Quotient Spaces

Let V be a vector space over the field F and let $W \subseteq V$ be a subspace. We now contruct a new vector space over F with a method analogous to that used for constructing the integers modulo n. We temporarily define a "congrunce relation" using the subspace W. We will say that two vectors $x, y \in V$ are *congruent modulo* W, and write $x \equiv y \mod W$ when $x - y \in W$. This gives an equivalence relation on V:

r. $x \equiv x \mod W$,

s. $x \equiv y \mod W$ implies $y \equiv x \mod W$,

t. $x \equiv y \mod W$ and $y \equiv z \mod W$ implies $x \equiv z \mod W$.

The relation being reflexive is just $x - x \in W$ which holds since 0 is in a subspace. The relation is symmetric since $x - y \in W$ implies $y - x \in W$ since a subspace is closed under negation.
The transitivity of the relation holds as $x - y \in W$ and $y - z \in W$ implies that $x - z = (x - y) + (y - z) \in W$, since any subspace is closed under addition.

Next note that the equivalence class of $v \in V$ under this equivalence relation is just the set

$$\begin{aligned} \text{class}(v) &= \{ u \in V \mid u \equiv v \mod W \} \\ &= \{ v + w \mid w \in W \} \\ &= v + W \end{aligned}$$

This follows as $u \equiv v \mod W$ means $u - v \in W$, that is $u - v = w$ for some $w \in W$. Thus $u = v + w$. Conversely any such u is congruent to $v \mod W$. We this define this set to be $v + W$ in analogy with our earlier notation for the sum of two subspaces. Such a subset is called a *coset* of W.

Recall that an equivalence relation on a set decomposes the set into a disjoint union of the equivalence classes. The quotient set is the set whose elements are these pieces. We now look at the set of these pieces:

We write V/W for the set of the equivalence classes of $V \mod W$:

$$V/W = \{ v + W \mid v \in V \} .$$

We now make this set into a vector space over F by defining addition and scalar multiplication:

$$\begin{aligned} (v_1 + W) + (v_2 + W) &= (v_1 + v_2) + W \\ a(v + W) &= (av) + W \end{aligned}$$

We must first show that these definitions make sense, or as a mathematician says, "that they are well-defined" (see the remark near the end of the handout on equivalence relations). There is a problem – the v in the $v+W$ may be any element of the set $v+W$, that is any representative of the coset. That is, any element $v+w$ for any $w \in W$. Note that then $v+W = v'+W$ if and only if $v-v' \in W$ (if and only if $v \equiv v' \mod W$).

Hence

$$\begin{array}{rcccc} v_1 + W & = & v_1' + W & \Longleftrightarrow & v_1 - v_1' \in W \\ v_2 + W & = & v_2' + W & \Longleftrightarrow & v_2 - v_2' \in W \\ \text{and adding gives} & & & & \\ (v_1+v_2) + W & = & (v_1'+v_2') + W & \Longleftrightarrow & (v_1+v_2) - (v_1'+v_2') \in W\ . \end{array}$$

A similar argument works for scalar multiplication:

$$\begin{array}{rcccc} v + W & = & v' + W & \Longleftrightarrow & v - v' \in W \\ \text{and multiplying by } a \text{ gives} & & & & \\ (av) + W & = & (av') + W & \Longleftrightarrow & av - av' \in W\ . \end{array}$$

It is now easy to check that V/W is a vector space over F using these definitions. For example, since $0 \in V$ is the zero in V, $0 + W = W$ is the zero of V/W. It then follows that $-(v+W) = (-v) + W$, since it behaves the correct way, and by uniqueness must be the only element that does.

There are several axioms to check, but the philosophy is simple: The corresponding result holds for V/W by using the definition and putting a number of instances of "$+W$" on the axiom for V. For example

$$\begin{aligned} a \cdot (b \cdot (v+W)) &= a \cdot ((bv) + W) \\ &= (a(bv)) + W \\ &= ((ab)v) + W \\ &= (ab) \cdot (v+W) \end{aligned}$$

Example 1. Consider $\mathbb{R}^2$, the Euclidean plane, and W a one-dimensional subspace (geometrically a line passing through the origin). It is easy to check that for $v \in \mathbb{R}^2$, $v+W$ is the line parallel to W that passes through v. The quotient space $\mathbb{R}^2/W$ consists of the set of all lines in $\mathbb{R}^2$ which are parallel to W.

The same sort of thing happens in higher dimensions (e.g., for W a plane through the origin in $\mathbb{R}^3$ and $v \in \mathbb{R}^3$, $v+W$ is just the plane parallel to W which contains v). For that reason these equivalence classes are sometime called *affine subspaces*. The quotient space consists of the set of all such affine subspaces parallel to the given W.

We next look at the function

$$p: \ V \longrightarrow V/W$$

which sends each vector in V to the coset (equivalence class) in which it lies, $p(v) = v + W$. Note that

- p is onto
- p is a linear transformation
- $\ker p = W$.

The first follows from the definition of V/W : it is the set of all such $p(v) = v + W$. The second is equivalent to the definition of addition and scalar multiplication in V/W :

$$\begin{aligned} p(u+v) &= (u+v)+W \\ p(u)+p(v) &= (u+W)+(v+W) \end{aligned}$$

and

$$\begin{aligned} p(av) &= (av)+W \\ ap(v) &= a(v+W)\ . \end{aligned}$$

For the last we have $\ker p = \{\, v \in V \mid p(v) = 0 \,\}$, but $p(v) = v + W = 0 + W$ just means that $v = v - 0 \in W$.

We finally consider the special role that this linear transformation $p :\ V \longrightarrow V/W$ plays for quotient spaces.

Theorem 2 (Universal Mapping Property for Quotient Spaces)**.** *Let V and U be vector spaces over the field F and W a subspace of V . For every linear transformation $t :\ V \longrightarrow U$ which satisfies $W \subseteq \ker t$, there exists a unique linear transformation $T :\ V/W \longrightarrow U$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "p"] \arrow[dr, "t"'] & V/W \arrow[d, dotted, "T"] \\ & U \end{tikzcd}$$

that is, $T \circ p = t$.

Remark 3. The term "commutes" will more generally mean that if one has a diagram with a number of objects (e.g., vector spaces, fields, or whatever) with a number of functions (arrows) between some of the objects, we say that the diagram *commutes* if for every pair of objects which can be connected by a path (all arrows pointing the same direction so that composition of the functions is possible) in more than one way, the compositions of the functions along the various possible paths must always be equal.

Hence an alternative way of stating the preceding theorm is

If the left triangle below commutes, then there exists a unique linear transformation $T:\ V/W \longrightarrow U$ making the right trangle commute.

$$\begin{tikzcd}
W \arrow[r] \arrow[dr, "0"'] & V \arrow[r, "p"] \arrow[d, "t"] & V/W \arrow[dl, dotted, "T"] \\
& U &
\end{tikzcd}$$

Proof. In outline, in almost all cases, proofs of universal mapping properties take the following form: first show uniqueness, next use the result of uniqueness (typically a formula) to show existence of the sought-after function, and finally verify that the function just constructed has all of the right properties.

Uniqueness: We show that there is only one linear transformation T that satisfies the equation $T \circ p = t$. If such a T exists, we have

$$T(v+W) = T(p(v)) = t(v) \tag{1}$$

by the definition of p and the definition of composition of functions. That is, the value of T is completely determined by the given function t.

Existence: The result of the previous part (equation (1)) is now used to define T. However, we must show that T is well-defined, since more than one vector can represent the coset $v+W$. If $v+W = v'+W$ then $v-v' \in W$, and we must show that $t(v)$ and $t(v')$ yield the same thing. But $v-v' \in W \subseteq \ker t$ so $t(v-v')=0$ and as t is a linear transformation we do indeed have $t(v)=t(v')$, the value defined for T on the coset $v+W$.

Properties: First, T is a linear transformation:

$$\begin{aligned}
T((u+W)+(v+W)) &= T((u+v)+W) \\
&= t(u+v) \\
T(u+W)+T(v+W) &= t(u)+t(v)
\end{aligned}$$

and

$$\begin{aligned}
T(a(v+W)) &= T((av)+W) \\
&= t(av) \\
aT(v+W) &= at(v).
\end{aligned}$$

In each case the last two are equal since t is a linear transformation. Finally note that $T \circ p = t$ holds since the equation (1) used to define T was exactly that condition. $\square$

Remark 4. Upon comparing the Universal Mapping Property(UMP) for Quotient Sets to the UMP for Quotient Spaces, one sees that the condition 't is constant on the fibers of $p\colon V \to V/W$' has been replaced by '$W \subseteq \ker t$'. Because t is a linear transformation, verification of the condition for a single fiber suffices: t is constant on all fibers of p if and only if $W \subseteq \ker t = t^{-1}(0) =$ the fiber over 0. Verify and explain!

Remark 5. If we had already proven Proposition 1 (the Universal Mapping Property for Quotient Sets) of the handout on Equivalence Relations, we could simply quote that result, as well as Remark 4 to obtain the existence and uniqueness of the function T above. We would then only need to verify that T is a linear transformation.

Remark 6. A universal mapping property such as the one just described always gives a one-to-one correspondence (bijection) between two collections of functions. In this case they are both sets of linear transformations (in fact, they are vector spaces over F):

$$\{\, t \in \operatorname{Hom}_F(V,U) \mid t(W) = 0 \,\} \longleftrightarrow \operatorname{Hom}_F(V/W,U)$$

where we write $\operatorname{Hom}_F(V,U)$ for the vector space of linear transformations from V to U.

The one-to-one arrow to the right is given by the theorem (existence and uniqueness). Further, it is onto, since given any $T \in \operatorname{Hom}_F(V/W,U)$ we can define the required t by $t = T \circ p$ (this gives the arrow pointing to the left). In fact, the given bijection is an isomorphism of vector spaces.

Exercises

QuoSpace 1. Let V be a vector space over the field F. Let X be a non-empty subset of V with the property that the set

$$Y = \{ x_1 - x_2 \mid x_i \in X \}$$

is closed under addition and scalar multiplication by elements of F. Show that this is equivalent to X being a coset of some subspace of V. What is Y in terms of this description?

QuoSpace 2. Let V be a vector space over the field F. Let v_1 and v_2 be two distinct elements of V. The *line through* v_1 *and* v_2 is the set $L \subseteq V$ given by

$$L = \{ rv_1 + sv_2 \mid r, s \in F,\ r + s = 1 \} \ .$$

Let X be a non-empty subset of V which contains all lines through two distinct elements of X. Show that X is a coset of some subspace of V. Describe the subspace. Relate this exercise to the preceding exercise.
Cosets of subspace are sometimes called *affine subspaces* of V in view of their geometric description. *Affine* indicates that geometrically the set is a translate of an actual subspace.

QuoSpace 3. Let V be a vector space over the field F. Give a careful description of the following quotient spaces and an isomorphism with a more naturally described vector space.

a.
$$Q_1 = V/0$$
where 0 denotes the zero subspace of V.

b.
$$Q_2 = V/V \ .$$

QuoSpace 4. Let U, V, W be vector spaces over a field F with $W \subseteq V$ a subspace. Let $T : V \longrightarrow U$ be a linear transformation whose kernel contains W. Show that there is a well-defined linear transformation $S : V/W \longrightarrow U$ given by $S(v + W) = T(v)$. Note that $T = S \circ \pi$ where $\pi : V \longrightarrow V/W$ is the natural quotient map $\pi(v) = v + W$. Is S the only linear map satisfying this property?

QuoSpace 5. Let U and V be vector spaces over the field F. Let $T : V \longrightarrow U$ be a linear transformation with kernel Z. Show that the image of T is isomorphic to the quotient space V/Z. The isomorphism given is to be natural, that is, not depend on a choice of basis.

QuoSpace 6. In this problem, you will show that the universal mapping property characterizes the quotient space up to unique isomorphism. Let $W \subseteq V$ be vector spaces over a field F. Suppose we have a vector space Q and a linear transformation

$\pi_Q : V \longrightarrow Q$ with $W \subseteq \ker(\pi_Q)$, and they have the property that for any linear transformation $T : V \longrightarrow U$ with $W \subseteq \ker(T)$ (where U is any vector space), there exists a unique linear transformation $T_Q : Q \longrightarrow U$ such that $T = T_Q \circ \pi_Q$. Prove that Q is isomorphic to the quotient space V/W.
(Hint: Use a universal mapping property 4 times!)

QuoSpace 7. Let $W \subseteq V$ be vector spaces over a field F and $T : V \longrightarrow V$ be a linear transformation such that $T(W) \subseteq W$. Then T induces a linear transformation $\overline{T} : V/W \longrightarrow V/W$ given by $\overline{T}(v + W) = T(v) + W$.

a. Show $\overline{T}$ is a well-defined linear transformation on V/W. If V is finite-dimensional and T an isomorphism, prove that $\overline{T}$ is an isomorphism.

b. Is (a.) necessarily true if V is not assumed to be finite-dimensional? Prove or provide a counterexample.

QuoSpace 8. Let V be a vector space over F and $W \subseteq V$ a subspace. Let $p : V \longrightarrow V/W$ be the linear transformation given by $p(v) = v + W$. Let X be the set of all subspaces of V which contain W. Let Y be the set of all subspaces of V/W. Prove that p induces a one-to-one correspondence between these two sets as follows:

- $L \in X$ is mapped to $p(L) = \{\, p(v) \mid v \in L \,\}$.
- $M \in Y$ is mapped to $p^{-1}(M) = \{\, v \in V \mid p(v) \in M \,\}$.

That is, show that these two are inverse correspondences.

QuoSpace 9. Let U and V be vector spaces over the field F, and $W \subseteq V$ a subspace. No assumptions on dimensions. All isomorphisms given are to be natural.

a. Let $A = \{\, T \in \mathrm{Hom}_F(V, U) \mid W \subseteq \ker(T) \,\}$. Show that A is a subspace of $\mathrm{Hom}_F(V, U)$. Prove $A \approx \mathrm{Hom}_F(V/W, U)$ and $\mathrm{Hom}_F(V, U)/A \approx \mathrm{Hom}_F(W, U)$ by constructing explicit isomorphisms.

b. Let $B = \{\, T \in \mathrm{Hom}_F(U, V) \mid \mathrm{im}(T) \subseteq W \,\}$. Show that B is a subspace of $\mathrm{Hom}_F(U, V)$. Prove $B \approx \mathrm{Hom}_F(U, W)$ and $\mathrm{Hom}_F(U, V)/B \approx \mathrm{Hom}_F(U, V/W)$ by constructing explicit isomorphisms.

QuoSpace 10. Let V be vector spaces over a field F and let W be a subspace. By Exercise 8, we know that the subspaces of V/W are in one-to-one correspondence with the subspaces of V which contain W. Now suppose U is a subspace of V which contains W, so that U/W is a subspace of the vector space V/W. Give a description of the vector space $(V/W)/(U/W)$ in terms of yet another quotient.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

and

Yuri Berest.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.