Direct Sums and Products

In the next few sections we will introduce several ways to construct new vector spaces from ones we already have. We then study carefully how these new vector spaces depend on the old ones. At the end of the discussion we will have found a powerful new way to think about such constructions. We now start with direct sums and direct products. All vector spaces in a given construction will be over the same field. We begin with the simplest case of two vector spaces.

Definition 1. Let V_1 and V_2 be two vector spaces over the same field F. Their *(external) direct sum*

$$V_1 \boxplus V_2 = \{\, (v_1, v_2) \mid v_i \in V_i \,\}$$

as a set is simply the cartesian product, the set of all ordered pairs. The set $V_1 \boxplus V_2$ is given the structure of a vector space over F be defining the sum by

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$

and scalar multiplication by

$$a(v_1, v_2) = (av_1, av_2) \ .$$

It is easy to check that $V_1 \boxplus V_2$ is a vector space with respect to these operations. The zero vector is just $(0, 0)$ and the negative of a vector is given by

$$-(v_1, v_2) = (-v_1, -v_2) \ .$$

Two elements of $V_1 \boxplus V_2$ are equal (by definition) if and only if their first coordinates are equal and their second coordinates are equal. Hence each of the equations one has to verify for $V_1 \boxplus V_2$ naturally breaks into a pair of equations, one for each coordinate. It should now be clear that $V_1 \boxplus V_2$ is a vector space as each of V_1 and V_2 are.

There are several natural linear transformations associated with $V_1 \boxplus V_2$. First there are

$$\begin{aligned} i_1 &: V_1 \longrightarrow V_1 \boxplus V_2 \\ i_2 &: V_2 \longrightarrow V_1 \boxplus V_2 \end{aligned}$$

which are given by

$$\begin{aligned} i_1(v_1) &= (v_1, 0) \\ i_2(v_2) &= (0, v_2) \ . \end{aligned}$$

These are clearly injective (one-to-one) and thus show that there is a subspace of $V_1 \boxplus V_2$ which "looks just like V_1" and one that "looks just like V_2".

There are also linear transformations

$$\begin{aligned} p_1 :\ & V_1 \boxplus V_2 \longrightarrow V_1 \\ p_2 :\ & V_1 \boxplus V_2 \longrightarrow V_2 \end{aligned}$$

given by

$$\begin{aligned} p_1(v_1, v_2) = &\quad v_1 \\ p_2(v_1, v_2) = &\quad v_2 . \end{aligned}$$

These are clearly surjective (onto) and are sometimes referred to as the natural (or canonical) *projections.*

Note that

$$\ker p_1 = \quad \operatorname{im} i_2 = \quad \{ (0, v_2) \mid v_2 \in V_2 \}$$

and

$$\ker p_2 = \quad \operatorname{im} i_1 = \quad \{ (v_1, 0) \mid v_1 \in V_1 \} .$$

One has the following equations

$$\begin{aligned} p_1 \circ i_1 = &\quad I_{V_1} \\ p_2 \circ i_2 = &\quad I_{V_2} \\ p_1 \circ i_2 = &\quad 0 \\ p_2 \circ i_1 = &\quad 0 \end{aligned}$$

and further

$$i_1 \circ p_1 + i_2 \circ p_2 = I_{V_1 \oplus V_2}$$

(see Exercise 3).

Let $W_1 = \operatorname{im} i_1$ and $W_2 = \operatorname{im} i_2$ be the subspaces of $V = V_1 \boxplus V_2$ just discussed. Note that these satisfy

$$W_1 + W_2 = V$$

and

$$W_1 \cap W_2 = 0 .$$

Definition 2. Let V be a vector space over a field F. Let V_1 and V_2 be subspaces. V is called the *(internal) direct sum* of the subspaces V_1 and V_2 if

$$V_1 \oplus V_2 = V$$

and

$$V_1 \cap V_2 = 0 .$$

Lemma 3. *Let V be a vector space over the field F which is the internal direct sum of the subspaces V_1 and V_2. There is a natural isomorphism of V with the external direct sum of V_1 and V_2*

$$\psi:\ V_1 \boxplus V_2 \longrightarrow V$$

given by $\psi(v_1, v_2) = v_1 + v_2$.

Proof. See Exercise 7. □

Remark 4. Let V be a vector space with subspaces V_i, $1 \leq i \leq k$. In order that V be the internal direct sum of the subspaces V_i one needs that V is the sum of all the subspaces

$$V = \sum_i V_i$$

and that for each i

$$V_i \cap \sum_{j \neq i} V_j = 0 \ .$$

It is not sufficient that the subspaces have pairwise trivial intersections for $k > 2$.

One may define the external direct sum of a finite number of vector spaces either inductively, or by using k-tuples. There is an analogous isomorphism between the internal and external versions here as well (see Exercise 8).

Remark 5. We will, as is commonly done, use the term *direct sum* and the symbol $\oplus$ to mean either the internal or external direct sum. It should be clear from context which is meant. As there is a natural isomorphism between the two, it is easy to convert from a statement about one case to the corresponding statement about the other.

Definition 6. Let V_i, $i \in I$, be an arbitrary collection of vector spaces over a field F. The *direct sum* of the collection of vector spaces $\{\, V_i \mid i \in I \,\}$ is the set of all functions

$$f:\ I \longrightarrow \bigcup_{i \in I} V_i$$

which have the property that $f(i) \in V_i$ and for which $f(i) \neq 0$ for only finitely many values of i in I. This set becomes a vector space over F by defining operations in a pointwise fashion

$$\begin{aligned} (f_1 + f_2)(i) &= \ f_1(i) + f_2(i) \\ (af)(i) &= \ af(i) \end{aligned}$$

for $i \in I$, f, f_1, f_2 functions, and $a \in F$. We denote this vector space by

$$\bigoplus_{i \in I} V_i \ .$$

Remark 7. a. For $I = \{1, 2\}$ there is an isomorphism between this definition and the earlier one given by

$$\phi : \bigoplus_{i\in I} V_i \longrightarrow V_1 \oplus V_2$$

where $\phi(f) = (f(1), f(2))$. That is, one of the functions f under consideration is completely determined by its values at 1 and 2. A similar statement holds for a finite number k of vector spaces V_i.

b. It should now be clear why $\bigoplus_{i\in I} V_i$ is a vector space over F: the sum and scalar multiple of functions with finite support have finite support, the zero vector is the function that has the value $0 \in V_i$ for each $i \in I$, the negative of a function f has $(-f)(i) = -f(i)$ and to verify that the definitions above make this a vector space requires checking equations for each $i \in I$, which are valid because V_i is a vector space over F.

c. As before, the description of V as an internal direct sum is given by a collection of subspaces V_i, $i \in I$, such that

$$V = \sum_i V_i$$

and that for each i

$$V_i \cap \sum_{j\neq i} V_j = 0 \ .$$

In general here although I may be infinite, an element is just a finite sum of elements from various V_i.

Definition 8. Let V_i, $i \in I$ be an arbitrary collection of vector spaces over a field F. The *direct product* of the collection of vector spaces $\{V_i \mid i \in I\}$ is the set of all functions

$$f : I \longrightarrow \bigcup_{i\in I} V_i$$

which have the property that $f(i) \in V_i$. This set becomes a vector space over F by defining operations in a pointwise fashion

$$\begin{aligned}(f_1 + f_2)(i) &= f_1(i) + f_2(i)\\ (af)(i) &= af(i)\end{aligned}$$

for $i \in I$, f, f_1, f_2 functions, and $a \in F$. We denote this vector space by

$$\prod_{i\in I} V_i \ .$$

Remark 9. a. Note that the only difference between the definition of $\prod_{i\in I} V_i$ and $\bigoplus_{i\in I} V_i$ is the latter has the extra condition requiring all functions to have finite support (be non-zero for only finitely many $i \in I$). Hence $\bigoplus_{i\in I} V_i$ is a subset of $\prod_{i\in I} V_i$, and, in fact, is a subspace since the operations are defined by the same formulas.

b. It is thus clear that the earlier remark about why the direct sum is a vector space applies in the case of direct product as well.

c. If I is a finite set, then the two are identical. For that reason many times one will sometimes see the two concepts referred to by either term, and denoted with either symbol.

d. For I infinite one should be extremly careful to distinguish between the two as they are definitely different.

e. In either case one can define linear transformations

$$i_j : V_j \longrightarrow \prod_{i\in I} V_i$$

$$i_j : V_j \longrightarrow \bigoplus_{i\in I} V_i$$

which are given by

$$i_j(v_j) = f$$

where f is the function with values $f(j) = v_j$ and $f(i) = 0$ for $i \neq j$. There are also linear transformations

$$p_j : \prod_{i\in I} V_i \longrightarrow V_j$$

$$p_j : \bigoplus_{i\in I} V_i \longrightarrow V_j$$

given by

$$p_j(f) = f(j) \ .$$

The i_j are injective and the p_j are surjective as before and there is an analogous list of subspaces, formulas, etc. as in the earlier discussion. However, not every such formula necessarily makes sense (see exercises).

f. Finally, when we begin to talk about what are called *universal mapping properties* we will introduce new definitions for sum and product for which the two (even in the finite case) will appear to be quite different.

We end this section with a question:

Question 10. Let W , V_1 , V_2 be vector spaces over the field F:

a. How does one determine all linear transformations

$$W \longrightarrow V_1 \oplus V_2 \ ?$$

b. How does one determine all linear transformations

$$V_1 \oplus V_2 \longrightarrow W \ ?$$

Exercises

SumProd 1. Verify that addition and scalar multiplication as given in Definition 1 indeed defines a vector space.

SumProd 2. Verify that the maps p_1, p_2, i_1 and i_2 in the discussion following Definition 1 are linear transformations, that the given equations hold, that p_1 and p_2 are surjective and i_1 and i_2 are injective.

SumProd 3. Verify that $i_1 \circ p_1 + i_2 \circ p_2$ is the identity map on $V_1 \boxplus V_2$. What is the analogous equation for the direct sum $V_1 \boxplus V_2 \boxplus \cdots \boxplus V_k$? What about an infinite direct sum? Is there such an equation which is valid for an arbitrary direct product?

SumProd 4. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let V_e be the subset of even functions, $f(-x) = f(x)$ and let V_o be the subset of odd functions, $f(-x) = -f(x)$.

a. Prove that V_e and V_o are subspaces of V.

b. Prove that $V_e \oplus V_o = V$.

c. Prove that $V_e \cap V_o = \{0\}$.

d. What conclusion can you now make?

e. Let F be an arbitrary field and V the vector space of all functions from F to F. Define V_e and V_o exactly the same way as above. Determine precisely when the exact same conclusions hold as held for $\mathbb{R}$.

SumProd 5. Let F be a field with characteristic unequal to 2 and let V be a vector space over F. Let $T : V \longrightarrow V$ be a linear transformation which satisfies $T^2 = I$, where I denotes the identity linear transformation. Define $V^+ = \{ v \in V \mid T(v) = +v \}$ and $V^- = \{ v \in V \mid T(v) = -v \}$. Show that $V = V^+ \oplus V^-$. [Hint: Note the characteristic of F. Nothing other than elementary manipulations is required, nor allowed, to solve this exercise.] What happens when $\operatorname{char} F = 2$?

Can you find analogous statement for transformation T such that $T^3 = I$? [You may put some mild restrictions on the field F if you want to.]

SumProd 6. a. Show that the operation of direct sum is "commutative": that is, there is a natural isomorphism

$$V_1 \boxplus V_2 \approx V_2 \boxplus V_1.$$

b. Explain the difference between the vector spaces $(V_1 \boxplus V_2) \boxplus V_3$ and $V_1 \boxplus (V_2 \boxplus V_3)$.

c. Show that the operation of direct sum is "associative": that is, there is a natural isomorphism

$$(V_1 \boxplus V_2) \boxplus V_3 \approx V_1 \boxplus (V_2 \boxplus V_3).$$

d. Give a definition of the direct sum of $k > 2$ vector spaces over F using k-tuples. Give an inductive definition assuming the case $k = 2$ is given. Verify that the two definitions give isomorphic vector spaces.

SumProd 7. Let V be the internal direct sum of the subspaces V_1 and V_2. Show that every element v of V can be written uniquely as a sum $v = v_1 + v_2$ for some $v_i \in V_i$. Verify the assertion of Lemma 3.

SumProd 8. Let V be the internal direct sum of the subspaces V_i, $1 \le i \le k$, $k > 2$. State and prove the appropriate generalization of the preceding exercise. Give examples to show that pairwise intersection being 0 is not sufficient for $k > 2$. (See Remark 4.)

SumProd 9. Prove or disprove: Given a subspace $W \subseteq V$, there exists a unique subspace W' such that $W + W' = V$ and $W \cap W' = 0$.

SumProd 10. Verify that Definition 8 indeed defines a vector space.

SumProd 11. Describe the direct sum and direct product of an "empty" collection of spaces (i.e., the index set I is empty).

SumProd 12. Let $I = I_1 \sqcup I_2$ be a partition of the index set I into two disjoint subsets. Define two natural isomorphisms from $\prod_{i\in I} V_i$ to $\left(\prod_{i\in I_1} V_i\right) \oplus \left(\prod_{i\in I_2} V_i\right)$ and from $\bigoplus_{i\in I} V_i$ to $\left(\bigoplus_{i\in I_1} V_i\right) \oplus \left(\bigoplus_{i\in I_2} V_i\right)$.

SumProd 13. Let $I = \bigsqcup_{j\in J} I_j$ be a partition of the index set I. Are there natural isomorphism between $\prod_{i\in I} V_i$ and $\bigoplus_{i\in I} V_i$ and any of the following spaces

$$a)\prod_{j\in j}\left(\prod_{i\in I_j} V_i\right) \qquad b)\bigoplus_{j\in j}\left(\prod_{i\in I_j} V_i\right) \qquad c)\prod_{j\in j}\left(\bigoplus_{i\in I_j} V_i\right) \qquad d)\bigoplus_{j\in j}\left(\bigoplus_{i\in I_j} V_i\right).$$

Can you order the four spaces above by inclusion?

SumProd 14. Show that if the index set I is infinite and all vector spaces V_i are non-trivial (i.e., they have elements different from 0), then the direct sum $\bigoplus_{i\in I} V_i$ is a proper subspace of the direct product $\prod_{i\in I} V_i$. (Proving this requires using the Axiom of Choice.)

SumProd 15. Let V be a vector space over the field F and let S be a set. Interpret the examples F^S, $F^{(S)}$, V^S, and $V^{(S)}$ from the section "Examples of Vector Spaces" in terms of the definitions given in this section.

SumProd 16. a. Show that $F^n \oplus F^m \approx F^{n+m}$.

b. Let the index set $I = \{1, 2, \ldots, n\}$ and each vector space $V_i = F$, the field F. Show that

$$\prod_{i=1}^{n} V_i \approx \bigoplus_{i=1}^{n} V_i \approx F^n.$$

c. Let the index set I be the set of natural numbers $\mathbb{N}$ and each vector space $V_i = F$, the field F. Show that

$$\prod_{i\in I} V_i \approx \quad F[[x]]$$

and

$$\bigoplus_{i\in I} V_i \approx \quad F[x],$$

here $F[x]$ is the vector space of formal polynomials over F and $F[[x]]$ is the vector space of formal power series with coefficients in F. The isomorphisms above are as vector spaces over F.

d. Let the index set I be the set of real numbers $\mathbb{R}$ and each vector space $V_i = \mathbb{R}$. What is $\prod_{i\in I} V_i$? (That is, give a description (give an isomorphism) using some other notation we've previously discussed.)

SumProd 17. Let $F = \mathbb{F}_p$ be the field of p elements with p a prime, and let $V \subseteq F^n$ be a subspace. How many subspaces W such that $V \cap W = 0$ and $V + W = F^n$ exist?

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

and

Yuri Berest.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on “Useful Definitions”, “Subobjects”, and “Universal Mapping Properties” rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn’s Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.