Tensor Products

Constructions of Free R-Modules

A quick summary of the construction of free R-modules is repeated here, for the convenience of the reader.

Definition 1. Let X be any set. A function $f: X \longrightarrow R$ is said to have *finite support*, if $f(x) \neq 0$ for only finitely many $x \in X$.

Definition 2. Let X be any set and let R be a commutative ring. The set R^X of all functions $f: X \longrightarrow R$ is an R-module with respect to pointwise addition:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \in R \text{ for all } x \in X$$

and scalar multiplication:

$$(rf)(x) = r(f(x)) \in R \text{ for all } r \in R, x \in X .$$

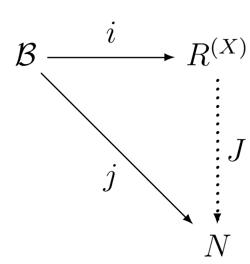
The subset consisting of all functions which have finite support is a submodule which is denoted by $R^{(X)}$. In case the set X is finite we have of course that $R^X = R^{(X)}$. For each $x \in X$ define $\delta_x \in R^{(X)}$ by

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases} .$$

The equation $f = \sum_{x \in X} f(x)\delta_x$ for any $f \in R^{(X)}$ shows that $\mathcal{B}$ both spans and is linearly independent.

Remark 3. The difference between R^X and $R^{(X)}$ is the same as the difference between direct sums and direct products.

Theorem 4. *The module $R^{(X)}$ is a free R-module with basis $\mathcal{B} = \{\, \delta_x \mid x \in X \,\}$, that is, we have the following universal mapping property: Let $i : \mathcal{B} \longrightarrow R^{(X)}$ be the inclusion map of $\mathcal{B}$ in $R^{(X)}$. For any R-module N and function $j : \mathcal{B} \longrightarrow N$, there exists a unique R-module homomorphism $J : R^{(X)} \longrightarrow N$ such that the following diagram commutes*



that is, $J \circ i = j$.

Proof. The proof is the same as that given earlier. □

Remark 5. Note that there is a one-to-one, onto function $X \longrightarrow \mathcal{B}$ given by $x \mapsto \delta_x$. For that reason $\mathcal{B}$ in this theorem could have been well replaced by the original set X. The theorem is commonly stated as " for an arbitrary set X, there exists a free module with basis X".

Corollary 6. *Given any R-module M, there exists a free R-module F and a surjective R-module homomorphism $h: F \longrightarrow M$.*

Proof. Let S be a set of generators of the R-module M (i.e., S spans M). For example, one could even take $S = M$. Let $R^{(S)}$ be the free R-module with basis $\mathcal{B} = \{\delta_s \mid s \in S\}$. By the universal mapping property there exists an R-module homomorphism $h: R^{(S)} \longrightarrow M$ such that $h(\delta_s) = s$ for all $s \in S$. This homomorphism is surjective because S generates M. In particular, each R-module M is an epimorphic image of a free R-module. If S is finite, then $R^{(S)} = R^S$ is a f.g. free R-module. □

Remark 7. Let M, N be R-modules. The set of all homomorphisms $\mathrm{Hom}_R(M,N)$ is an R-module with addition and scalar multiplication given by

$$\begin{aligned}(\alpha+\beta)(m) &= \alpha(m)+\beta(m)\\ (r\alpha)(m) &= r\cdot\alpha(m)\ .\end{aligned}$$

Bilinear and Multilinear Functions

Definition 8. Let R be a commutative ring and let M_1, M_2, N be R-modules. A function $f: M_1 \times M_2 \longrightarrow N$ is called a *bilinear function*, if it satisfies the following:

$$\begin{aligned}f(m_1+m_1',m_2) &= f(m_1,m_2)+f(m_1',m_2)\\ f(m_1,m_2+m_2') &= f(m_1,m_2)+f(m_1,m_2')\end{aligned}$$

and

$$\begin{aligned}f(rm_1,m_2) &= rf(m_1,m_2)\\ f(m_1,rm_2) &= rf(m_1,m_2)\end{aligned}$$

for all $r \in R$ and $m_i, m_i' \in M_i$. The set of all bilinear functions is denoted by $\mathrm{Hom}_R(M_1,M_2;N)$ and is an R-module with pointwise addition and scalar multiplication.

Remark 9. That is, a bilinear function f is just a linear function (i.e., homomorphism) when considered as a function of one variable with the other variable replaced by a fixed element. These functions of one variable are usually denoted by $f(-,m_2)$ for a fixed $m_2 \in M_2$ and $f(m_1,-)$ for a fixed $m_1 \in M_1$. That is, the function $f(-,m_2)$ evaluated at $m \in M_1$ is $f(m,m_2)$. This yields homomorphisms

$$f_1: M_1 \longrightarrow \mathrm{Hom}_R(M_2,N)$$

given by $m_1 \mapsto f(m_1, -)$ and

$$f_2 : M_2 \longrightarrow \mathrm{Hom}_R(M_1, N)$$

given by $m_2 \mapsto f(-, m_2)$. These then yield natural isomorphisms (see Exercise 1)

$$\begin{array}{rcl} \mathrm{Hom}_R(M_1, M_2; N) & \longrightarrow & \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, N)) \\ \mathrm{Hom}_R(M_1, M_2; N) & \longrightarrow & \mathrm{Hom}_R(M_2, \mathrm{Hom}_R(M_2, N)) \ . \end{array}$$

Example 10. 1. Let R be a commutative ring and consider ordinary multiplication

$$R \times R \longrightarrow R$$

given by $(a, b) \mapsto ab$. Bilinearity follows from the two distributive laws and commutativity of multiplication.

2. Similarly let R be a commutative ring and consider multiplication of matrices

$$R^{m\times n} \times R^{n\times p} \longrightarrow R^{m\times p}$$

given by $(A, B) \mapsto AB$. Bilinearity follows from the two distributive laws for matrix multiplication together with the fact that $rA = Ar$ for $r \in R$.

3. For two vector spaces V and W over F a pairing $V \times W \longrightarrow F$ is a bilinear function.

4. Let R be a commutative ring and define

$$f : R^{2\times 1} \times R^{2\times 1} \longrightarrow R$$

by

$$f\left(\begin{bmatrix} r_{11} \\ r_{21} \end{bmatrix}, \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix} \right) = r_{11}r_{22} - r_{12}r_{21} \ .$$

Then f is bilinear.

5. Let R be a commutative ring and $n > 0$ an integer. The dot product

$$f : R^n \times R^n \longrightarrow R$$

given by

$$f((r_1, r_2, \ldots, r_n), (s_1, s_2, \ldots, s_n)) = r_1s_1 + \cdots + r_ns_n$$

is a bilinear function. This function is commonly written as

$$(r_1, r_2, \ldots, r_n) \cdot (s_1, s_2, \ldots, s_n)) = r_1s_1 + \cdots + r_ns_n \ .$$

Similarly one can define multilinear functions:

Definition 11. Let R be a commutative ring. Let M_1, $M_2, \ldots, M_k$, N be R-modules where $k > 0$ is an integer. A function $f : M_1 \times M_2 \times \cdots \times M_k \longrightarrow N$ is called a *multilinear function* if it satisfies the following:

$$f(m_1, m_2, \ldots, m_i + m_i', \ldots, m_k) = f(m_1, m_2, \ldots, m_i, \ldots, m_k) + f(m_1, m_2, \ldots, m_i', \ldots, m_k)$$

and

$$f(m_1, m_2, \ldots, rm_i, \ldots, m_k) = rf(m_1, m_2, \ldots, m_i, \ldots, m_k)$$

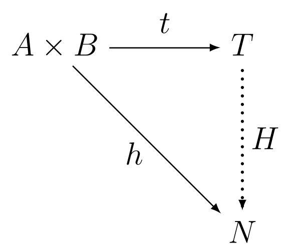
for all $r \in R$, and $m_i, m_i' \in M_i$ for $1 \le i \le k$.

The set of all multilinear maps is denoted by $\mathrm{Hom}_R(M_1, \ldots, M_k; N)$. It is an R-module under pointwise addition and scalar multiplication. As in the case of bilinear functions there is a natural isomorphism

$$\mathrm{Hom}_R(M_1, \ldots, M_k; N) \cong \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, \ldots, M_k; N))$$

Tensor Products

Definition 12. Let A, B be two R-modules over the commutative ring R. A *tensor product* (T, t) *of* A, B *over* R consists of an R-module T and a bilinear function $t : A \times B \longrightarrow T$ such that for each bilinear function $h : A \times B \longrightarrow N$ to an R-module N there exists a unique $H \in \mathrm{Hom}_R(T, N)$ such that the following diagram commutes:



i.e., $h = H \circ t$. This is equivalent to saying that the bilinear function t induces a natural isomorphism of $\mathrm{Hom}_R(A, B; N)$ with $\mathrm{Hom}_R(T; N)$ for any R-module N.

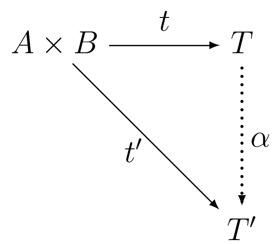
Remark 13. The idea behind the tensor product is to find a module T such that there is a natural isomorphism between $\mathrm{Hom}_R(A, B; N)$ and $\mathrm{Hom}_R(T, N)$ for any module N.

Theorem 14 (Uniqueness of tensor product)**.** *Let (T, t) and (T', t') be two tensor products of the R-modules A and B. Then there exists a unique R-module isomorphism*

$$\alpha : T \longrightarrow T'$$

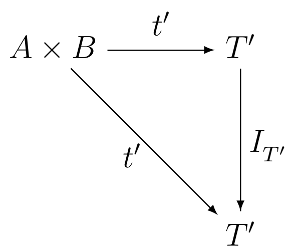
such that $\alpha \circ t = t'$.

Proof. The universal mapping property of T and the bilinearity of t' imply the existence of $\alpha \in \operatorname{Hom}_R(T, T')$ such that



i.e., $t' = \alpha \circ t$. Analogously, there exists a unique $\beta \in \operatorname{Hom}_R(T', T)$ such that $t = \beta \circ t'$.

Certainly the diagram



commutes.

Thus the uniqueness in the definition of the tensor product gives that $\alpha \circ \beta = I_{T'}$. Analogously $\beta \circ \alpha = I_T$, thus α and β are inverse isomorphisms. □

Theorem 15 (Existence of Tensor Product)**.** *For each pair A, B of R-modules a tensor product exists. It is denoted $(A \otimes_R B, t)$. The tensor product is uniquely determined up to R-module isomorphism.*

Remark 16. The tensor map t is usually denoted by $t(a, b) = a \otimes b$.

Remark 17. The idea behind the proof is simple and is used in many different situations: We don't want a module that is too large, as it probably wouldn't satisfy the uniqueness property of universal, so we take one generated by all formal elements (a, b) (the ones we must have). We then put in enough "relations" so that the elements satisfy all the equations we want to hold. But we're careful to make sure that we don't put in too many relations, or we'll get something too small to be universal.

How do we put in the realtions, that is, how do we force equations to be valid? We just take the submodule generated by the things that should be 0 (move all elements to one side of the equation) and go modulo the submodule which forces the equations to be valid for the corresponding elements of the quotient.

One can always do such things, but sometimes all one gets is a complicated description of the zero module. So after we've made our construction, we'll carry out a number of computations to show that we've indeed constructed something that is not always zero, and in fact has many nice properties.

Proof. Uniqueness holds by Theorem 14. In order to show existence we will construct a module T and a map t with the desired properties.

Let let $R^{(A\times B)}$ be the free module generated by $A\times B$. Note that $A\times B$ is usually infinite and one has to be careful to take only the functions with finite support. We simplify the notation and replace the function $\delta_{(a,b)}$ by simply (a,b) for the duration of the proof. Then the set $\mathcal{B} = A\times B$ will be the basis and every element of the free module can be written uniquely in the form $\sum_{i=1}^{k} r_i(a_i,b_i)$ for some $a_i\in A$, $b_i\in B$, and $r_i\in R$.

Let $\mathcal{R}$ be the R-submodule of $R^{(A\times B)}$ generated by:

(a) $(a_1+a_2,b)-(a_1,b)-(a_2,b)$

(b) $(a,b_1+b_2)-(a,b_1)-(a,b_2)$

(c) $(ra,b)-r(a,b)$

(d) $(a,rb)-r(a,b)$

for all $a,a_1,a_2\in A$, $b,b_1,b_2\in B$ and $r\in R$,

Let $T = R^{(A\times B)}/\mathcal{R}$ be the quotient module and define $a\otimes b = (a,b)+\mathcal{R}$. We write $\pi:\ R^{(A\times B)}\longrightarrow T$ for the natural surjection so $\pi((a,b)) = a\otimes b$.

We then have

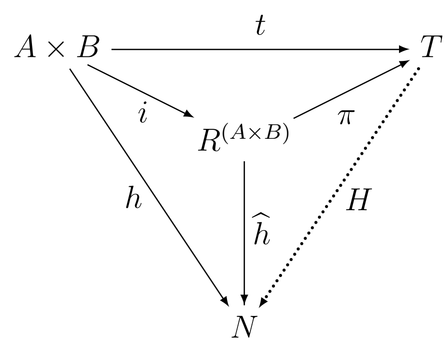
$$\begin{aligned}(a_1+a_2)\otimes b &= a_1\otimes b + a_2\otimes b\\ a\otimes(b_1+b_2) &= a\otimes b_1 + a\otimes b_2\\ (ra)\otimes b &= r(a\otimes b)\\ a\otimes(rb) &= r(a\otimes b)\end{aligned}$$

as the third follows from (c) since $(ra,b)-r(a,b)\in\mathcal{R}$ and similarly for the other three equations.

The tensor map $t:\ A\times B\longrightarrow T$ is now defined by $t(a,b) = a\otimes b$. That t is bilinear follows from the four properties of $a\otimes b$.

Let N be an R-module and let $h:\ A\times B\longrightarrow N$ be a bilinear function. As $A\times B$ is a basis of the free R-module $R^{(A\times B)}$, there exists a unique homomorphism $\widehat{h}:\ R^{(A\times B)}\longrightarrow N$ such that $\widehat{h}((a,b)) = h(a,b)\in N$ for all $(a,b)\in A\times B$.

The bilinearity of h and the properties of $a\otimes b$ yield that $\mathcal{R}\subseteq\ker(\widehat{h})$. The universal mapping property of the quotient R-module $T = R^{(A\times B)}/\mathcal{R}$ asserts the existence of a unique R-module homomorphism $H:\ T\longrightarrow N$ such that $H\circ\pi = \widehat{h}$, i.e., $H(\pi(a\otimes b)) = \widehat{h}((a,b))$ for all $(a,b)\in A\times B$. One concludes that the following diagram



commutes:

The left triagle and right triangle each commute via the universal mapping properties, and the top triangle commutes by the definition of t. It follows then (by substitution) that $h = H \circ t$. $\square$

Notation 18. The element $a \otimes b$ (equal to $t(a,b)$) is called the *tensor product* of $a \in A$ and $b \in B$ and satisfies the equations listed below.

Corollary 19. *Let $A \otimes_R B$ be the tensor product of the R-modules A and B. Then the tensor map $\otimes$ satisfies:*

(a) $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$,

(b) $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$,

(c) $ra \otimes b = a \otimes rb = r(a \otimes b)$,

(d) $0 \otimes b = 0 = a \otimes 0$.

Proof. As $a \otimes b = t(a,b)$ all assertions are immediate from our earlier discussion, or simply as consequences of the bilinearity of t. $\square$

Corollary 20. *Let $(A \otimes_R B, t)$ be the tensor product of the R-modules A, B with tensor map $t\colon A \times B \longrightarrow A \otimes_R B$. Then each element of $A \otimes_R B$ can be written as*

$$\sum_{i=1}^{k} a_i \otimes b_i.$$

for some $a_i \in A$, $b_i \in B$, $1 \le i \le k$.

Proof. This is immediate as every element of the free module $R^{(A\times B)}$ can be written uniquely as $\sum_{i=1}^{k} r_i(c_i, b_i)$ which was noted earlier: Since π is onto, an arbitrary element is then $\pi(\sum_{i=1}^{k} r_i(c_i,b_i)) = \sum_{i=1}^{k} r_i\pi(c_i,b_i) = \sum_{i=1}^{k} r_i(c_i \otimes b_i) = \sum_{i=1}^{k}((r_ic_i) \otimes b_i)$. Finally, let $a_i = r_ic_i$. Note that the elements a_i, b_i and the number k are not necessarily uniquely determined for a given element (see Exercise 2). $\square$

Example 21. 1. $R \otimes_R M \approx M$ for every R-module M

This may be seen in several different ways:

The map given by $(r, m) = rm$ is bilinear inducing an isomorphism $r \otimes m \mapsto rm$ which has an inverse given by $m \mapsto 1 \otimes m$. That is, the tensor map is just $t(r, m) = rm$.

There is a one-to-one correspondence between bilinear functions $R \times M \longrightarrow M$ and homomorphisms $M \longrightarrow M$ given by $g \mapsto h$, where $h(m) = g(1, m)$ and $g(r, m) = h(rm)$.

Another way to see this is via the isomorphisms

$$\mathrm{Hom}_R(R, M; N) \approx \mathrm{Hom}_R(R, \mathrm{Hom}_R(M, N) \approx \mathrm{Hom}_R(M, N) \; .$$

2. $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = 0$ if m and n are relatively prime.

   Here $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. It is enough to show that $a \otimes b = 0$ for all $a \in \mathbb{Z}_m$, $b \in \mathbb{Z}_n$. As $\gcd(m, n) = 1$, there exist $r, s \in \mathbb{Z}$ such that $rm + sn = 1$. Now $m \cdot a = 0$ for all $a \in \mathbb{Z}_m$ and similarly $n \cdot b = 0$ for all $b \in \mathbb{Z}_n$. Hence

$$\begin{aligned} a \otimes b &= (rm + sn)(a \otimes b) \\ &= r(ma \otimes b) + s(a \otimes nb) \\ &= r(0 \otimes b) + s(a \otimes 0) \\ &= 0 + 0 \end{aligned}$$

   This example can be generalized to show the tensor product of two cyclic R-modules R/I and R/J is isomorphic to $R/I \otimes_R R/J \approx R/(I + J)$, in particular $R/I \otimes_R R/J = 0$ if the ideals I and J are relatively prime (comaximal) (see Exercise 3).

3. $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ for $n \neq 0$.

   It is enough to show that $a \otimes b = 0$ for all $a \in \mathbb{Z}_n, b \in \mathbb{Q}$. Since $na = 0$ for all $a \in \mathbb{Z}_n$ and for any $b \in \mathbb{Q}$ there exists $c \in Q$ such that $nc = b$ we get

$$a \otimes b = a \otimes (nc) = n(a \otimes c) = (na) \otimes c = 0 \otimes c = 0.$$

4. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \approx \mathbb{Q}$.

   First note that any generator is of the form $p \otimes q$ for some $p, q \in \mathbb{Q}$. Choose a non-zero integer m so that $mp \in \mathbb{Z}$. Then

$$\begin{aligned} p \otimes q &= p \otimes m(q/m) \\ &= mp \otimes q/m \\ &= 1 \otimes (mp)(q/m) \\ &= 1 \otimes pq \; . \end{aligned}$$

   Hence a typical element is just a sum of the form

$$1 \otimes q_1 + 1 \otimes q_2 + \cdots + 1 \otimes q_k$$

which is just

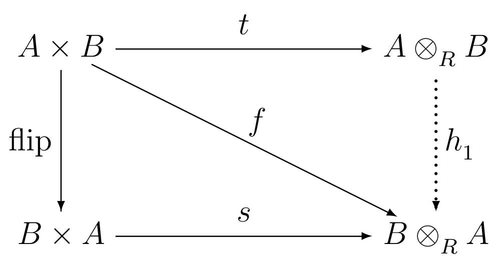
$$1 \otimes (q_1 + q_2 + \cdots + q_k) ,$$

that is, of the form $1 \otimes q$.

The map $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{Q}$ arising from the bilinear functions multiplication $(p, q) \mapsto pq$ is onto. By the above computation, the homomorphism $\mathbb{Q} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ given by $q \mapsto 1 \otimes q$ is its inverse.

Theorem 22. *Let A and B be R-modules, then $A \otimes_R B \approx B \otimes_R A$.*

Proof. Let $(A \otimes_R B, t)$ and $(B \otimes_R A, s)$ be the two tensor products. Define a bilinear function $f : A \times B \longrightarrow B \otimes_R A$ by $f = s \circ \text{flip}$ where $\text{flip} : A \times B \longrightarrow B \times A$ is $\text{flip}(a, b) = (b, a)$, $f(a, b) = b \otimes a$. By definition this map is bilinear. Now the universal mapping property of the tensor product implies that there exists a unique $h_1 \in \text{Hom}_R(A \otimes_R B, B \otimes_R A)$ such the the diagram



commutes: $h_1(a \otimes b) = b \otimes a$. Similarly, one find an $h_2 \in \text{Hom}_R(B \otimes_R A, A \otimes_R B)$ such that $h_2(b \otimes a) = a \otimes b$. Finally the uniqueness in the universal mapping properties shows that h_1 and h_2 are inverse to each other.

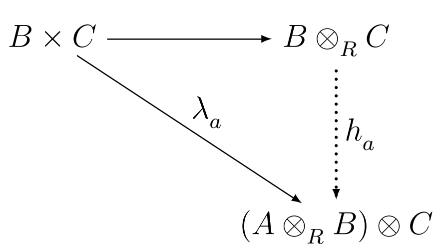
Another way to see this isomorphism it to notice that flip gives a natural isomorphism $\text{Hom}_R(A, B; N) \approx \text{Hom}_R(B, A; N)$. □

Theorem 23. *Let A, B and C be R-modules, then $(A \otimes_R B) \otimes_R C \approx A \otimes_R (B \otimes_R C)$.*

Proof. By Theorem 15 the tensor products $([A \otimes_R B] \otimes_R C, t)$ and $(A \otimes_R [B \otimes_R C], b)$ exist.

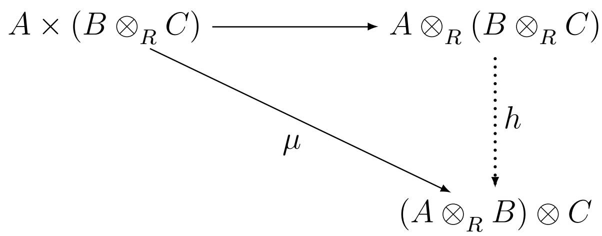
Let $a \in A$ be fixed. Define $\lambda_a : B \times C \longrightarrow (A \otimes_R B) \otimes_R C$ by $\lambda_a(b, c) = (a \otimes b) \otimes c$. The map λ_a is bilinear as it is a composition of bilinear maps.

Now by definition of the tensor product $B \otimes_R C$, there will exist a unique $h_a \in \text{Hom}_R(B \otimes_R C, (A \otimes_R B) \otimes C)$ such that



commutes, that is, $h_a(b \otimes c) = \lambda_a(b, c) = (a \otimes b) \otimes c$.

Define $\mu : A \times (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes_R C$ by $\mu(a, u) = h_a(u)$ for $u \in B \otimes_R C$. Now μ is clearly linear in u since h_a is linear. Further μ is linear in a as h_a varies linearly with a due to the linearity of the tensor product (of the three terms) in the first variable, a. Hence there exists a unique $h : A \otimes_R (B \otimes_R C) \longrightarrow (A \otimes_R B) \otimes C$ such that the diagram



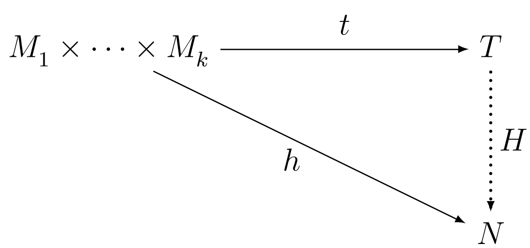
commutes, that is,

$$h(a \otimes (b \otimes c)) = \mu(a, b \otimes c) = (a \otimes b) \otimes c .$$

Repeating the construction, we obtain another R-module homomorphism $g : (A \otimes_R B) \otimes C \longrightarrow A \otimes_R (B \otimes_R C)$ satisfying $g((a \otimes b) \otimes c) = a \otimes (b \otimes c)$.

Two more applications of the universal mapping property show that g and h are inverse to each other and hence isomorphisms. □

Remark 24. One can define the tensor product of k modules $M_1, \ldots, M_k$ as (T, t) where $t : M_1 \times \cdots \times M_k \longrightarrow N$ is a multilinear map and is universal: for each multilinear map $h : M_1 \times \cdots \times M_k \longrightarrow N$ to an R-module N there exists a unique $H \in \mathrm{Hom}_R(T, N)$ such that the following diagram commutes:



that is, $h = H \circ t$. An equivalent statement is that the map t induces an isomorphism $\mathrm{Hom}_R(M_1, \ldots, M_k; N) \longrightarrow \mathrm{Hom}_R(T; N)$ by $h \mapsto H$ for an R-module N.

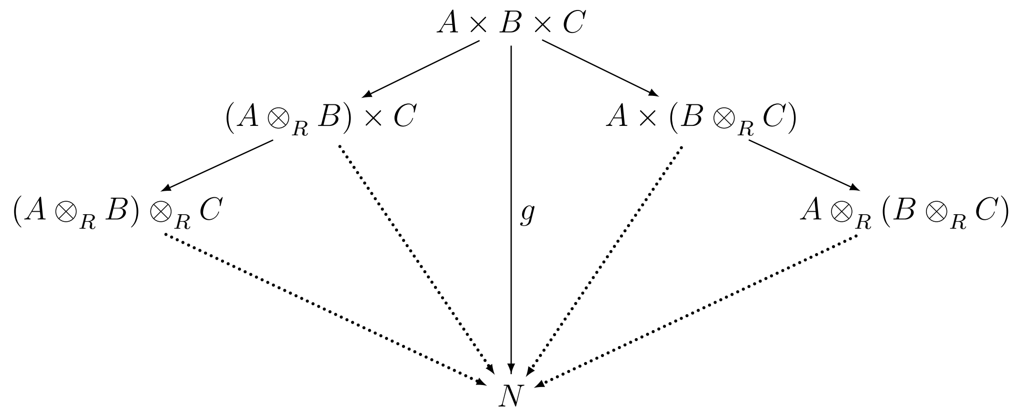
Using a method similar to that used earlier, one can directly prove the existence and uniqueness of such tensor products, which we denote by $M_1 \otimes_R \cdots \otimes_R M_k$. However, one can also construct them by an iterative process.

Theorem 23 and its proof imply that there exist natural isomorphisms

$$(A \otimes_R B) \otimes_R C \approx A \otimes_R B \otimes_R C \approx A \otimes_R (B \otimes_R C)$$

which arise from the isomorphisms (see the diagram)

$$\mathrm{Hom}_R(A, B; \mathrm{Hom}_R(C, N)) \approx \mathrm{Hom}_R(A, B, C; N) \approx \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C; N)) .$$



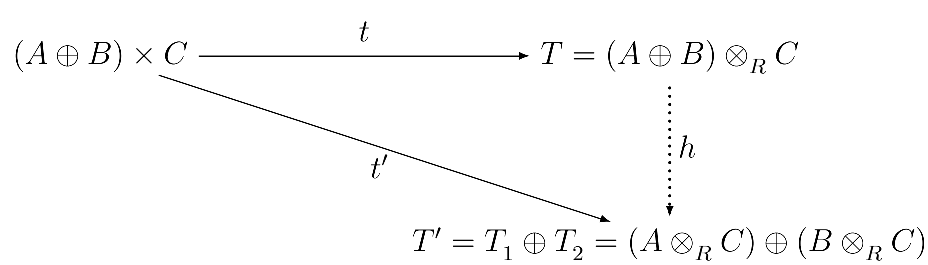
Theorem 25. *Let A, B and C be R-modules. Then $(A \oplus B) \otimes_R C \approx (A \otimes_R C) \oplus (B \otimes_R C)$.*

Proof. The tensor products $T_1 = (A \otimes_R C, t_1)$, $T_2 = (B \otimes_R C, t_2)$ and $T = ((A \oplus B) \otimes_R C, t)$ exist by Theorem 15.

Let $T' = T_1 \oplus T_2$. Define a bilinear map $t' : (A \oplus B) \times C \longrightarrow T'$ by

$$t'((a,b),c) = (t_1(a,c), t_2(b,c)) \in T_1 \oplus T_2 = T' ,$$

where (a,b) denotes an arbitrary element in $A \oplus B$. Since (T,t) is a tensor product, there exists a unique $h \in \mathrm{Hom}_R(T,T')$ such that

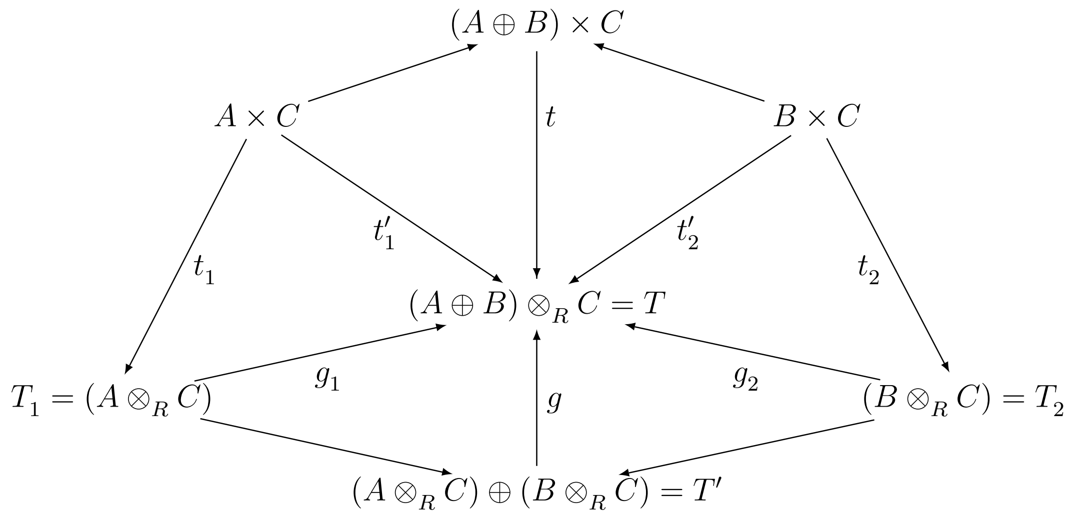


commues, i.e., $t' = h \circ t$ and $(t_1(a,c), t_2(b,c)) = h(t(a+b,c))$ for all $a \in A$, $b \in B$ and $c \in C$.

Let t_1', t_2' be the restriction of t to $(A \oplus 0) \times C$ and $(0 \oplus B) \times C$. The maps t_i' are bilinear because they are restrictions of the bilinear map t. Hence there exist uniquely defined homomorphisms $g_i \in \mathrm{Hom}_R(T_i, T)$ such that $t_i' = g_i \circ t_i$, i.e.,

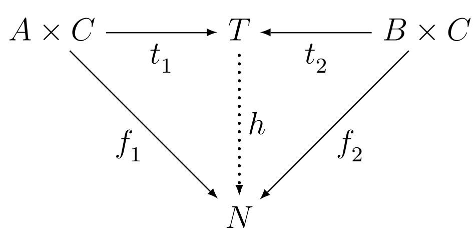
$$g_1(t_1(a,c)) = t((a,0),c) \qquad g_2(t_2(b,c)) = t((0,b),c)$$

We combine these two homomorphisms into a homomorphism $g \in \mathrm{Hom}_R(T',T)$ by $g(u_1,u_2) = g_1(u_1) + g_2(u_2)$ where $u_i \in T_i$. The diagram below explains the construction of the map g.

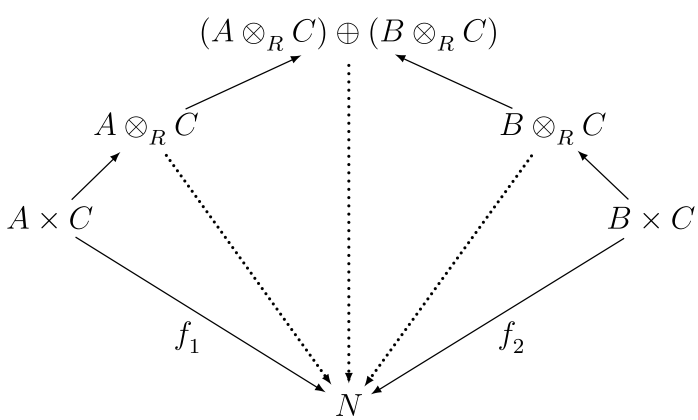


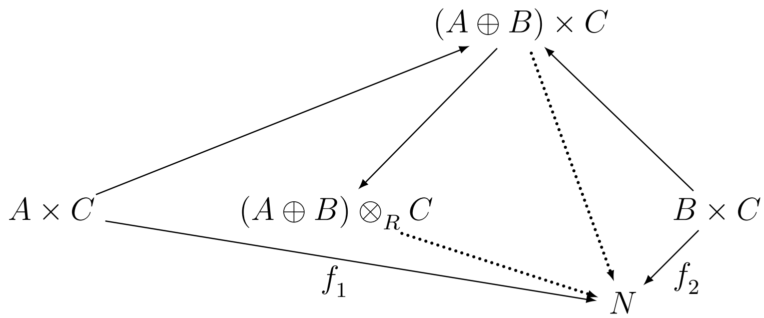
A direct computation shows that $g \circ h \circ t = t$ and by the universal mapping property for (T, t) we have that $g \circ h = I_T$. Similarly $h \circ g = I_{T'}$, thus g and h are inverse isomorphisms. $\square$

Remark 26. Another way to define the product $(A \oplus B) \otimes_R C \approx (A \otimes_R C) \oplus (B \otimes_R C)$ is as a module with the following universal mapping property: a module T and bilinear maps $t_1 : A \times C \longrightarrow T$ and $t_2 : B \times C \longrightarrow t$ such that for any module N and any two bilinear maps $f_1 : A \times C \longrightarrow N$ and $f_2 : B \times C \longrightarrow N$, there exists a unique $h : T \longrightarrow N$ such that $f_j = h \circ t_j$.



It is possible to construct (see the diagram) such a universal object in two different ways which yield the isomorphisms with $(A \oplus B) \otimes_R C$ and $(A \otimes_R C) \oplus (B \otimes_R C)$, respectively.





Remark 27. The proof of Theorem 25 can be generalized (see Exercise 9) to show that the tensor product preserves direct sums, i.e.,

$$\left(\bigoplus_I M_i\right)\otimes_R N \approx \bigoplus_I (M_i\otimes_R N)\ .$$

In general the tensor product does not preserve the direct product, but there is a canonical injective map

$$\left(\prod_I M_i\right)\otimes_R N \longrightarrow \prod_I (M_i\otimes_R N)\,,$$

which is an isomorphism if N is a finitely generated module or the index set is finite. We introduce the notation $\mathcal{A}\otimes\mathcal{B}=\{\, a\otimes b \mid a\in\mathcal{A},\ b\in\mathcal{B}\,\}$ for the next statement. Observe that it has no independent meaning.

Corollary 28. *Let M and N be free R-modules with bases $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $M\otimes_R N$ is a free R-module with basis $\mathcal{A}\otimes\mathcal{B}$.*

Proof. This follows at once from Theorem 25, because $M=\bigoplus_{i=1}^n Rm_i$ and $N=\bigoplus_{j=1}^m Rn_j$ and the basis will be $\left\{\, m_i\otimes n_j \,\right\}$. □

Corollary 29. *Let V and W be two finite dimensional F-vector spaces. Then*

$$\dim_F(V\otimes_F W)=(\dim_F V)(\dim_F W).$$

Tensor Products of Homomorphisms

Theorem 30. *Let A, A', B, B' be R-modules. Then for each pair $\alpha\in\operatorname{Hom}_R(A,A')$ and $\beta\in\operatorname{Hom}_R(B,B')$, there exists a unique $\alpha\otimes\beta\in\operatorname{Hom}_R(A\otimes_R B, A'\otimes_R B')$ such that*

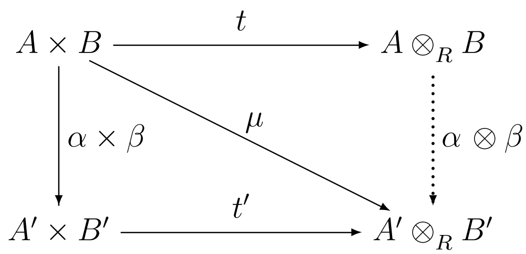
$$(\alpha\otimes\beta)(a\otimes b)=\alpha(a)\otimes\beta(b)$$

for $a\in A$, $b\in B$.

Proof. Let $(A \otimes_R B, t)$ and $(A' \otimes_R B', t')$ be the tensor products of A, B and A', B', respectively. Define $\mu : A \times B \longrightarrow A' \otimes_R B'$ by $\mu = t' \circ (\alpha \times \beta)$ where $\alpha \times \beta : A \times B \longrightarrow A' \times B'$ is the product of the maps α and β, i.e.,

$$\mu(a, b) = \alpha(a) \otimes \beta(b) \in A' \otimes_R B'$$

for $(a, b) \in A \times B$. Since α and β are R-linear, μ is bilinear. Hence there exists a unique $\alpha \otimes \beta$ so that the diegram



commutes, i.e.,

$$\begin{aligned} (\alpha \otimes \beta)(a \otimes b) &= (\alpha \otimes \beta)(t(a, b)) \\ &= \mu(a, b) \\ &= t'((\alpha \times \beta)(a, b)) \\ &= t'(\alpha(a), \beta(b)) \\ &= \alpha(a) \otimes \beta(b) \ . \end{aligned}$$

□

Remark 31. By Theorem 30 there exists a map

$$p : \operatorname{Hom}_R(A, A') \times \operatorname{Hom}_R(B, B') \longrightarrow \operatorname{Hom}_R(A \otimes_R B, A' \otimes_R B') \ .$$

This map is bilinear and thus can be extended to a map:

$$\otimes : \operatorname{Hom}_R(A, A') \otimes_R \operatorname{Hom}_R(B, B') \longrightarrow \operatorname{Hom}_R(A \otimes_R B, A' \otimes_R B') \ .$$

This map also preserve compositions, i.e., if A, A', A'' and B, B', B'' are R-modules and

$$\alpha \in \operatorname{Hom}_R(A, A') \quad \alpha' \in \operatorname{Hom}_R(A', A'') \quad \beta \in \operatorname{Hom}_R(B, B') \quad \beta' \in \operatorname{Hom}_R(B', B'')$$

are R-module homomorphisms, then

$$(\alpha' \otimes \beta') \circ (\alpha \otimes \beta) = (\alpha' \circ \alpha) \otimes (\beta' \circ \beta) \ .$$

Definition 32. Let A, A', B, B' be free R-modules with ordered bases given by

$$\begin{aligned} \mathcal{A} &= \{ a_1, a_2, \ldots, a_n \} \qquad & \mathcal{A}' &= \{ a'_1, a'_2, \ldots, a'_m \} \\ \mathcal{B} &= \{ b_1, b_2, \ldots, b_s \} \qquad & \mathcal{B}' &= \{ b'_1, b'_2, \ldots, b'_t \} \end{aligned}$$

Let $\alpha: A \longrightarrow A'$ and $\beta: B \longrightarrow B'$ be R-module homomorphisms with matrices which respect to these bases $[\alpha]_{\mathcal{A},\mathcal{A}'} = (x_{ij}), [\beta]_{\mathcal{B},\mathcal{B}'} = (y_{k\ell})$. By Corollary 28, the tensor products $A \otimes_R B$ and $A' \otimes_R B'$ have bases $\mathcal{A} \otimes \mathcal{B}$ and $\mathcal{A}' \otimes \mathcal{B}'$. In order to write the matrix of $\alpha \otimes \beta$ we need to fix an ordering of the basis elements. Let

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B} = \{ a_1 \otimes b_1, \ldots, a_1 \otimes b_s, a_2 \otimes b_1, \ldots, a_n \otimes b_s \}$$

and

$$\mathcal{D} = \mathcal{A}' \otimes \mathcal{B}' = \{ a_1' \otimes b_1', \ldots, a_1' \otimes b_t', a_2' \otimes b_1, \ldots, a_m' \otimes b_t' \}$$

be the ordered bases of the free R-modules $A \otimes_R B$ and $A' \otimes_R B'$. Then

$$[\alpha \otimes \beta]_{\mathcal{C},\mathcal{D}} = (x_{ij} y_{k\ell}) := [\alpha]_{\mathcal{A},\mathcal{A}'} \otimes [\beta]_{\mathcal{B},\mathcal{B}'}$$

is called the *Kronecker product* of $[\alpha]_{\mathcal{A},\mathcal{A}'}$ and $[\beta]_{\mathcal{B},\mathcal{B}'}$.

Example 33. Let $X = \begin{bmatrix} 12 \\ 30 \end{bmatrix}$, $Y = \begin{bmatrix} 123 \\ 021 \end{bmatrix}$. The Kronecker product $X \otimes Y$ of the matrices X and Y is

$$X \otimes Y = \left[\begin{array}{c|c} 1 \cdot Y & 2 \cdot Y \\ \hline 3 \cdot Y & 0 \cdot Y \end{array}\right] = \left[\begin{array}{c|c} 123 & 246 \\ 021 & 042 \\ \hline 369 & 000 \\ 063 & 000 \end{array}\right].$$

$$Y \otimes X = \left[\begin{array}{c|c|c} 1 \cdot X & 2 \cdot X & 3 \cdot X \\ \hline 0 \cdot X & 2 \cdot X & 1 \cdot X \end{array}\right] = \left[\begin{array}{c|c|c} 12 & 24 & 36 \\ 30 & 60 & 90 \\ \hline 00 & 24 & 12 \\ 00 & 60 & 30 \end{array}\right].$$

In particular note that $X \otimes Y \neq Y \otimes X$.

Exercises

Tensor 1. Show that if $f: M_1 \times M_2 \longrightarrow N$ is a bilinear function, then the functions $f(-, m_2)$ and $f(m_1, -)$ are R-module homomorphisms. Show also that $f_1: M_1 \longrightarrow \mathrm{Hom}_R(M_2, N)$ and $f_2: M_2 \longrightarrow \mathrm{Hom}_R(M_1, N)$ are R-module homomorphisms. Use the correspondences $f \leftrightarrow f_1$ and $f \leftrightarrow f_2$ to show that there are natural isomorphisms

$$\mathrm{Hom}_R(M_1, M_2; N) \approx \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, N)) \approx \mathrm{Hom}_R(M_2, \mathrm{Hom}_R(M_1, N)).$$

Generalize these isomorphisms to the case of multilinear maps.

Tensor 2. Construct an example showing that the elements a_i, b_i and k in Corollary 20 are NOT uniquely determined by the element u.

Tensor 3. Prove that for any two cyclic modules R/I and R/J there exists an isomorphism $R/I \otimes_R R/J \approx R/(I+J)$. What is the tensor map t?

Tensor 4. Let R be any commutative domain with field of fractions $F = \{ a/b \mid a, b \in R, b \neq 0 \}$ (recall your earlier exercises). Show that:

a. $F \otimes_R F \approx F$.

b. $F \otimes_R R/I \approx 0$ for each non-zero ideal $I \subseteq R$.

Tensor 5. Let R be a commutative ring with an identity element 1 and let I be an ideal in R.

a. Let N be any R-module. Prove that $R/I \otimes_R N \approx N/IN$. Here IN is the submodule of N consisting of all sums of elements of the form $i \cdot n$ for $i \in I$, $n \in N$. Show that N/IN is a module over R/I. If N is a free R-module, what happens to a basis under this isomorphism? Prove your statement.

b. Let N be a finitely generated free R-module. Let $\mathcal{B}_1 = \{ n_1, n_2, \ldots, n_k \}$ and $\mathcal{B}_2 = \{ p_1, p_2, \ldots, p_\ell \}$ be two bases of N. Show that $k = \ell$.
Hint: Use part a) for some maximal ideal $M \subseteq R$.

Tensor 6. a. Let I and J be two ideals in the ring R. Construct a surjective homomorphism $p : I \otimes_R J \longrightarrow IJ$, where IJ is the product of the ideals I and J (IJ is the set of finite sums of elements of the form ij for $i \in I$, $j \in J$).

b. Prove that if I (or J) is a principal ideal and R is a domain, then p is an isomorphism.

c. Show that if $R = \mathbb{Z}[x]$ and $I = J = (2, x)$, then p is NOT an isomorphism. Compute the kernel of p.

Tensor 7. Prove the analogs of Theorems 14 and 15 for tensor products of finitely many modules.

Tensor 8. Prove that the universal object described in the remark on page 9 exists and show that it is isomorphic (as an R-module) to both $(A \oplus B) \otimes_R C$ and $(A \otimes_R C) \oplus (B \otimes_R C)$. What are the tensor maps t_1 and t_2?

Tensor 9. Let M_i be an arbitrary collection of R-modules.

a. Construct an isomorphism

$$\left(\bigoplus_I M_i \right) \otimes_R N \qquad \text{with} \qquad \bigoplus_I (M_i \otimes_R N) \ .$$

b. Construct an injective map

$$\left(\prod_I M_i \right) \otimes_R N \longrightarrow \prod_I (M_i \otimes_R N) \ .$$

c. Find an example where the map in part b. is not an isomorphism.

Tensor 10 (Extension of Scalars). Let $F \subseteq K$ be two fields.

a. Let V be a vector space over F. Show that $V_K = V \otimes_F K$ is a vector space over K. Prove that $\dim_F V = \dim_K V_K$.

b. Construct an isomorphism of the vector spaces over K

$$(V \otimes_F K) \otimes_K (W \otimes_F K) \approx (V \otimes_F W) \otimes_F K .$$

c. Let V and W be two finite dimensional vector spaces over K. Show that $V \otimes_F W$ and $V \otimes_K W$ are not isomorphic as vector spaces if K is a finite dimensional vector space over F and $\dim_F K \neq 1$.

d. Construct an example where $V \otimes_F W$ and $V \otimes_K W$ are isomorphic as vector spaces over F for $F \neq K$.

Tensor 11. Let M' and N' be submodules of the R-modules M and N.

a. Use the tensor product of two homomorphisms to construct an R-module homomorphism $i: \ M' \otimes_R N' \longrightarrow M \otimes_R N$.

b. Use the tensor product of two homomorphisms to construct an R-module homomorphism $p: \ M \otimes_R N \longrightarrow (M/M') \otimes_R (N/N')$. Show that p is always surjective. What is the kernel of p?

c. Show that if R is a field, then show that i is one-to-one.

d. Construct an example showing that i is not always one-to-one.

Tensor 12. Show that the map

$$p: \ \mathrm{Hom}_R(A, A') \times \mathrm{Hom}_R(B, B') \longrightarrow \mathrm{Hom}_R(A \otimes_R B, A' \otimes_R B')$$

constructed in Theorem 30 is bilinear and use the universal mapping property of the tensor product to construct a map

$$\otimes: \ \mathrm{Hom}_R(A, A') \otimes_R \mathrm{Hom}_R(B, B') \longrightarrow \mathrm{Hom}_R(A \otimes_R B, A' \otimes_R B') .$$

Tensor 13. Let M be a free R-module, let M^* be the dual module $\mathrm{Hom}_R(M, R)$.

a. Show that there exists an injective R-module homomorphism $p: \ M^* \otimes_R N \longrightarrow \mathrm{Hom}_R(M, N)$ such that $(p(f \otimes n))(m) = f(m)n$.

b. Show that p is an isomorphism if M is finitely generated.

c. Show that p is an isomorphism if N is finitely generated.

d. Explain why p is not an isomorphism if both M and N are infinitely generated.

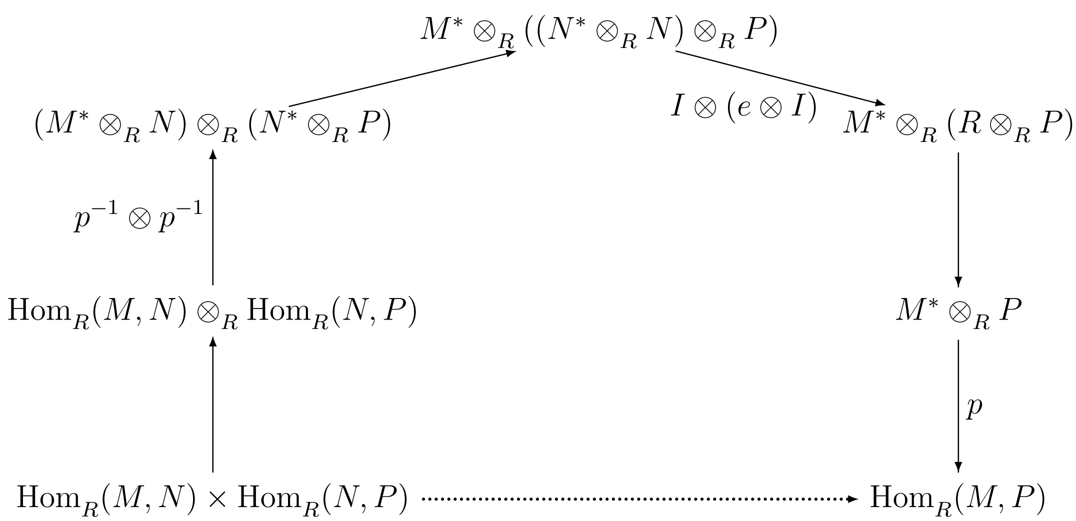
e. Let R be a field. Describe the image of p.

f. Use part d. to show that the ring $\mathrm{Hom}_F(V,V)$ contains a non-trivial two-sided ideal if V is an infinite dimensional vector space over F.

g. Let $M = N$. Construct a map $e:\ M^* \otimes_R M \longrightarrow R$ such that $e(f \otimes n) = f(n)$.

h. If $M = N$ is a finitely generated free module using p and e we can construct $t:\ \mathrm{Hom}(M,M) \longrightarrow R$ by $t = e \circ p^{-1}$. You have seen this map before, what is it?

i. Let M, N, P be finitely generated free modules over R. Consider the diagram



Describe all unlabeled arrows and show that the dotted arrow corresponds to the composition of the homomorphisms.

Tensor 14. Let A and B be two square matrices. Prove that the Kronecker products $A \otimes B$ and $B \otimes A$ are similar matrices.

Tensor 15. Let V and W be vector spaces over a field F.

a. Construct an injective linear transformation $\varphi:\ V^* \otimes_F W^* \longrightarrow (V \otimes_F W)^*$.

b. Show that φ is an isomorphism if both V and W are finite dimensional.

c. Show that φ is an isomorphism if one of V and W is finite dimensional.

Tensor 16. Let $F \subseteq K$ be fields, let V be a vector space over F, and let W be a vector space over K. Recall that W is a vector space over F as well. This is called *restriction of scalars*; elements of F act on W since they are in K.

a. To go in the other direction, show that the F-vector space $V \otimes_F K$ has a natural structure of a vector space over K. This is called *extension of scalars*, since V now has "new" scalars that can act on it.

b. There is a similar construction for Hom. Show that $\operatorname{Hom}_F(K, W)$ is an F-vector space, and that it is isomorphic to W with the F-vector space structure from restriction of scalars.

c. One may expect that, for all vector spaces V over F, that extension of scalars of V, followed by restriction of scalars of $V \otimes_F K$, is isomorphic to V as F-vector spaces. Show that in fact this is never the case unless $F = K$. (Hint: Dimension!)

d. There is a close relationship between extension and restriction, however. Show that there is a natural isomorphism of F-vector spaces

$$\operatorname{Hom}_K(V \otimes_F K, W) \approx \operatorname{Hom}_F(V, \operatorname{Hom}(K, W)).$$

Use the universal mapping property of $\otimes$ carefully, since the linear maps on the left hand side of the equation *are over* K not F.

Tensor 17. Let R be a commutative ring with unit, and let M be an R-module. The *annihilator of* M is the set

$$\operatorname{Ann}_R M = \{ r \in R \mid rm = 0 \text{ for all } m \in M \} \subseteq R$$

a. Show that $\operatorname{Ann}_R M$ is an ideal of R.

b. Recall that $R' := R / \operatorname{Ann}_R M$ is a commutative ring. Show that there is a natural R'-module structure on M. Compute $\operatorname{Ann}_{R'}(M)$.

c. Let A, B be R-modules. Show that $\operatorname{Ann}_R(A \otimes_R B) \supseteq \operatorname{Ann}_R(A) + \operatorname{Ann}_R(B)$.

d. State and prove a similar statement for $\operatorname{Hom}_R(A, B)$. Use it to prove that if $\operatorname{Ann}_R(A)$ and $\operatorname{Ann}_R(B)$ are comaximal (i.e., $\operatorname{Ann}_R(A) + \operatorname{Ann}_R(B) = R$ as ideals), then $\operatorname{Hom}_R(A, B) = 0$.

e. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be a short exact sequence of R-modules. Show that $\operatorname{Ann}_R(B) \subseteq \operatorname{Ann}_R(A) \cap \operatorname{Ann}_R(C)$. Show by an example that this inclusion can be a proper inclusion.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

and

Yuri Berest.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.