Matrices and Row Reduction

Exercises

Determinants are not necessary (and should NOT be used) to solve these problems (except where a determinant is specifically requested).

RowRed 1. Let

$$A = \begin{bmatrix} 0 & 0 & 3 & 3 \\ 4 & -8 & 3 & -1 \\ -1 & 2 & 2 & 3 \\ 1 & -2 & 2 & 1 \end{bmatrix}$$

where the entries lie in the field $\mathbb{Q}$. For which (columns) (y_1, y_2, y_3, y_4) does the matrix equation $AX = Y$ have a solution?

RowRed 2. Let

$$A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

where the entries lie in the field $\mathbb{Q}$. For which (columns) (y_1, y_2, y_3, y_4) does the matrix equation $AX = Y$ have a solution?

RowRed 3. Prove that the interchange of two rows of a matrix can be accomplished by a (short) finite sequence of elementary row operations of the other two types. Write a matrix equation which implements this explicitly in the 2×2 case.

RowRed 4. An $n \times n$ matrix A is called *upper-triangular* if $A_{ij} = 0$ for $i > j$, that is, if every entry below the main diagonal is 0. Prove that an upper-triangular (square) matrix with entries in a field is invertible if and only if every entry on its main diagonal is different from 0.

RowRed 5. Prove the following: If A is an $m \times n$ matrix, B is an $n \times m$ matrix and $n < m$, then AB is not invertible. All entries are assumed to be in a field.

RowRed 6. Let A be an $n \times n$ (square) matrix with entries in a field. Prove the following two statements:

(a) If A is invertible and $AB = 0$ for some $n \times n$ matrix B, then $B = 0$.

(b) If A is not invertible, then there exists an $n \times n$ matrix B such that $AB = 0$ but $B \neq 0$.

RowRed 7. Let A be an $m \times n$ matrix with entries in a field. Show that by means of a finite number of elementary row and/or column operations one can go from A to a matrix R which is both 'row-reduced echelon' and 'column-reduced echelon' i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1$, $1 \leq i \leq r$, $R_{ii} = 0$ if $i > r$. Show that $R = PAQ$, where P is an invertible $m \times m$ matrix and Q is an invertible $n \times n$ matrix.

RowRed 8. Let

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

be a $2n \times 2n$ block matrix where A, B, C and D are $n \times n$ matrices with entries in a field. Assume that the matrix A is invertible. Show that M is invertible if and only if the $n \times n$ matrix $D - CA^{-1}B$ is invertible.

RowRed 9. Let

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}.$$

be a 2×2 matrix with entries in a field. When it is possible to find 2×2 matrices A and B such that $C = AB - BA$. Prove that such matrices can be found if and only if $C_{11} + C_{22} = 0$.

RowRed 10. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

a. If the entries of A are in the field $\mathbb{Q}$, row reduce to the row reduced echelon form and compute the row rank of A.

b. If the entries of A are in the field $\mathbb{F}_2$, row reduce to the row reduced echelon form and compute the row rank of A.

RowRed 11. a. Explicitly describe all possible row reduced echelon forms of 3×3 matrices (for 2×2 you should find 4 "shapes").

b. Determine the number of "shapes" of possible row reduced echelon forms of $n \times n$ matrices for all $n \geq 1$. What about $n \times m$ matrices? (Is this equivalent to some other problem?)

RowRed 12. Let X_n denote the $n \times n$ matrix whose (i, j) entry is $\frac{1}{i+j-1}$. Using row operations put X_4 in row reduced echelon form. Compute the inverse of X_4 and show in particular that all of its entries are integers. Here we have

$$X_n = \begin{bmatrix} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

RowRed 13. Let X_n denote the $n \times n$ matrix whose (i, j) entry is $\frac{1}{i+j-1}$. For all $n \geq 1$ do the following:

a. Show that X_n has an inverse.

b. Give an explicit formula for X_n^{-1}.

c. Conclude that the entries of X_n^{-1} are integers.

d. Give an explicit formula for $\det X_n$ and $(\det X_n)^{-1}$.

e. Show that $\det X_n$ is the reciprocal of an integer and $(\det X_n)^{-1}$ is an integer.

RowRed 14. Let $n > 1$ be an integer and F a field. Let $E_{ij}(r)$ denote the $n \times n$ matrix with $r \in F$ in the position (i, j) for $i \neq j$, 1 in all diagonal positions, and 0 elsewhere. Find nice explicit formulas for the following: ["Nice" means that the answer is a product of a tiny number k of elementary matrices; e.g., $k = 0$ (the identity matrix), $k = 1$ (an elementary matrix),...]

a.
$$E_{ij}(r)E_{ij}(s) = ?$$

b. Let a, b be invertible $n \times n$ matrices. Their *commutator* is the matrix $[a, b] = aba^{-1}b^{-1}$. Assume $i \neq j$ and $k \neq l$. Show that there is an explicit (and nice) formula for
$$[E_{ij}(r), E_{kl}(s)] = ?$$
in 3 out of the 4 cases:
Case 1. i, j, k distinct (so $n \geq 3$)
$$[E_{ij}(r), E_{jk}(s)] = ?$$
What are the other 3 cases? Which case doesn't give a nice formula? [Hint: Solve first for $n = 3$ and $n = 4$.]

RowRed 15. Let $A = (a_{ij})$ be an $n \times n$-matrix whose entries are real numbers. Assume the absolute values of the entries $|a_{ij}|$ satisfy the following n inequalities:

$$\sum_{j=1,\, j\neq i}^{n} |a_{ij}| < |a_{ii}|$$

for $1 \leq i \leq n$. Show that A has row rank n.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

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Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.