alpha_j$. That is, every coefficient $\alpha_j = 1$ for all $j \in I$. However, the span of $\mathcal{B}'$ is the set of finite sums so if $\mathcal{B}$ infinite, $\theta \notin \mathrm{Span}_F(\mathcal{B}')$. □

Remark 7. In fact, if V is a vector space with countable dimension, then V^* has uncountable dimension. See Exercise 29.

We will now look a bit more carefully at the finite dimensional case. Let $\mathcal{B} = \{ e_1, \ldots, e_n \}$ be an ordered basis for V. Then as noted in Lemma 6, $\mathcal{B}^*$ is a basis for V^*.

Theorem 8. *Let V be a finite dimensional vector space over F with ordered basis $\mathcal{B} = \{ e_1, \ldots, e_n \}$. Then the dual set $\mathcal{B}^*$ is the unique basis for V^* which satisfies*

$$e_i^*(e_j) \;=\; \delta_{ij}\ .$$

Further the following equations are valid:

1. $f = \sum_{i=1}^{n} f(e_i)e_i^*$ *for* $f \in V^*$,

2. $v = \sum_{i=1}^{n} e_i^*(v)e_i$ *for* $v \in V$.

Proof. As noted previously for $\dim V$ finite, $\mathcal{B}^*$ is a basis for V^*. Equation 1 was verified in the proof of Lemma 6: the coefficient of the dual basis element b^* is precisely $f(b)$.

The second equation is verified by a similar ("dual") argument: Write $v \in V$ in terms of the basis $\mathcal{B}$ as $v = \sum_{i=1}^{n} \alpha_i e_i$. Now apply e_j^* to the equation yielding

$$e_j^*(v) \;=\; \alpha_j e_j^*(e_j) + 0$$

as all but one term on the right hand side must be 0 with the remaining term $\alpha_j \cdot 1$. That is, $\alpha_j = e_j(v)$ as asserted in equation 2. □

Remark 9. 1. In the finite dimensional case the basis $\mathcal{B}^*$ is called the *dual basis.*

2. The linear functionals e_j^* are nothing other than the *coordinate functions* with respect to the ordered basis $\mathcal{B}$.

Remark 10. The dual spaces V^* may seem a bit mysterious on first encounter, but in fact there is a simple way to think about them. Let $\mathcal{B}$ be a basis for V. Then there is an isomorphism

$$[\quad]_{\mathcal{B}} : V \longrightarrow F^{(\mathcal{B})}$$

given by taking coordinates with respect to $\mathcal{B}$. That is, each vector $v \in V$ corresponds to a unique function $[v]_{\mathcal{B}}$ which is non-zero on only finitely many elements of $\mathcal{B}$.

Similarly via the Universal Mapping Property for Bases, each linear transformation on V, and in particular, each linear functional $f : V \longrightarrow F$, is uniquely determined by what it does on the basis $\mathcal{B}$: that is, by the function $[\![f]\!]_{\mathcal{B}} : \mathcal{B} \longrightarrow F$ given by $[\![f]\!]_{\mathcal{B}}(b) = f(b)$ for each $b \in \mathbf{B}$ or equivalently, $[\![f]\!]_{\mathcal{B}}$ is f restricted to $\mathcal{B}$. This yields an isomorphism

$$[\![\quad]\!]_{\mathcal{B}} : V^* \longrightarrow F^{\mathcal{B}}$$

given by the evaluation on the basis $\mathcal{B}$. Note that if $\mathcal{B}$ is infinite, then $F^{\mathcal{B}}$ is all possible functions on $\mathcal{B}$, so is different from $F^{(\mathcal{B})}$.

If $\mathcal{B}$ is finite, then in fact we have $[f]_{\mathcal{B}^*} = [\![f]\!]_{\mathcal{B}}$ by Theorem 8, first equation.

So in summary, if one fixes a basis $\mathcal{B}$ for V, then one may think of V is "functions on $\mathcal{B}$ with values in F which have finite support" and of V^* as "all functions on $\mathcal{B}$ with values in F".

Example 11 (Lagrange Interpolation)**.** Let $\mathcal{P}_n \subseteq F[x]$ be the subspace of polynomials of degree less than the positive integer n. Let $c_1, \dots, c_n$ be n distinct elements of F. Define $\pi(x) = (x - c_1)(x - c_2) \cdots (x - c_n)$ as the product of all $x - c_i$, $\pi_i(x) = \pi(x)/(x - c_i)$ and $p_i(x) = \pi_i(x)/\pi_i(c_i)$. Let $\mathcal{A} = \{ p_1, \dots, p_n \}$.

Consider evaluation of the linear functionals $\mathcal{E} = \left\{ E_{c_1}, E_{c_2}, \dots, E_{c_n} \right\} \subseteq {\mathcal{P}_n}^*$. Then

$$\begin{aligned} E_{c_i}(p_j) &= p_j(c_i) \\ &= \pi_j(c_i)/\pi_j(c_j) \\ &= \delta_{ij} . \end{aligned}$$

Hence $\mathcal{E} = \mathcal{A}^*$ and $\mathcal{A}$ is linearly independent by Lemma 6. Since it has $n = \dim \mathcal{P}_n$ elements, it is a basis. Thus we have dual bases $\mathcal{A}^* = \mathcal{E}$.

Now we also have the standard basis $\mathcal{B} = \{ 1, x, x^2, \dots, x^{n-1} \}$ for $\mathcal{P}_n$. What is the change of basis matrix $P = P(\mathcal{A}, \mathcal{B})$? From what we did earlier we have the formulas

$$\begin{aligned} P &= P(\mathcal{A}, \mathcal{B}) \\ &= \big[[p_1]_{\mathcal{B}} \; [p_2]_{\mathcal{B}} \cdots [p_n]_{\mathcal{B}} \big] \end{aligned}$$

and

$$\begin{aligned} P^{-1} &= P(\mathcal{B}, \mathcal{A}) \\ &= \big[[1]_{\mathcal{A}} \; [x]_{\mathcal{A}} \cdots [x^{n-1}]_{\mathcal{A}} \big] . \end{aligned}$$

Now the inverse of P is easy to compute using the dual basis of $\mathcal{A}$ (the coordinate functions with respect to $\mathcal{A}$) since $\mathcal{A}^* = \mathcal{E}$. We have for the j-th column:

$$[x^{j-1}]_{\mathcal{A}} = \begin{bmatrix} E_{c_1}(x^{j-1}) \\ E_{c_2}(x^{j-1}) \\ \vdots \\ E_{c_n}(x^{j-1}) \end{bmatrix} = \begin{bmatrix} {c_1}^{j-1} \\ {c_2}^{j-1} \\ \vdots \\ {c_n}^{j-1} \end{bmatrix} .$$

And finally the matrix of P^{-1}

$$\begin{bmatrix} 1 & c_1 & c_1^2 & \cdots & c_1^{n-1} \\ 1 & c_2 & c_2^2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^{n-1} \end{bmatrix} .$$

This invertible $n \times n$ matrix is called the *Vandermonde matrix*. Equation 2 in Theorem 8 applied to the basis $\mathcal{A}$ is called the *Lagrange interpolation formula*: for $f \in \mathcal{P}_n$

$$\begin{aligned} f &= \sum_{i=1}^{n} E_{c_i}(f) p_i \\ &= \sum_{i=1}^{n} f(c_i) p_i . \end{aligned}$$

Annihilators

Let V be any vector space over the field F and let $S \subseteq V$ be any subset. The *annihilator* of S, S°, is the subset of linear functionals that vanish on every element of S:

$$\mathrm{S}^\circ \;=\; \{\, f \in V^* \mid f(s) = 0 \text{ for all } s \in S \,\} \;.$$

Lemma 12. *Let $S \subseteq V$ be a subset of the vector space V over F. Then*

1. S° *is a subspace of* V^*.
2. $\mathrm{S}^\circ = \mathrm{Span_F(S)}^\circ$.

Proof. This is left as an easy exercise. □

Theorem 13. *Let V be a finite dimensional vector space over F. Let $W \subseteq V$ be a subspace. Then*

$$\dim_F W + \dim_F \mathrm{W}^\circ = \dim_F V \;.$$

Proof. The basic idea is the same that was used to give the dimension formulas for short exact sequences, quotient spaces, and the one with the kernel and image of a linear transformation. Start with a basis for W: $\{ v_1, \ldots, v_\ell \}$. Enlarge to a basis for V: $\mathcal{B} = \{ v_1, \ldots, v_\ell, v_{\ell+1}, \ldots, v_{\ell+k} \}$. So $\dim V = \ell + k$ and $\dim W = \ell$. Then $\mathcal{B}^* = \{ v_1^*, \ldots, v_\ell^*, v_{\ell+1}^*, \ldots, v_{\ell+k}^* \}$.
Now by the definition of the dual basis for $1 \le i \le \ell$ and for $\ell + 1 \le j \le \ell + k$ we have $e_j^*(e_i) = 0$. Hence $\{ v_{\ell+1}^*, \ldots, v_{\ell+k}^* \} \subseteq \mathrm{W}^\circ$.
Next let $f \in \mathrm{W}^\circ \subseteq V^*$. By the first equation in Theorem 8 we have for any $f \in V^*$

$$f = \sum_{i=1}^{\ell+k} f(v_i) v_i^* \;.$$

However since actually $f \in \mathrm{W}^\circ$ we have $f(v_i) = 0$ when $i \le \ell$ and thus:

$$f = \sum_{i=\ell+1}^{\ell+k} f(v_i) v_i^* \;,$$

that is, $f \in \mathrm{Span}(\{ v_{\ell+1}^*, \ldots, v_{\ell+k}^* \})$ showing that this set is a basis for W°. Finally, $\dim V = \ell + k = \dim W + \dim \mathrm{W}^\circ$. □

Remark 14. See Exercise 13 for two alternate proofs.

Proposition 15. *Let V be a non-trivial vector space over F and let $v \in V$ be non-zero. Then there exists an $f \in V^*$ so that $f(v) = 1$.*

Proof. We use the Main Theorem on Bases: Let v be the non-zero vector in V. Since $\{v\}$ is linearly independent, we may enlarge it to a basis $\mathcal{B}$ containing v and take the dual set $\mathcal{B}^*$. Then $v^* \in \mathcal{B}^*$ and $v^*(v) = 1$. □

Double Dual

Next we repeat the process of taking the dual and find a striking *natural* relationship with the original vector space. Let V be a vector space over the field F. We define the *double dual* to be $V^{**} = \big(V^*\big)^* = \mathrm{Hom}_F(V^*, F)$. It is easy to find some elements of V^{**}: for $v \in V$ define

$$E_v(f) \;=\; f(v)$$

for $f \in V^*$. The function E_v is called *evaluation at* v. It is easy to check that $E_v : V^* \longrightarrow F$ is a linear functional:

$$\begin{aligned} E_v(\alpha f) &= (\alpha f)(v) \\ &= \alpha f(v) \\ &= \alpha E_v(f) \end{aligned}$$

and for $f, g \in V^*$

$$\begin{aligned} E_v(f+g) &= (f+g)(v) \\ &= f(v) + g(v) \\ &= E_v(f) + E_v(g) \;, \end{aligned}$$

That is, E_v is a linear functional precisely because of the definitions of addition and scalar multiplication for functions.

Further, we can consider E itself as the function

$$E : V \longrightarrow V^{**}$$

which sends v to E_v. We'll keep using the notation E_v (rather than the more common $E(v)$). Note that this function E is also a linear transformation, that is, it satisfies

$$E_{\alpha v} = \alpha E_v$$

and

$$E_{u+v} = E_u + E_v$$

for all $\alpha \in F$ and $u, v \in V$. This is also easy to check, namely

$$\begin{aligned} E_{\alpha v}(f) &= f(\alpha v) \\ &= \alpha f(v) \\ &= \alpha E_v(f) \end{aligned}$$

and

$$\begin{aligned} E_{u+v}(f) &= f(u+v) \\ &= f(u)+f(v) \\ &= E_u(f)+E_v(f) \end{aligned}$$

hold for all $f \in V^*$ because each f is a linear transformation. Hence we have equality of functions $E_{\alpha v} = \alpha E_v$ and $E_{u+v} = E_u + E_v$ as they give the same value for each element of their domain.

Next note that the linear transformation E is always one-to-one. To see this we need to compute $\ker E$. A vector $v \in V$ is in $\ker E$ precisely when $E_v(f) = 0$ for all $f \in V^*$. But by Proposition 15 if v is not 0, there will exist an f with $f(v) = 1$. Hence $v \in \ker E$ must be 0.

For V finite dimensional, it is now clear that $E : V \longrightarrow V^{**}$ is an isomomorphism since $\dim V = \dim V^* = \dim V^{**}$. We now show that this is the only case for which E is an isomorphism.

Proposition 16. *Let V be a vector space over the field F. If $E : V \longrightarrow V^{**}$ is an isomorphism, then*

1. *For every $L \in V^{**}$ there exists a unique vector $v \in V$ such that $E_v = L$.*
2. *For every basis $\mathcal{A}$ of V^* there exists a unique basis $\mathcal{B}$ of V so that $\mathcal{B}^* = \mathcal{A}$.*
3. $\dim V = \dim V^* = \dim V^{**}$.
4. *V has finite dimension.*

Proof. The first statement is just that E is one-to-one and onto.

The second assertion contains the real content of the proposition. Let $\mathcal{A} = \{ f_i \mid i \in I \}$ be a basis for V^*. Consider $\mathcal{A}^* \subseteq V^{**}$ and let $\mathcal{B}_0 = E^{-1}(\mathcal{A}^*)$ be the corresponding subset of V. So we can write $\mathcal{B}_0 = \{ v_i \mid i \in I \}$ where $v_i \in V$ is given by $E_{v_i} = f_i^* \in \mathcal{A}^* \subseteq V^{**}$. Now $\mathcal{A}^*$ is linearly independent, and as E and E^{-1} are isomorphisms by our assumption, $\mathcal{B}_0$ is a linearly independent subset of V. Enlarge $\mathcal{B}_0$ to a basis $\mathcal{B}$ for V and consider $\mathcal{B}^* \subseteq V^*$.

Note that we have the following equations for our dual sets:

$$f_j^*(f_i) = \delta_{ij}$$

and

$$b^*(a) = \delta_{ab}$$

for $a, b \in \mathcal{B}$. Hence as $E_{v_j} = f_j^*$ we have

$$\begin{aligned} f_i(v_j) &= E_{v_j}(f_i) \\ &= f_j^*(f_i) \\ &= \delta_{ij} \\ &= v_i^*(v_j) \ . \end{aligned}$$

We show that $\mathcal{B}_0 = \mathcal{B}$ is a basis for V as follows: assume there exists an element $b \in \mathcal{B}$ which is not in $\mathcal{B}_0$. Then $b^*(v_i) = 0$ for all $v_i \in \mathcal{B}_0$. Recall that $\mathcal{A}$ is a basis for V^* by our original assumption. Thus there exist elements $\alpha_i \in F$, only finitely many non-zero, so that

$$b^* = \sum_{i \in I} \alpha_i f_i .$$

Applying this equation to any $v_j \in \mathcal{B}_0$ yields

$$\begin{aligned} b^*(v_j) &= \sum_{i \in I} \alpha_i f_i(v_j) \\ 0 &= \alpha_j f_j(v_j) + 0 \\ 0 &= \alpha_j \end{aligned}$$

as $b \notin \mathcal{B}_0$ by assumption together with the usual argument for the right hand side. Thus all coefficients α_j are 0 and hence $b = 0$, yielding a contradiction as bases never contain 0. Thus no b which is independent of $\mathcal{B}_0$ can exist. That is, $\mathcal{B}_0 = \mathcal{B}$ is a basis for V. Further we see from the equations above that $f_i(v_j) = v_i^*(v_j) = \delta_{ij}$ for all $i, j \in I$. Hence $f_i = v_i^*$ for all i since they agree on all basis elements in $\mathcal{B}$. That is, $\mathcal{B}^* = \mathcal{A}$ which is the second statement.

The sets $\mathcal{B}$, $\mathcal{A}$ and $E(\mathcal{B}) = \mathcal{A}^*$ all have the same cardinality as I, $\mathcal{B}$ is a basis for V, $\mathcal{A}$ is a basis for V^* and as E is an isomorphism, $E(\mathcal{B}) = \mathcal{A}^*$ is a basis for V^{**}. Thus all three vector spaces have the same dimension as given in the third statement.

Finally, the fourth assertion follows from Lemma 6 since if V has infinite dimension, $\mathcal{B}^*$ is not a basis for V^*. □

We summarize the final result of this discussion.

Theorem 17. *Let V be a vector space over F. The evaluation map $E : V \longrightarrow V^{**}$ is a one-to-one linear transformation which is an isomorphism if and only if V is finite dimensional.*

Proof. By the discussion just before Proposition 16, E is always injective. If V is finite dimensional, this is enough to ensure that E is an isomorphism. If E is an isomorphism, the previous proposition shows that V is finite dimensional. □

Remark 18 (Identification)**.** It is common, although perhaps slightly confusing at first, to consider elements of V and V^{**} to be the same if they correspond under the natural isomorphism E. We will do so but at the beginning will point out where the identification is being made.

It is also clear that E preserves essentially everything. For example if $X \subseteq Y$ are subsets of V, then $E(X) \subseteq E(Y)$. If W_1 and W_2 are subspaces, $E(W_1 + W_2) = E(W_1) + E(W_2)$.

Proposition 19. *Let V be a finite dimensional vector space with subspace W. Then $(\mathrm{W}^\circ)^\circ = W$.*

Proof. As for every element w of W and all $f \in W^\circ$ we have $f(w) = 0$, it is clear that we have:

$$\begin{aligned} W &\subseteq \{ v \in V \mid f(v) = 0 \; \forall f \in W^\circ \} \\ &= \{ v \in V \mid E_v(f) = 0 \; \forall f \in W^\circ \} \\ &= E^{-1}((W^\circ)^\circ) . \end{aligned}$$

So we have (via the Remark) that $E(W) \subseteq (W^\circ)^\circ$. Now

$$\begin{aligned} \dim W + \dim W^\circ &= \dim V \\ \dim W^\circ + \dim (W^\circ)^\circ &= \dim V^* \end{aligned}$$

but as $\dim V = \dim V^*$ we have $\dim W = \dim (W^\circ)^\circ$. As E is an isomorphism, it follows then that $E(W) = (W^\circ)^\circ$. □

Remark 20. One normally writes $W^{\circ\circ}$ instead of $(W^\circ)^\circ$.

If the vector space V does not have finite dimension, the first computation still shows that $E(W) \subseteq W^{\circ\circ}$.

Transpose

Let V and W be vector spaces over the field F and let $T: V \longrightarrow W$ be a linear transformation. We define $T^t: W^* \longrightarrow V^*$ by $T^t(f) = f \circ T$ for $f \in W^*$, that is T^t is just "compose with T". Further note that T^t is itself a linear transformation:

$$\begin{aligned} T^t(f_1 + f_2) &= T^t(f_1) + T^t(f_2) \\ T^t(\alpha f) &= \alpha T^t(f) \end{aligned}$$

for all $f, f_1, f_2 \in W^*$ and all $\alpha \in F$. One must really check such things. By definition, $T^t(f_1 + f_2) = (f_1 + f_2) \circ T$ and we need to prove a left distributive law. But more generally one has both distributive laws

$$\begin{aligned} (S_1 + S_1) \circ T &= S_1 \circ T + S_2 \circ T \\ S \circ (T_1 + T_2) &= S \circ T_1 + S \circ T_2 \end{aligned}$$

for any linear transformations S, S_1, S_2, T, T_1, T_2 for which these compositions are defined. One easily verifies this by computing the value of each expression on a vector v in the domain, using the definition of composition, addition of functions, and for the second equation, the fact that S is a linear transformation.

To verify the second equation for the transpose one can verify similarly, and more easily, the more general version

$$\begin{aligned} (\alpha S) \circ T &= \alpha(S \circ T) \\ S \circ (\alpha T) &= \alpha(S \circ T) \end{aligned}$$

which holds in general for the first equation, and requires that S be a linear transformation for the second. The linear transformation T^t is called the *transpose* of T.

Thus we have shown that $T^t \in \mathrm{Hom}_F(W^*, V^*)$. This now gives a map called the *transpose* and denoted τ:

$$\begin{aligned} \tau: \mathrm{Hom}_F(V, W) &\longrightarrow \mathrm{Hom}_F(W^*, V^*) \\ T &\mapsto T^t \end{aligned}$$

We next note that τ is also a linear transformation, that is, it satisfies

$$\begin{aligned} (T_1 + T_2)^t &= T_1^t + T_2^t \\ (\alpha T)^t &= \alpha(T^t) . \end{aligned}$$

But for $f \in W^*$ the first equation is equivalent to

$$\begin{aligned} (T_1 + T_2)^t(f) &= f \circ (T_1 + T_2) \\ &= f \circ T_1 + f \circ T_2 \\ &= T_1^t(f) + T_2^t(f) \end{aligned}$$

as observed earlier. Since this holds for all $f \in W^*$ it follows that the two functions are the same, yielding the first equation for τ. Proof of the second is similar via our second earlier observation.

Theorem 21. *Let V and W be vector spaces over the field F. Then*

1. *The map transpose $\tau : \operatorname{Hom}_F(V, W) \longrightarrow \operatorname{Hom}_F(W^*, V^*)$ is always one-to-one.*
2. *If V and W are finite dimensional, then τ is an isomorphism.*

Proof. Statement 1 implies statement 2 since the dimensions of the two vector spaces are equal by our earlier results as $\dim V = \dim V^*$ and $\dim W = \dim W^*$:

$$\begin{aligned} \dim \operatorname{Hom}_F(V, W) &= \dim V \cdot \dim W \\ \dim \operatorname{Hom}_F(W^*, V^*) &= \dim W^* \cdot \dim V^* \,. \end{aligned}$$

So we'll now verify the first part, which means we need to compute $\ker \tau$. For $T \in \operatorname{Hom}_F(V, W)$, $T \in \ker \tau$ means that $T^t(f) = f \circ T = 0$ for every $f \in W^*$. That is, $f(T(v)) = 0$ for all $v \in V$ or that f is 0 on $\operatorname{im} T$.

But by Proposition 15 this can only happen for all $f \in W^*$ if $\operatorname{im} T = 0$. That is, $T = 0$ and thus τ is one-to-one. □

Remark 22. It is also interesting to see how $\ker T^t$, $\operatorname{im} T^t$, $(\ker \mathrm{T})^\circ$ $(\operatorname{im} \mathrm{T})^\circ$ relate to one another. See Exercise 18 for details.

We now give an explicit version of the transpose in the case of finite dimensional vector spaces. Let V be a vector space with ordered basis $\mathcal{A} = \{ v_1 \ldots, v_n \}$ and let W be a vector space with ordered basis $\mathcal{B} = \{ w_1, \ldots, w_m \}$. For a linear transformation $T : V \longrightarrow W$ there exist scalars $a_{ij} \in F$ so that $T(v_j) = \sum_{i=1}^m a_{ij} w_i$. Then we have

$$[T]_{\mathcal{A},\mathcal{B}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in F^{m \times n} \,.$$

Now W^* has dual basis $\mathcal{A}^*$ and V^* has dual basis $\mathcal{B}^*$. We will compute the matrix of the transpose T^t with respect to these dual bases. That is, we need to determine scalars $b_{\ell,k} \in F$ so that $T^t(w_k^*) = \sum_{\ell=1}^n b_{\ell,k} v_\ell^*$. Thus we compute

$$\begin{aligned} \big(T^t(w_k^*)\big)(v_\ell) &= w_k^* \circ T(v_\ell) \\ &= w_k^*\big(\sum_{i=1}^m a_{i\ell} w_i\big) \\ &= 0 + a_{k\ell} \cdot 1 \end{aligned}$$

That is, we now have computed the coefficient of v_ℓ^* in $T(w_k^*)$ as $a_{k\ell}$. That is,

$$T^t(w_k^*) = \sum_{\ell=1}^n a_{k\ell} v_\ell^*$$

and hence

$$[T^t]_{\mathcal{B}^*,\mathcal{A}^*} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \ddots \\ a_{1n} & a_{2m} & \cdots & a_{mn} \end{bmatrix} \in F^{n\times m} \;;$$

that is, we have

$$[T^t]_{\mathcal{B}^*,\mathcal{A}^*} = [T]^t_{\mathcal{A},\mathcal{B}} \tag{2}$$

where the transpose on the right hand side (which we will denote by τ), is the transpose of matrices. In fact, this equation is the reason matrix transpose is defined as it is.

In summary, we have a commutative diagram with all horizontal and vertical arrows isomorphisms:

$$\begin{array}{ccc} \operatorname{Hom}_F(V,W) & \xrightarrow{\;\tau\;} & \operatorname{Hom}_F(W^*,V^*) \\ \Big\downarrow [\ \ ]_{\mathcal{A},\mathcal{B}} & & \Big\downarrow [\ \ ]_{\mathcal{B}^*,\mathcal{A}^*} \\ F^{m\times n} & \xrightarrow{\;\tau\;} & F^{n\times m} \end{array}$$

Let $T: V \longrightarrow W$ be a linear transformation. Then the rank of T is defined by $\operatorname{rank} T = \dim \operatorname{im} T$. In case $M \in F^{m\times n}$, then L_M; $F^{n\times 1} \longrightarrow F^{m\times 1}$ is the linear transformation given by $L_M(C) = MC$. Now $\operatorname{im} L_M = \{ MC \mid C \in F^{n\times 1} \}$ and if we write $M = (C_1, C_2, \ldots, C_n)$ for C_i the columns of M we have

$$M \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + \cdots + x_nC_n \; .$$

Thus $\operatorname{im} L_M \subseteq F^{m\times 1}$ is the column space of the matrix M, and $\operatorname{rank} L_m$ is the dimension of the column space of M, that is the column rank of M. If we use the standard bases $\mathcal{A}$ for $F^{n\times 1}$ and $\mathcal{B}$ for $F^{m\times 1}$, then

$$[L_M]_{\mathcal{A},\mathcal{B}} = M \; .$$

Applying the result from above, we have

$$[L_M^t]_{\mathcal{B}^*,\mathcal{A}^*} = M^t \; .$$

But transpose is an isomorphism so $\operatorname{rank} L_M = \operatorname{rank} L_M^t$, so the column rank of M equals the row rank of M.

Double Transpose

Let $T:\ V \longrightarrow W$ be a linear transformation. Upon taking the transpose we obtain a linear transformation $T^t:\ W^* \longrightarrow V^*$ and upon taking the transpose again obtain a linear transformation $(T^t)^t:\ V^{**} \longrightarrow W^{**}$. We write T^{tt} instead of $(T^t)^t$. In the diagram below, each evaluation map E should really be labelled to denote the vector space on which it is defined.

Lemma 23. *The diagram*

$$\begin{array}{ccc} V & \xrightarrow{\quad T \quad} & W \\ \Big\downarrow E & & \Big\downarrow E \\ V^{**} & \xrightarrow{\quad T^{tt} \quad} & W^{**} \end{array} .$$

commutes, i.e., $E \circ T = T^{tt} \circ E$.

Proof. The proof is an exercise in using the definitions (multiple times) and is left to the reader. See Exercise 28. □

Proposition 24. *The map* $\tau \circ \tau$: $\mathrm{Hom}_F(V,W) \longrightarrow \mathrm{Hom}_F(V^{**},W^{**})$ *is always one-to-one. It is an isomorphism if both* V *and* W *have finite dimension.*

This Proposition follows immediately from the corresponding result for transpose.

If V and W are finite dimensional vector spaces the maps E are isomorphisms. These isomorphisms allows us to consider T^{tt} (more precisely $E^{-1} \circ T^{tt} \circ E$) as a linear transformation from V to W. The lemma above asserts that after this identification $T^{tt} = T$.

Lattices

We will give a very brief description of what is called the *lattice* of subspaces of a vector space. We begin with the notion of a *partially ordered set.*

Definition 25. Let X be a non-empty set. A subset $\mathcal{O} \subseteq X \times X$ is called a *partial order* on X if it satisfies the following three properties:

1. *Reflexive:*
   For all $x \in X$, $(x,x) \in \mathcal{O}$.
2. *Anti-Symmetric:*
   If $(x,y) \in \mathcal{O}$ and $(y,x) \in \mathcal{O}$, then $x = y$.
3. *Transitive:*
   If $(x,y),(y,z) \in \mathcal{O}$, then $(x,z) \in \mathcal{O}$.

One typically denotes the relation by some symbol, such as $\preccurlyeq$, so that $x \preccurlyeq y$ means the same thing as $(x, y) \in \mathcal{O}$. The three conditions above then mean that $\preccurlyeq$ has the same sort of properties as $\le$ for real numbers or $\subseteq$ for sets. We call a set X with a partial order $\preccurlyeq$ a *partially ordered set.*

Definition 26. An element u of a partially ordered set X is an *upper bound* of a subset Y if $y \preccurlyeq u$ for all $y \in Y$. An upper bound u of Y is a *least upper bound* if $u \preccurlyeq v$ for every upper bound v of Y. If a least upper bound exists, then it is unique by the anti-symmetry property. It is denoted by $\sup Y$.

One defines *lower bound*, *greatest lower bound* and $\inf Y$ in a similar fashion.

Definition 27. A partially ordered set L is called a *lattice* if any two elements of L have a least upper bound and a greatest lower bound. A lattice is called *complete* if every subset has a least upper bound and a greatest lower bound.

If L is a lattice and $a, b \in L$, then one normally denotes $\sup\{a, b\}$ by $a \vee b$ ("join") and $\inf\{a, b\}$ by $a \wedge b$ ("meet").

Proposition 28. *Let L be a lattice. The following hold:*

L1. $a \vee b = b \vee a\,, \quad a \wedge b = b \wedge a\,.$

L2. $(a \vee b) \vee c = a \vee (b \vee c)\,, \quad (a \wedge b) \wedge c = a \wedge (b \wedge c)\,.$

L3. $a \vee a = a\,, \quad a \wedge a = a\,.$

L4. $(a \vee b) \wedge a = a\,, \quad (a \wedge b) \vee a = a\,.$

Conversely, if L is a set with two binary operations $\vee$ and $\wedge$ which satisfy L1, L2, L3, and L4, then there is a definition of a partial order $\preccurlyeq$ on L such that $a \vee b$, $a \wedge b$, are the sup *and* inf *of a and b in this lattice.*

As this proposition is not an integral part of our discussion, we will not prove it. One can either work out the proof oneself or consult one of the many textbooks on abstract algebra or lattice theory. Note that the axioms remain unchanged if we switch $\wedge$ and $\vee$. Thus the two binary operations satisfy the *principle of duality*, that is, for any statement S which can be deduced from the axioms, then the dual statment S° (where $\wedge$ and $\vee$ are switched) can be deduced as well.

Given a partially ordered set X one can reverse the ordering (i.e., use $\mathcal{O}^\circ = \{\,(y, x) \mid (x, y) \in \mathcal{O}\,\}$) to get a new partially ordered set, the dual of X denoted X°. Similarly, if L is a lattice, we obtain the dual lattice, L°.

An *isomorphism of lattices* is a function $\ell : L_1 \longrightarrow L_2$ from the lattice L_1 to the lattice L_2 which is one-to-one, onto and preserves the lattice structure $\ell(a \vee b) = \ell(a) \vee \ell(b)$ and $\ell(a \wedge b) = \ell(a) \wedge \ell(b)$.

Although there are many examples of lattices, we will only consider one:

Example 29 (The Lattice of Subspaces of a Vector Space)**.** Let V be a vector space over the field F. We denote by $\mathbf{Lat}(V)$ the set of all subspaces of V. First note that $\mathbf{Lat}(V)$ is a partially ordered set under $\subseteq$. It is easy to see that

$$\begin{aligned}\sup\{W_i \mid i \in I\} &= \operatorname{Span}\big(\{W_i \mid i \in I\}\big)\\ &= \sum_{i\in I} W_i\end{aligned}$$

and

$$\inf\{W_i \mid i \in I\} = \bigcap_{i\in I} W_i$$

for any collection of subspaces $\{W_i \mid i \in I\}$ of V. In particular we have for subspaces $W_1, W_2 \subseteq V$

$$\begin{aligned}W_1 \vee W_2 &= W_1 + W_2\\ W_1 \wedge W_2 &= W_1 \cap W_2 .\end{aligned}$$

Note that $\mathbf{Lat}(V)$ is a complete lattice.

Theorem 30. *Let V be a finite dimensional vector space over the field F. There is a natural duality between the vector spaces V and V^*:*

1. *Let $\mathbf{B}(V)$ denote the set of bases of V. Then $\mathcal{B} \mapsto \mathcal{B}^*$ gives a bijection: ${}^*: \mathbf{B}(V) \longrightarrow \mathbf{B}(V^*)$.*
2. *Let $\mathbf{Lat}(V)$ denote the lattice of subspaces of V. Then $W \mapsto \mathrm{W}^\circ$ gives a duality ${}^\circ: \mathbf{Lat}(V) \longrightarrow \mathbf{Lat}(V^*)$. That is, the inverse ${}^\circ: \mathbf{Lat}(V^*) \longrightarrow \mathbf{Lat}(V)$ gives a lattice isomorphism between $\mathbf{Lat}(V^*)$ and $\mathbf{Lat}(\mathrm{V})^\circ$*

Proof. The first statement was proved earlier (see Theorem 8). In Proposition 19 it was shown that $\mathrm{W}^{\circ\circ} = W$, that is, $^\circ$ is its own inverse. So the map $^\circ$ is one-to-one and onto.

To prove $^\circ$ gives a lattice isomorphism with the dual (preserving $\vee$ and $\wedge$) is the same as saying that $^\circ$ switches $\vee$ and $\wedge$ going from $\mathbf{Lat}(V)$ to $\mathbf{Lat}(V^*)$. This is just

$$\begin{aligned}(\mathrm{W}_1 + \mathrm{W}_2)^\circ &= \mathrm{W}_1^\circ \cap \mathrm{W}_2^\circ\\ (\mathrm{W}_1 \cap \mathrm{W}_2)^\circ &= \mathrm{W}_1^\circ + \mathrm{W}_2^\circ\end{aligned}$$

which is left as Exercise 20 for the reader. □

Remark 31. Calling $^\circ$ "its own inverse" is a bit sloppy as there are really two different maps.

Note in particular that $^\circ$ reverses the partial ordering, that is, it switches $\subseteq$ and $\supseteq$.

Geometry

Let V be a vector space over a field F and let $f \in V^*$ be a non-zero linear functional. The subspace $\ker f \subseteq V$ (which is not equal to V) is called a *hyperplane*. Alternatively it is referred to as having *codimension* 1. For a subspace $W \subseteq V$, we define $\operatorname{codim} W = \dim V/W$, so in the case of $\ker f$, $\operatorname{codim} \ker f = \dim V/\ker f = \dim F = 1$. An alternate way to think of codimension 1 is that $\ker f$ has a one-dimensional complementary subspace as is shown in the next lemma. Yet another way to think of a hyperplane is that it is a *maximal subspace* of V, that is, it is not equal to V and there are no subspace of V which properly contain it: If $W \subseteq V$ is maximal and $W \subseteq U \subseteq V$ then $U = W$ or $U = V$ are the only possibilities.

Lemma 32. *Let V be a vector space over the field F. Let $f \in V^*$ be a non-zero linear functional. Then for $v_0 \in V$ such that $f(v_0) \neq 0$, $V = \ker f \oplus Fv_0$.*

Proof. If $u \in \ker f \cap Fv_0$, then $u = av_0$ for some $a \in F$. As $f(u) = af(v_0)$, it must be the case that $a = 0$, that is, $u = 0$.

Let $f(v_0) = c \neq 0$. If $v \in V$ and $f(v) = b$, then $v_1 = v - bc^{-1}v_0$ and $f(v_1) = b - bc^{-1}c = 0$. Thus $v = v_1 + bc^{-1}v_0$ is in $\ker f + Fv_0$. □

If one wishes to think of a hyperplane W as a maximal subspace or as a subspace having codimension 1, then V/W is a vector space of dimension 1 (so isomorphic to F) and $p\colon V \longrightarrow V/W$ gives W as the kernel of a non-zero linear functional. Such a linear functional is unique up to multiplication by a non-zero scalar.

Lemma 33. *Let V be a vector space over F and let $f, g \in V^*$. Then g is a scalar multiple of f if and only if $\ker f \subseteq \ker g$. In particlar, if $\ker f = \ker g \neq V$, Then $g = af$ for some non-zero $a \in F$.*

Proof. Clearly if $g = af$, we have $\ker f \subseteq \ker g$ and if $f = 0$ then $g = 0$ as well. So assume $f \neq 0$ and $\ker f \subseteq \ker g$. If $g = 0$, then $g = 0 \cdot f$. Assume $g \neq 0$ as well. Then for $g(v_0) \neq 0$ we also have $f(v_0) \neq 0$. The linear functional $g(v_0)f - f(v_0)g$ will be 0 on $\ker f \subseteq \ker g$ as well as v_0 and hence is 0 on all of V. Thus $a = g(v_0)/f(v_0)$ is the required scalar. □

Proposition 34. *Let V be a vector space over F and let $f_1, \ldots, f_k, g \in V^*$. Then g is a linear combination of $f_1, \ldots, f_k$ if and only if $\bigcap_{i=1}^{k} \ker f_i \subseteq \ker g$.*

Proof. Clearly if g is a linear combination of the f_i, then $g(v) = 0$ for all $v \in V$ with $f_i(v) = 0$ for every i.

We prove the result by induction on k with $k = 1$ being the preceeding lemma. Write W for the intersection of the $\ker f_i$ and $U = \ker f_k$ and let $\overline{g}, \overline{f}_1, \ldots, \overline{f}_{k-1}$ denote the restrictions of $g, f_1, \ldots, f_{k-1}$ to U. They are linear functionals on U. For $u \in U$ such that $f_i(u) = 0$ for $i = 1, \ldots, k-1$, then $u \in W \subseteq \ker g$. Hence $\overline{g}(u) = 0$. By the induction hypothesis, $\overline{g}$ is a linear combination of the $\overline{f}_i$, $1 \leq i \leq k-1$: $\overline{g} = a_1\overline{f}_1 + \cdots + a_{k-1}\overline{f}_{k-1}$.

Now consider $g - a_1 f_1 - \cdots - a_{k-1} f_{k-1}$. It is 0 on U and by the preceeding lemma is a scalar multiple of f_k, say $a_k f_k$, Thus $g = a_1 f_1 + \cdots + a_k f_k$. □

Proposition 35. *Let V be a vector space over F of dimension n and let $W \subseteq V$ be a subspace of dimension m. Then W is the intersection of $n - m$ hyperplanes.*

Proof. We follow the same idea as that used in the proof of Theorem 13. Let $\{ e_1, \ldots, e_m \}$ be a basis for W and enlarge to a basis for V: $\left\{ e_1, \ldots, e_m, e_{m_1}, \ldots, e_n \right\}$. Then $\mathrm{W}^\circ = \mathrm{Span}_F \left\{ {e_{m+1}}^*, \ldots, {e_n}^* \right\}$ and $dim\, \mathrm{W}^\circ = n - m$. By Proposition 19

$$\begin{aligned} W &= \mathrm{W}^{\circ\circ} \\ &= \{ v \in V \mid f(v) = 0 \forall f \in \mathrm{W}^\circ \} \\ &= \left\{ v \in V \;\middle|\; {e_j}^*(v) = 0,\ j = m+1, \ldots, n \right\} \\ &= \bigcap_{j=m+1}^{n} \ker {e_j}^* . \end{aligned}$$

□

The basic idea geometrically is that for suitable hyperplanes (in "general position") each time one intersects with an additional hyperplane, the dimension of the resulting subspace goes down by exactly 1.

Exercises

DualSpace 1. If we have a pair $\mathcal{B} \subseteq V$ and $\mathcal{B}' \subseteq V^*$ which are dual, then show that $U = \bigcap_{f \in \mathcal{B}'} \ker f$ is a complement to $W = \operatorname{Span}(\mathcal{B})$ in V, i.e., V is the internal direct sum of U and W. Is this always true, or does it require that $\dim V$ be finite?

DualSpace 2. Let n be a positive integer.

a. Let $\operatorname{Tr} : F^{n\times n} \longrightarrow F$ denote the trace function on $n \times n$ matrices over the field F. Show $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any $A, B \in F^{n\times n}$. Show that Tr is never the zero linear functional.

b. Show that there do not exist $A, B \in F^{n\times n}$ for F the field of complex numbers, such that $AB - BA = I$. What happens for an arbitrary field?

c. Let $T : V \longrightarrow V$ be a linear transformation where V is a finite dimensional vector space over a field F. Choose a basis $\mathcal{B}$ for V and define $\operatorname{Tr}(T) = \operatorname{Tr}([\,T\,]_{\mathcal{B},\mathcal{B}})$. Show that the definition does not depend on the choice of basis $\mathcal{B}$.

d. Let $f : F^{n\times n} \longrightarrow F$ be a linear functional which satisfies $f(AB) = f(BA)$ for all $A, B \in F^{n\times n}$. Prove that $f = a \cdot \operatorname{Tr}$ for some scalar $a \in F$. Further show that if the characteristic of F is 0, then $f = \operatorname{Tr}$ precisely when $f(I) = n$. What happens if the characteristic is not 0? Can you find another way to decide if $f = \operatorname{Tr}$ by computing a single value?

e. Let $S = \operatorname{Span}_F(\{\, AB - BA \mid A, B \in F^{n\times n} \,\})$. Prove that $S = \ker \operatorname{Tr}$.
Hint: Compute the dimension of $\ker \operatorname{Tr}$, and find a basis for S using some well-known matrices.

f. Let A be a fixed $n \times n$ matrix with entries in F. Define $\operatorname{ad}_A : F^{n\times n} \longrightarrow F^{n\times n}$ by $\operatorname{ad}_A(B) = AB - BA$. Recall that Tr is a linear functional on $F^{n\times n}$, and so we may apply $\operatorname{ad}_A{}^t$ to it. What is $\operatorname{ad}_A{}^t(\operatorname{Tr})$?

g. Let $A \in \mathbb{R}^{n\times n}$. Show that $A = 0$ if and only if $A^t \cdot A = 0$.

DualSpace 3. Let V be a finite-dimensional vector space over the field F and let W be a subspace of V. If f is a linear functional on W, prove that there is a linear functional h on V such that $h(w) = f(w)$ for all $w \in W$.

DualSpace 4. Let F be a field with $\operatorname{char} F = 0$. Let $A \in F^{n\times n}$. Assume that A is not similar to any matrix with 0 as its $(1,1)$ entry. Prove that A must be a non-zero scalar matrix. [Hint: Use the assumption to find one vector in V^* and carefully extend to a basis.]

DualSpace 5. Let F be a field with $\operatorname{char} F = 0$. Let $A \in F^{n\times n}$. Assume that $\operatorname{Tr}(A) = 0$. Prove that there exist $X, Y \in F^{n\times n}$ with $A = XY - YX$. [Hint: You may as well assume A is not 0 because .... Also ok if $n = 1$ because .... Induct on n. Now A is not a scalar matrix because ...and thus the matrix ? has 0 as its $(1,1)$ entry. Apply the induction hypothesis to the lower right-hand corner A' of ? Further we can assume that the matrix X' has an inverse by replacing it by $cI + X'$ for some $c \in F$ (why?). Then ...]

DualSpace 6. Let F be a field. Let $C \in F^{n\times n}$. Define $T_C \in (F^{n\times n})^*$ by

$$T_C(A) = \mathrm{Tr}(AC)$$

and define

$$T:\ F^{n\times n} \longrightarrow (F^{n\times n})^*$$

by $T(C) = T_C$.

a. Show that T is a linear transformation.

b. Prove that T is one-to-one.

c. Prove that T is an isomorphism.

d. Given $f \in (F^{n\times n})^*$ find an explicit formula for the unique matrix $C \in F^{n\times n}$ for which $T(C) = f$.

DualSpace 7. Let F be a field. Let $A \in F^{n\times n}$. Assume the characteristic polynomial of A factors completely into linear factors in $F[x]$. Then $\mathrm{Tr}(A)$ is the sum of characteristic values of A (with appropriate multiplicities).

DualSpace 8. Let F be a finite-dimensional vector space over the field F and let $T : V \longrightarrow V$ be a linear transformation. Let c be a scalar and suppose there is a non-zero vector v in V such that $T(v) = cv$. Prove that there is a non-zero linear functional f on V such that $T^t(f) = cf$.

DualSpace 9. Let V be the vector space of all polynomials over the real numbers (considered as functions in the usual way). For real numbers $a < b$ define the linear functional γ on V by

$$\gamma(f) = \int_a^b f(t)dt\ .$$

Let $D : V \longrightarrow V$ denote differentiation. What is $D^t\gamma$?

DualSpace 10. Compute the change of basis matrix $P = P(\mathcal{A}, \mathcal{B})$ matrix in Example 11.

DualSpace 11. Let

$$0 \longrightarrow U \xrightarrow{S} V \xrightarrow{T} W \longrightarrow 0$$

be a short exact sequence of vector spaces over the field F. Show that

$$0 \longrightarrow W^* \xrightarrow{T^t} V^* \xrightarrow{S^t} U^* \longrightarrow 0$$

is a short exact sequence as well.

DualSpace 12. Let F be a field and let $s = a_i$ for $i = 1, \ldots, \infty$ be any sequence of elements of F. Define $f_s :\ F[x] \longrightarrow F$ on the usual basis for formal polynomials by $f_s(x^i) = a_i$. Show that $f_s \in F[x]^*$ and every element of the dual is given by such a sequence.

DualSpace 13. Let V be a vector space over the field F and let $W \subseteq V$ be a subspace. Make no assumptions on the field F nor the dimension of V.

a. Show that there is a natural isomorphism $\big(V/W\big)^* \longrightarrow \mathrm{W}^\circ$.

b. Show that there is a natural isomorphism $V^*/\,\mathrm{W}^\circ \longrightarrow W^*$.

c. Let $V = W_1 \oplus W_2$ be an internal direct sum of vector spaces. Then $\mathrm{W}_\mathrm{i}^\circ \subseteq V^*$. Give natural isomorphisms $W_i{}^* \approx \mathrm{W}_\mathrm{j}^\circ$ for $i \neq j$.

d. In the case V is finite dimentional, for each of the first two parts, derive an alternate proof of Theorem 13

DualSpace 14. Verify the following statements for duals of direct sums.

a. Let $V = V_1 \oplus \cdots \oplus V_n$ be an external direct sum of vector spaces. There is a natural isomorphism $V^* \approx V_1{}^* \oplus \cdots \oplus V_n{}^*$

b. Let $V = \bigoplus_{i=1}^\infty V_i$. Show that $V^* \approx \prod_{i=1}^\infty V_i$, and that $\dim V < \dim V^*$.

DualSpace 15. Let V be a vector space over the field F and let $W \subseteq V$ be a subspace. Make no assumptions on the field F nor the dimension of V. Define $\mathrm{Res} : V^* \longrightarrow W^*$ to be the restriction map; for $f \in V^*$ and $w \in W$, define $\mathrm{Res}(f)(w) = f(w)$. Show

a. $\mathrm{Res}(f) \in W^*$,

b. Res is a linear transformation, and

c. Res is onto.

d. Compute $\ker \mathrm{Res}$.

DualSpace 16. Let F be a finite field (such as, for example, $\mathbb{F}_p$). Let $0 \leq m \leq n$ be integers. Show that the number of subspaces of $V = F^n$ of dimension m is exactly the same as the number of subspaces of dimension $n - m$.

DualSpace 17. Let $W_1, W_2 \subseteq V$ be subspaces of V over F. Make no assumptions on F nor on the dimension of V.

1. If $W_1 \neq W_2$ show that there is a linear functional $f \in V^*$ such that f is 0 on one of the W_i but not the other.

2. Show that in all cases $W_1 = W_2$ if and only if $\mathrm{W}_1^\circ = \mathrm{W}_2^\circ$.

DualSpace 18. Let V and W be vector spaces over the field F and let $T : V \longrightarrow W$ be a linear transformation.

a. $\ker T^t = (\mathrm{im\,T})^\circ$.

b. If V and W have finite dimension, then

   i. $\operatorname{rank} T^t = \operatorname{rank} T$.

   ii. $\operatorname{im} T^t = (\ker \mathrm{T})^\circ$.

c. What happens in the previous part in the case of infinite dimensional vector spaces?

DualSpace 19. Let F be a field.

a. Assume F is an infinite field and let V be a finite-dimensional vector space over F. If $v_1, \ldots, v_m$ are finitely many vectors in V, each non-zero, prove that there exists a linear functional f on V such that $f(v_i) \neq 0$ for all i.

b. Describe precisely what happens if F is finite with q elements (i.e., relate n to q).

DualSpace 20. Let V be a finite dimensional vector space over the field F. Let W_1 and W_2 be subspaces of V.

a. Show that $(\mathrm{W}_1 + \mathrm{W}_2)^\circ = \mathrm{W}_1^\circ \cap \mathrm{W}_2^\circ$.

b. Show that $(\mathrm{W}_1 \cap \mathrm{W}_2)^\circ = \mathrm{W}_1^\circ + \mathrm{W}_2^\circ$.

c. Determine what happens in each case if V has infinite dimension.

DualSpace 21. Let V be a non-zero vector space over the field F. Let f, g be two linear functionals on V. Show that if $|F| > 2$, then $\theta(v) = f(v)\, g(v)$ cannot be a linear functional unless at least one of f or g is 0. Is it possible that θ is a linear functional in the case where F has two elements?

DualSpace 22. Let F be a field and X a non-empty set. Let $V = F^X$ denotes the vector space of all functions from X to F. If W is any m-dimensional subspace of V, show that there exist elements $x_1, \ldots, x_m$ in X and functions $f_1, \ldots, f_m$ in W with $f_i(x_j) = \delta_{ij}$.

DualSpace 23. Let n be a positive integer and let F be a field whose characteristic is either 0 or greater than $2n$.

a. Let $\mathcal{P}_n \subseteq F[x]$ be the vector space of polynomials of degree less than n. Let $\mathcal{B} = \left\{ E_{-1}, E_{-2}, \ldots, E_{-n} \right\} \in {\mathcal{P}_n}^*$ be the evaluation maps for $\{ -1, -2, \ldots, -n \} \in F$. Give the basis $\mathcal{A} = \{ p_1, \ldots, p_n \} \subseteq \mathcal{P}_n$ which is dual to $\mathcal{B}$.

b. Let $\mathcal{C} = \left\{ \frac{1}{x+1}, \frac{1}{x+2}, \ldots, \frac{1}{x+n} \right\} \subseteq F(x)$ where $F(x)$ is the field of fractions of $F[x]$. Let $\mathcal{Q}_n = \operatorname{Span}_F(\mathcal{C}) \subseteq F(x)$. Give a natural isomorphism $m : \mathcal{Q}_n \longrightarrow \mathcal{P}_n$ and conclude that $\mathcal{Q}_n$ has dimension n and that $\mathcal{C}$ is a basis. [Hint: What does $\pi(x) = (x+1)(x+2)\cdots(x+n) = (x+n)^{(n)}$ have to do with $\mathcal{P}_n$? [$(x+n)^{(n)}$ is notation of example in the section on the "Matrix of a Linear Transformation")].

c. Define E_i for $i \in \{1, 2, \ldots, n\}$ by

$$E_i\left(\frac{1}{x+j}\right) = \frac{1}{i+j}.$$

Show that $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ is a basis for ${\mathcal{Q}_n}^*$ and find the basis $\mathcal{D} = \{f_1, f_2, \ldots, f_n\}$ of $\mathcal{Q}_n$ to which it is dual. If $m(f_i) = q_i \in \mathcal{P}_n$ give a formula for f_i explicitly and determine q_i. ??

d. Compute the change of basis matrix $P = P(\mathcal{C}, \mathcal{D})$. This part should be easy. Compute $P^{-1} = P(\mathcal{D}, \mathcal{C})$. This is more interesting and should answer a hard question you've seen before.

DualSpace 24. Prove Proposition 28.

DualSpace 25. Let U and V be arbitrary vector spaces over the field F. Explicitly give a natural isomorphism $(U \oplus V)^* \longrightarrow U^* \oplus V^*$ and its inverse. Include complete proofs.

DualSpace 26. Consider the linear transformation Der, X on the vector space $F[x]$ defined as follows:

$$\mathrm{Der}(x^n) = nx^{n-1} \qquad X(x^n) = x^{n+1}.$$

a. Show that these linear transformations satisfy $\mathrm{Der} \circ X - X \circ \mathrm{Der} = \mathrm{Id}$, i.e., for any $f \in F[x]$ we have $\mathrm{Der}(X(f)) - X(\mathrm{Der}(f)) = f$.

b. Construct analogs of the linear transformations Der and X, which act on the vector space $C^\infty(\mathbb{R})$, which also satisfy the equation $\mathrm{Der} \circ X - X \circ \mathrm{Der} = \mathrm{Id}$.

c. Construct analogs $\tilde{D}$ and $\tilde{X}$ of the linear transformations Der and X, which act on the vector space $C(\mathbb{R})$ and satisfy the equation $\tilde{D} \circ \tilde{X} - \tilde{X} \circ \tilde{D} = \mathrm{Id}$.
Hint: You can not use any form of differentiation.

d. Does there exist a finite dimensional vector space V over a field F and two linear transformations D and X from V to itself, such that $D \circ X - X \circ D = \mathrm{Id}$.
Hint: What is $\mathrm{char}\, F$?

e. Does there exist a finite dimensional vector space V over a field F and two invertible linear transformations from V to itself, such that $D \circ X - X \circ D = \mathrm{Id}$.

DualSpace 27. A linear transformation $D \in \mathrm{Hom}_F(V, V)$ is called *nilpotent* if there exists some N such that $D^N = 0$. Here D^k denote the composition of D with itself k times, also $D^0 = \mathrm{Id}$. The transformation is called *locally nilpotent* if for every vector $v \in V$ there exists n_v such that $D^{n_v}(v) = 0$.

a. Show that if V is finite dimensional then any locally nilpotent transformation is also nilpotent.

b. Show that the differential operator $\text{Der} : F[x] \longrightarrow F[x]$ from the previous exercise defined by $\text{Der}(x^n) = nx^{n-1}$ is locally nilpotent.

c. Show that $\text{Der} : F[x] \longrightarrow F[x]$ (from part b.) is nilpotent if and only if $\operatorname{char} F > 0$.

d. Let D be a locally nilpotent transformation. Let π_D be the map from $F[[t]]$ to $\text{Hom}_F(V, V)$ defined as follows

$$\left(\pi_D(\sum a_i t^i)\right)(v) = \sum a_i D^i(v)$$

Verify that this map is well defined, i.e., all infinite sums make sense.

e. Show that π_D is an algebra homomorphism, i.e.

$$\pi_D(f+g) = \pi_D(f) + \pi_D(g) \quad \pi_D(\lambda f) = \lambda \pi_D(f) \quad \pi_D(f \cdot g) = \pi_D(f) \circ \pi_D(g) \ .$$

f. Let Cen_D denote the set of all linear transformations $T \in \text{Hom}_F(V, V)$ which commute with D, i.e.

$$\text{Cen}_D = \{ T \in \text{Hom}_F(V, V) \mid T \circ D = D \circ T \}$$

Show that Cen_D is a subspace of $\text{Hom}_F(V, V)$.

g. Show that if D is locally nilpotent then $\operatorname{im} \pi_D \subset \text{Cen}_D$. **Hint:** This follows immediately from part e.

h. Show $\operatorname{im} \pi_{\text{Der}} \subset \text{Cen}_{\text{Der}}$, i.e., any operator which commutes with Der can be expressed as a formal power series. **Hint:** Look at the constant term of $T(x^n)$ for $T \in \text{Cen}_{\text{Der}}$.

i. Construct an example of a locally nilpotent derivation, such that $\operatorname{im} \pi_{\text{Der}}$ is a proper subspace of Cen_{Der}.

j. Find a necessary and sufficient conditions for $\operatorname{im} \pi_{\text{Der}} = \text{Cen}_{\text{Der}}$. **Hint:** Look at $\ker D$.

DualSpace 28. . Give a proof for Lemma 23

DualSpace 29. Let V be a vector space over the field F. We demonstrate a method to construct a set which is dual to a given linearly independent subset of V^* by using Proposition 34. Assume that V has countable dimension (i.e., the basis can be put into one-to-one correspondence with the positive integers). We prove that V^* has uncountable dimension.

Here is an outline of the exercise: Assume that $\mathcal{F}$ is a countable basis for V^*. Construct a linearly independent subset $\mathcal{A}$ of V which is dual to $\mathcal{F}$, and show that in fact it must be a basis for V. But this contradicts the fact (proved earlier) that the dual ($\mathcal{F}$) of an infinite basis ($\mathcal{A}$) of V cannot span V^*.

a. Let $\mathcal{F} = \{ f_1, f_2, \dots \}$ be a sequence of linearly independent elements in V^*. Show that there exists a non-zero $v_1 \in V$ with $f_1(v_1) = 1$.

b. Show that there exists an element $v_2 \in V$ with $f_2(v_2) = 1$ and $f_1(v_2) = 0$.

c. Give an induction argument to show that for any positive k there exist vectors $v_1, \dots, v_k \in V$ with $f_i(v_i) = 1$ and $f_i(v_j) = 0$ for $j < i$. Show that the set of v_i is linearly independent.

d. Let $\mathcal{A}$ be the infinite set of v_i constructed inductively by this process. Assume that $\mathcal{F}$ is a basis for V^*. Let $\mathcal{A} \subseteq \mathcal{B}$ be a basis for V containing $\mathcal{A}$. If $v \in \mathcal{B}$ is not in $\mathcal{A}$, then let v^* in V^* be the linear functional that is 1 on v and 0 on the other elements of $\mathcal{B}$. Show that v^* is not a finite linear combination of the elements of $\mathcal{F}$.

e. Conclude that $\mathcal{A}$ must be a basis for V. Let $\theta \in V^*$ be the linear functional that is 1 on every element of $\mathcal{A}$. Show that θ is not in the span of $\mathcal{F}$. Conclude that $\mathcal{F}$ is not a basis and hence that V^* does not have countable dimension.

f. For a fixed k show that by modifing the v_i one can even construct a set $u_1, \dots, u_k$ with $f_i(u_j) = \delta_{ij}$.

DualSpace 30. Assume that F is a field whose characteristic is not 2. Let V be the vector space of all polynomials of degree less than or equal to three over F. Define the linear functionals L_i, $i = 1, 2, 3, 4$ by

$$\begin{aligned} L_1(f) &= f(0) \\ L_2(f) &= f(1) \\ L_3(f) &= (Df)(0) \\ L_4(f) &= (\Delta f)(1) \end{aligned}$$

where D denotes the derivative and Δ denotes the difference operator (defined by $\Delta(f(x)) = f(x+1) - f(x)$).

a. Prove that these four linear functionals form a basis $\mathcal{B}^*$ for V^* and find the basis $\mathcal{B}$ of V to which it is dual.

b. Express 1, x, x^2, x^3 in terms of $\mathcal{B}$.

c. Let W be the subspace of V spanned by 1 and x. Find a basis for W° expressed in terms of $\mathcal{B}^*$.

d. Assuming that F is the field of real numbers, express the linear functional "the definite integral of f between 0 and 1" in terms of $\mathcal{B}^*$.

DualSpace 31. Let $\mathbb{F}_q$ be a finite field with q elements and having characteristic p. Prove the assertions you make to answer the following questions.

a. Let m, n be positive integers. Give a formula for the number of elements in $\mathbb{F}_q^n$ and $\mathbb{F}_q^{m\times n}$.

b. Give a formula for the number of different ordered bases of $\mathbb{F}_q^n$.

c. Give a formula for the number of invertible matrices in $\mathbb{F}_q^{n\times n}$.

d. Give a formula for the number of ordered, linearly independent sequences of vectors with m elements in $\mathbb{F}_q^n$.

e. Give a formula for the number of matrices of $\mathbb{F}_q^{m\times n}$ with rank m.

f. For a non-negative integer k determine the number of subspaces of dimension k of $\mathbb{F}_q^n$.

g. For a non-negative integer k determine the number of ordered sequences of k vectors in $\mathbb{F}_q^n$ that span.

DualSpace 32. Let U, V and W be vector spaces over the field F.

a. Let $T : U \longrightarrow V$ and $S : V \longrightarrow W$ be linear transformations. Give a formula for $(S \circ T)^t$ and verify that it is correct.

b. If the vector spaces all have finite dimension, choose bases $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for U, V and W having m, n and p elements, respectively. Construct a commutative diagram with $\mathrm{Hom}_F(V,W) \times \mathrm{Hom}_F(U,V) \longrightarrow F^{p\times n} \times F^{n\times m}$ as its top row and $\mathrm{Hom}_F(U,W) \longrightarrow F^{p\times m}$ as its bottom row. The two horizontal arrows should be isomorphisms and the two vertical arrows should be "composition" and "matrix multiplication", respectively. (See pages 2 and 6 of the section "The Matrix of a Linear Transformation" for things very closely related to this.)

c. Continue with the notation of the preceeding part with finite bases $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for U, V and W. Construct the diagram of the previous part and a similar one for the dual spaces. Construct three diagrams of the type given on page 11 of this section. Use these three to "glue" together the two just constructed into a larger diagram. Which faces of the diagram commute from results already proven? What can you conclude about the remaining face(s)?

DualSpace 33. Let $\mathcal{P}_n$ be the vector space of all polynomials of degree less than n over a field F of characteristic 0. Consider the linear transformation $\Theta : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ defined by

$$\Theta\big(p(x)\big) = p(x+1) - 2p(x) + p(x-1)$$

Construct a basis $\mathcal{A}$ of $\mathcal{P}_n$ such that the matrix $[\,\Theta\,]_{\mathcal{A}}$ of the linear transformation Θ with respect to this basis has exactly $n-2$ non-zero coefficients. Explain what changes if the characteristic of the field F is finite.

DualSpace 34. a. Prove that the F-algebra $\mathrm{Hom}_F(V,V)$ is simple (i.e., it does not contain any non-trivial 2-sided ideals) if V is finite dimensional vector space over F.

b. Construct a non-trivial 2-sided ideal if $\dim_F V$ is infinite.

c. Generalize to give a tower distinct of 2-sided ideals in case $\dim_F V > \aleph_0$.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

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Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.