Universal Mapping Properties

Universal Mapping Properties are used in a number of different ways. First of all they give a way of specifying an object (together with maps) that will be the "best" solution to a certain type of problem. There are many types of questions that can be expressed in such terms, but not all will have solutions. So our first problem will be to prove that the given problem does have a solution (the object and necessary maps exist) and then to determine if the solution is unique, or if not, determine "how unique" it is. Secondly, such problems usually descibe how to construct more complicated things (objects or functions) from simpler things. In many cases it will turn out that we get a complete and precise description of a more complicated situation in terms of a simpler one. In our typical application this means that we obtain a one-to-one correspondence between the collection of all the functions satisfying some simple conditions and the collection of functions satisfying some more complicated conditions. This vague description should used as a guide in understanding the concrete examples of Universal Mapping Properties given below, and elsewhere in the course.

The statements below give a natural identification of one collection of functions, usually denoted $\operatorname{Hom}(A, B)$, with another. Here we use subscripts to denote what type of functions are meant. This note should probably really be a bit more formal and use the words "category", "object", and "morphism" but it won't. Nevertheless, that is really the topic.

Informally, the "objects" are the things like sets, vector spaces, modules, groups, etc. while the functions considered in the given context (the "category") are the "morphisms" - ordinary functions, F-linear transformations, R-homomorphisms, group homomorphisms, etc. For convenience **Set**, F-Mod, R-Mod, **Grp** will denote the given context (category) below. One could for example consult an edition of Lang's *Algebra* for a more formal treatment.

After each universal mapping property (UMP), we will give the correspondence of sets of functions one obtains as a result. The UMPs that we will discuss below fall into two classes, those that allow one to define functions out of quotient objects, and those that allow one to extend certain functions from a set to an algebraic object to functions *between* algebraic objects that preserve the algebraic structure (so-called freeness properties).

Quotients

We begin with the quotient construction which is frequently used in mathematics, and in particular, in this course. Let X be a non-empty set and let $\mathcal{E} \subseteq X \times X$ be an equivalence relation (see the section on "Equivalence Relations").

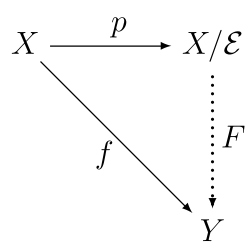
If the equivalence relation $\mathcal{E}$ behaves "nicely" with respect to the structure of X, then $X/\mathcal{E}$ will have the same sort of structure, and the function $p: \; X \longrightarrow X/\mathcal{E}$

will preserve it. We now elaborate on this idea. The idea is used in the construction of quotient spaces, quotient rings, fields of fractions, tensor products, and others. It gives a method for the construction of "universal" objects via the following:

Theorem 1. *Let $\mathcal{E}$ be an equivalence relation on the set X. If $f:\ X \longrightarrow Y$ is a function which is constant on the fibers of $p:\ X \longrightarrow X/\mathcal{E}$, then there exists a unique function $F:\ X/\mathcal{E} \longrightarrow Y$ such that $f = F \circ p$.*

By "constant on the fibers" we mean $f(x) = f(x')$ whenever $p(x) = p(x')$ or equivalently, $x \sim x'$ for $x \in X$ (i.e., when $(x, x') \in \mathcal{E}$).

We will typically state the condition $f = F \circ p$ by saying that the following diagram commutes:

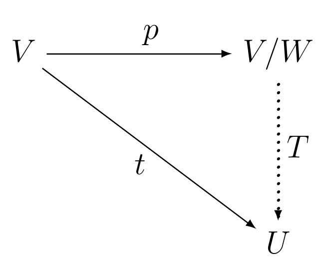


The basic idea introduced in Theorem 1 will be used many times. This gives us a one-to-one correspondence

$$\{\, f \in \mathrm{Hom}_{\mathbf{Set}}(X, Y) \mid f(\mathrm{class}_{\mathcal{E}}(x)) = \mathrm{const} \,\} \longleftrightarrow \mathrm{Hom}_{\mathbf{Set}}(X/\mathcal{E}, Y)\ .$$

When the equivalence class $\mathcal{E}$ is defined algebraically, often one is able to put an algebraic structure on $X/\mathcal{E}$, the surjection p preserves this structure, and one has a stronger version of the theorem above; namely that F preserves the algebraic struture. We will conclude this section with several examples of this, each of which takes place in a different context that we will need at some point in the semester. For the definitions of ring, module, etc, see the section "Some Useful Definitions".

Theorem 2 (UMP for Quotient Spaces)**.** *Let V be a vector space over the field F and let W be a subspace. For any vector space U and linear transformation $t:\ V \longrightarrow U$ such that $W \subseteq \ker t$, then there exists a unique linear transformation $T:\ V/W \longrightarrow U$ such that the following diagram commutes:*



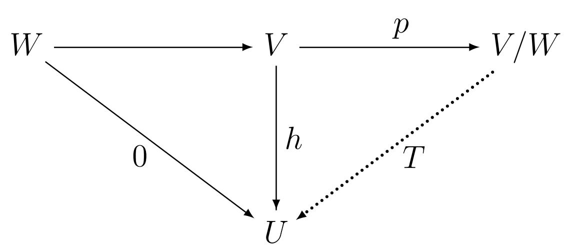
that is, $T \circ p = t$.

Theorem 2 gives a bijective correspondence

$$\{ f \in \mathrm{Hom}_F(V,U) \mid f(W)=0 \} \longleftrightarrow \mathrm{Hom}_F(V/W,U)\ .$$

In fact, the correspondence is an isomorphism of vector spaces!

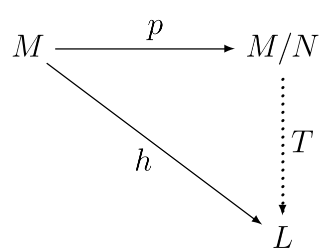
The condition that $W \subseteq \ker t$ is often depicted via the diagram



The hypothesis of the theorem becomes 'the left triangle with solid arrows commutes', and the conclusion becomes 'there exists a unique dotted arrow making the diagram commute'. This theorem is also sometimes called the universal property of cokernels.

One has a version of Theorem 2 for modules over a ring R as well (in fact, Theorem 2 is a special case of this more general version).

Theorem 3 (UMP for Quotient Modules)**.** *Let M be a module over the ring R and let N be a submodule. For any R-module L and R-homomorphism $t:\ M \longrightarrow L$ such that $N \subseteq \ker t$, then there exists a unique R-homomorphism $T:\ M/N \longrightarrow L$ such that the following diagram commutes:*



that is, $T \circ p = t$.

As before, one gets a one-to-one correspondence

$$\{ f \in \mathrm{Hom}_R(M,L) \mid f(N)=0 \} \longleftrightarrow \mathrm{Hom}_R(M/N,L)$$

which is an isomorphism of R-modules (provided each is indeed an R-module).

One can make similar statements for other algebraic structures, such as rings. Special cases of such constructions have already occurred in the exercises: Problems EqRel 9 and EqRel 12 deal with quotient rings. In particular, the field $\mathbf{F}_p$ is usually described by this type of construction. See Exercise 3 at the end.

A different type of structure appears in EqRel 11 which constructs the field of fractions of a commutative domain and gives a Unversal Mapping Property for that construction.

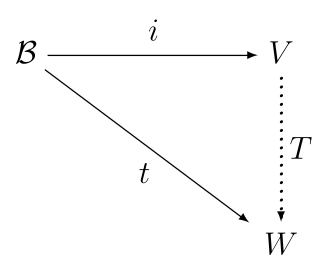
One such example is the field of rational functions which already appeared as an example in the section on "Fields". In most calculus courses one sees such fractions as well. There one learns about partial fractions in order to integrate rational functions, $f(x)/g(x)$ where $f(x), g(x) \in \mathbf{R}[x]$ and $g(x) \neq 0$.

Freeness Properties

In many situations it is useful to describe every element in some object by constructing it from some small, fixed set, of elements. In really nice situations each element can be constructed in only one way. In such a situation, the fixed set is usually referred to as a "basis" for the object, and the object itself is referred to as "free". The "free" simply means that there are no "dependence relations". As we saw earlier for the case of vector spaces, this can be stated in terms of a universal mapping property. We then reverse the process and use this to define such situations.

Every vector space V has a basis $\mathcal{B}$. The following theorem identifies linear transformations from V to W with (arbitrary!) functions from a basis $\mathcal{B}$ of V to W. This provides an easy way to define linear transformations, provided one has a basis. This result is also stated in terms of a universal mapping property below.

Theorem 4 (UMP for Bases of Vector Spaces)**.** *Let V and W be vector spaces over F, and let $\mathcal{B}$ be a basis for V. Let $i: \mathcal{B} \longrightarrow V$ be the inclusion map. Given any function $t: \mathcal{B} \longrightarrow W$, there exists a unique linear transformation $T: V \longrightarrow W$ such that the following diagram commutes:*



that is, $T \circ i = t$.

For F-vector spaces we have a bijective correspondence

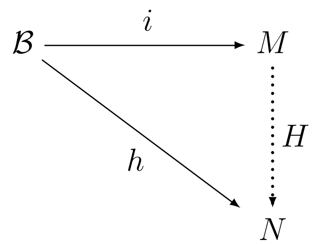
$$\mathrm{Hom}_{\mathbf{Set}}(\mathcal{B}, W) \longleftrightarrow \mathrm{Hom}_F(V, W) .$$

Note that the correspondence depends on the existence of a basis $\mathcal{B}$ and will be different for different choices of bases.

If R is now a ring (with identity), it is no longer true that an arbitrary module M over R has a basis (consider $\mathbb{Z}/n\mathbb{Z}$ as a module over the ring $\mathbb{Z}$). In this case, the universal mapping property becomes a definition:

Definition 5 (Free Module)**.** Let M be a module over a ring R and let $\mathcal{B}$ be a subset of M. M is a *free R-module* with basis $\mathcal{B}$ if for any R-module N and any function

$h: \mathcal{B} \longrightarrow N$, there exists a unique R-module homomorphism $H: M \longrightarrow N$ such that the following diagram commutes:



that is, $H \circ i = h$.

Again this yields one-to-one correspondence

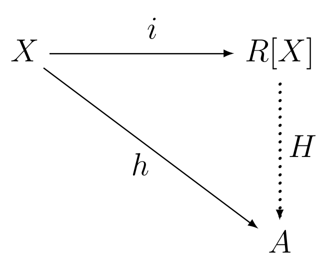
$$\mathrm{Hom}_{\mathbf{Set}}(\mathcal{B}, N) \longleftrightarrow \mathrm{Hom}_R(M, N) \ .$$

The basis exists by assumption, and as before, the correspondence is dependent upon the choice of basis $\mathcal{B}$.

The previous freeness theorems were for 'linear' objects, that is, objects that did not carry an intrinsic multiplication. We now assume our ring R is commutative and give an example of a universal mapping property for R-algebras. Recall than an R-algebra is a ring A with identity that is also an R-module and for which the ring multiplication and scalar multiplication are compatible: $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and all $a, b \in A$.

Here $R[X]$ denotes the ordinary (formal) (commutative) polynomial ring for the set of variables X. By definition, this means (as it does for fields) that the various monomials in X form a basis of the free R-module $R[X]$. This theorem asserts something you already believe: "it is possible to evaluate polynomials by inserting values for the variables".

Theorem 6 (UMP for Free Commutative Algebras)**.** *Let A be a commutative R-algebra, and let X be any set. Then there exists a commutative R-algebra (denoted $R[X]$) such that for any function $h\colon X \to A$, there exists a unique homomorphism $H: R[X] \longrightarrow A$ of R-algebras such that the following diagram commutes:*



that is, $H \circ i = h$.

The set X is the "basis" in this situation - every element of $R[X]$ can be constructed (by addition, multiplication and scalar multiplication, using R and X) and the resulting expressions uniquely represent the elements.

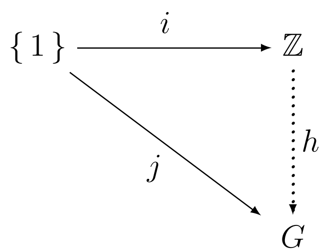
The universal mapping property provides a bijection

$$\mathrm{Hom}_{\mathbf{Set}}(X, A) \longleftrightarrow \mathrm{Hom}_{R\text{-Alg}}(R[X], A)\ .$$

Note that there is a non-commutative version of this theorem as well (meaning that the R-algebras A are allowed to be non-commutative rings). The replacement for $R[X]$ in the theorem is $R\langle X\rangle$, the (formal) ring of non-commutative polynomials with coefficients in R: that is, the variables commute with the coefficients in the ring, but not with each other (e.g., xy and yx are part of the basis for $R\langle x, y\rangle$). The ring $R\langle X\rangle$ is often referred to as the "free associative R-algebra on X. See Exercise 6.

The following theorems that describe how to give all homomorphisms from $\mathbb{Z}$ or $\mathbb{Z}_n$ to an arbitrary group G are also examples of universal mapping properties:

Theorem 7 (UMP for $\mathbb{Z}$)**.** *Let G be any group. Let $i : \{1\} \longrightarrow \mathbb{Z}$ be the inclusion map. Given any $x \in G$ define $j : \{1\} \longrightarrow G$, by $j(1) = x$. Then there exists a unique group homomorphism $h : \mathbb{Z} \longrightarrow G$ such that the following diagram commutes:*

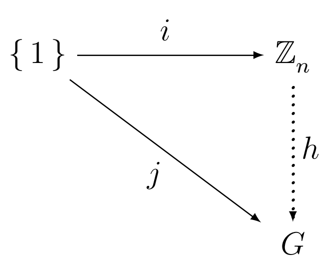


that is, $h \circ i = j$.

Note that this in fact just says that $\mathbb{Z}$ is a free (abelian) group with $\{1\}$ as a basis. Therefore, this universal mapping property identifies group homomorphisms from $\mathbb{Z}$ to G with set maps from $\{1\}$ to G, which may further be identified with G itself.

Now let n be a positive integer, and $\mathbb{Z}_n$ the additive group of integers mod n.

Theorem 8 (UMP for $\mathbb{Z}_n$)**.** *Let G be any group. Let $i : \{1\} \longrightarrow \mathbb{Z}_n$ be the inclusion map. Given $x \in G$, then there exists a group homomorphism $h : \mathbb{Z}_n \longrightarrow G$ satisfying $h(1) = x$ if and only if $o(x)|n$. If h exists, then it is the unique group homomorphism such that the following diagram commutes:*



that is, $h \circ i = j$.

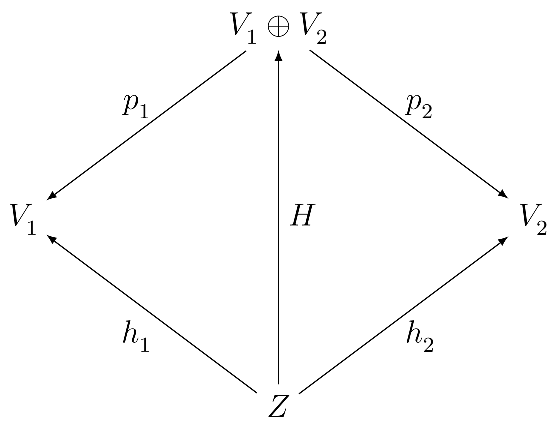
Remark 9. Since 'abelian group' is really the same thing as '$\mathbb{Z}$-module' (see Exercise 10), the previous two theorems have interpretations in the setting of abelian groups. Indeed, if one restricts Theorem 7 to the case when G is abelian, then one simply obtains that $\mathbb{Z}$ is a free $\mathbb{Z}$-module with basis $\{1\}$, as in Definition 5.

Furthermore, in the case of abelian groups, Theorem 8 is simply a special case of Theorem 3.

Products and Coproducts

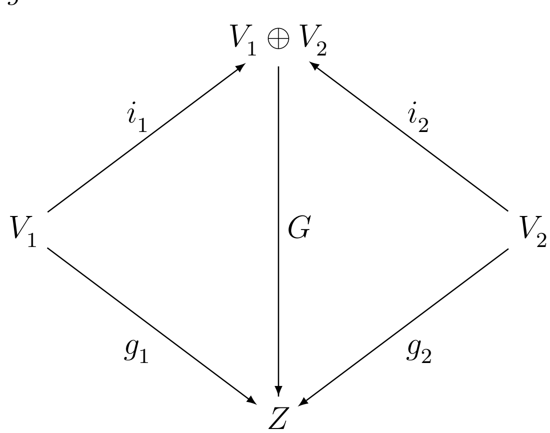
Now we look at some standard results in linear algebra.

Theorem 10 (Product Property)**.** *Let F be a field and V_1 and V_2 vector spaces over F. The linear transformations $p_1 : V_1 \oplus V_2 \longrightarrow V_1$ and $p_2 : V_1 \oplus V_2 \longrightarrow V_2$ are such that for any F-vector space Z and linear transformations $h_1 : Z \longrightarrow V_1$ and $h_2 : Z \longrightarrow V_2$ there exists a unique linear transformation $H : Z \longrightarrow V_1 \oplus V_2$ that makes the following diagram commute:*



that is, $p_1 \circ H = h_1$ and $p_2 \circ H = h_2$.

Theorem 11 (Coproduct Property)**.** *Let F be a field and V_1 and V_2 vector spaces over F. The linear transformations $i_1 : V_1 \longrightarrow V_1 \oplus V_2$ and $i_2 : V_2 \longrightarrow V_1 \oplus V_2$ are such that for any vector space Z and linear transformations $g_1 : V_1 \longrightarrow Z$ and $g_2 : V_2 \longrightarrow Z$ there exists a unique linear transformation $G : V_1 \oplus V_2 \longrightarrow Z$ that makes the following diagram commute:*



that is, $G \circ i_1 = g_1$ and $G \circ i_2 = g_2$.

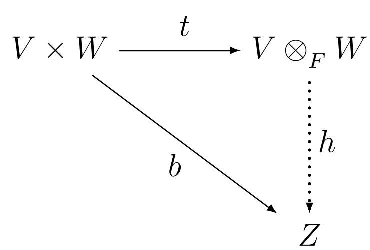
The proofs of these two "theorems" are of course easy: one takes $H(z) = (h_1(z), h_2(z))$ in the first and $G(v, w) = g_1(v) + g_2(w)$ in the second.

In the general case one makes these properties the definition: A product is denoted by $V_1 \Pi V_2$ and a coproduct by $V_1 \amalg V_2$. For two vector spaces it turns out that $V_1 \oplus V_2$ and the canonical projection and inclusion maps gives a vector space (plus relevant maps) so that it has both properties. Something similar happens for the product or coproduct of any finite number of vectors spaces. However, for an infinite collection of vector spaces there are two distinct vector spaces (with relevant maps) that are not even isomorphic. See Exercise 11.

Tensor Product

Another example from linear algebra:

Definition 12 (Tensor product)**.** Let F be a field and let V, W, Z be vector spaces over F. A vector space $V \otimes_F W$ together with a bilinear function $t: V \times W \longrightarrow V \otimes_F W$ is called a *tensor product* if for every bilinear function $b: V \times W \longrightarrow Z$ there exists a unique linear transformation $h: V \otimes_F W \longrightarrow Z$ such that the following diagram commutes:



that is, $h \circ t = b$.

Note that the above theorem establishes a bijection between the set of F-bilinear maps from $V \times W$ to Z, and ordinary F-linear maps from $V \otimes_F W$ to Z.

$$\mathrm{Hom}_{\mathbf{Bilin}}(V \times W, Z) \longleftrightarrow \mathrm{Hom}_F(V \otimes_F W, Z) \ .$$

We will consider further things like $V \otimes_F V \otimes_F \cdots \otimes_F V$ (n times) and quotients of these to construct something called *exterior powers* which play a major role in the study of determinants,

In fact, even though we're primarily interested in vector spaces, we must use the versions of these theorem for modules over commutative rings. The primary applications will be to any commutative ring since we need determinants for commutative rings, such as $F[x]$. Additionally $F[x]$ allows the study of a linear transformation $T: V \longrightarrow V$ by using T to think of V as an $F[x]$-module.

Exercises

UMP 1. Prove Theorem 1.

UMP 2. Carefully state the exercise EqRel 11 in the section "Equivalence Relations" as a Universal Mapping Property. Explain how the different parts of the exercise give a proof of the UMP and that this proof also follows the general outline of a proof of a UMP.

UMP 3. Let R be a ring and let $I \subseteq R$ be a two-sided ideal (see "Some Useful Definitions"). State and prove the Universal Mapping Property for Quotient Rings, that is, the ring R/I. (See exercises EqRel 9 and EqRel 12 which deal with two special cases of this construction. Note that by using some earlier UMP theorem you might be able to avoid some work!)

UMP 4. Let R be a commuative ring and $R[x]$ the ring of polynomials with coefficients in R.

a. Show that $R[x]$ is a free R-algebra with one generator. Hence $R[x]$ is in particular a free commutative R-algebra with 1 generator.

b. Note that $R[x_1, \ldots, x_n, x_{n+1}] = (R[x_a, \ldots, x_n])[x_n + 1]$. For $n > 1$ show that $R[x_1, \ldots, x_n]$ is a free commutative R-algebra on n generators.

c. Show that $R[x_1, \ldots, x_n]$ is not a free R-algebra on n generators.

UMP 5. Let R be an arbitrary commutative ring with identity. Generalize the previous problem to construct the free commutative R-algebra on an arbitrary set X.

UMP 6. Let F be a field. Let $n \geq 2$ be an integer. Let $F\langle x_a, \ldots, x_n\rangle$ be the vector space over F whose basis consists of all monomials in the x_i where the variables are not allowed to commute. Define the degree of a monomial h to be the total number of variables that appear (e.g., $h = x_3^2 x_1 x_2^4 x_3$ has degree 8). An element of this ring is a finite linear combination of these monomials. Two are multiplied via $h_1 \cdot h_2 = h_1 h_2$ (juxtapose – combine any adjacent equal variables by adding the expontents, otherwise do nothing – e.g., $x_1 x_2 \cdot x_2 x_1 = x_1 x_2^2 x_1$ and $x_1 x_2 \cdot x_3 x_1 x_2 = x_1 x_2 x_3 x_1 x_2$). Elements of F commute with the x_i and the multiplication is extended by the distributive law. This gives a ring structure on $F\langle x_1, \ldots, x_n\rangle$ and in fact, it becomes an F-algebra. For any non-zero f in this ring, $\deg f$ is defined to be the maximum of the degrees of the monomials with non-zero coefficients which appear in it. In the following you will show that the informal description above leads to a formal construction

a. Count the number of monomials of degree exactly k for $k \geq 0$.

b. Let H_m be the collection of all polynomials for degree less than m together with 0. It is a subspace. What is its dimension?

c. State and prove the Universal Mapping Property analogous to Theorem 6 where the F-algebra A is now allowed to be non-commutative.
Here is an outline of the construction of the free F-algebra: Let M be the vector space with basis $\mathcal{B}$ the set of all non-commuting monomials h in n non-commuting variables as described above. Let $\text{End}_F(M)$ denote the ring of endomorphisms of M. Using the UMP for the basis of M define the linear transformation T_i, $1 \leq i \leq n$, by as "right multiplication by x_i"

$$T_i(h) = h \cdot x_i$$

where the multiplication of monomials is as described above. [Careful: There are two cases to consider depending on whether or not h has x_i as the last variable on the right.] Let $A \subseteq \text{End}_F(M)$ be the sub-F-algebra generated by all of the T_i. Note that A contains the identity linear transformation by the definition of F-algebra. Hence A contains the set $\mathcal{B}$ of all monomials (note that the identity is the monomial where all degrees on all terms equal 0). Now prove that A is the free (non-commutative) F-algebra on $\mathcal{A} = \{ x_1, x_2, \ldots, x_n \}$.

UMP 7. Let R be an arbitrary ring with identity. Combine the ideas in the two problems Exercise 4 and Exercise 6 to construct the free (non-commutative) R-algebra in n variables for n a positive integer.

UMP 8. Let R be an arbitrary ring with identity. Generalize the previous problem to construct the free (non-commutative) R-algebra on an arbitrary set X.

UMP 9. Give a definition of a 'free group', and state and prove the universal property that it satisfies.
A brief description of the ideas required: Let X be an arbitrary set. Let $X \dot{\cup} \overline{X}$ denote the disjoint union of two sets, the first being X and the second being the set of all symbols $\overline{x}$ for all $x \in X$. Let Z be the set of all finite sequences of of elements in $X \dot{\cup} \overline{X}$ (including the empty sequence $[\ ]$). Let $\mathcal{S}(Z)$ denote the group of all permutations of the set Z. For any $x \in X$ define a "right multiplication" r_x of x on Z analogous to what was done in Exercise 6 (think of $\overline{x}$ as denoting the inverse of x with the product of the two in either order cancelling out). This definition of r_x will basically involve two cases. Let F_X denote the subgroup of $\mathcal{S}(Z)$ generated by all the r_x for $x \in X$. Prove that F_X is the free group on the set X.

UMP 10. Let $n > 1$ be an integer.

a. Show that A being a module over $\mathbb{Z}$ is equivalent to A being an abelian group.

b. Let A be an abelian group. Show that A is a module over the ring $\mathbb{Z}_n$ if and only if $n \cdot a = 0$ for all $a \in A$. [Note that this generalizes one of your early exercises on fields.]

c. Show that $\mathbb{Z}^m$ is a free $\mathbb{Z}$-module for all $m > 0$. Describe all free $\mathbb{Z}$-modules, e.g., those with a basis $\mathcal{B}$.

d. Show that $\mathbb{Z}_n^m$ is a free $\mathbb{Z}_n$-module for all $m > 0$. Describe all free $\mathbb{Z}_n$-modules, e.g., those with a basis $\mathcal{B}$.

UMP 11. Let F be a field and let $\{ V_i \mid i \in I \}$ be a collection of vector spaces over F.

a. If $I = \{ 1, \ldots, n \}$ is a finite set, define the product $\prod_{i=1}^n V_i$ via a universal mapping property.

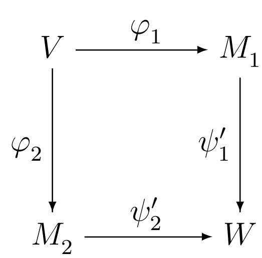
b. Verify that the usual direct sum of the V_i with appropriate maps gives the product when $I = \{ 1, \ldots, n \}$ is a finite set.

c. If $I = \{ 1, \ldots, n \}$ is a finite set, define the coproduct $\coprod_{i=1}^n V_i$ via a universal mapping property.

d. Verify that the usual direct sum of the V_i with appropriate maps gives the coproduct when $I = \{ 1, \ldots, n \}$ is a finite set.

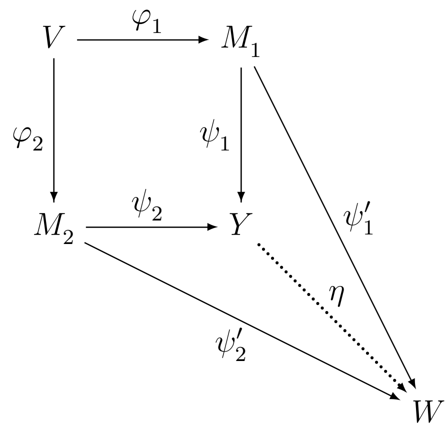
e. If I is arbitrary, define the product $\prod_{i \in I} V_i$ and the coproduct $\coprod_{i \in I} V_i$ via universal mapping properties. Consider the vector space $\bigoplus_{i \in I} V_i$. Does it together with some collection of maps give either the product or coproduct? Answer the same question for $\prod_{i \in I} V_i$ as defined in the section on "Direct Sums and Products". [Hint: See the exercises at the end of the setion on "Bases and Coordinates".]

UMP 12 (Pushout)**.** Let V, M_1, M_2 be vector spaces over the field F and $\varphi_i : V \to M_i$ linear transformations. Let W be a vector space over F and let $\psi_i' : M_i \to W$ be linear transformations that make the following diagram commute,



that is, $\psi_1' \circ \varphi_1 = \psi_2' \circ \varphi_2$.

a. Show that there exists a vector space Y and linear transformations $\psi_i : M_i \to Y$ which makes the diagram commute, and that is universal among all such; that is, there exists a unique $\eta \in \mathrm{Hom}_F(Y, W)$ such that the following diagram commutes:



b. Show that Y is determined uniquely up to isomorphism by V, M_1, M_2 and the linear transformations φ_1, φ_2.

c. Consider the special case when both φ_1 and φ_1 are zero. Explain why and how the universal object constructed in part a is related to a construction considered earlier.

UMP 13 (Pullback)**.** Start with the same diagram as in the previous problem, and define the *pullback* of the diagram as the object (and arrows) in the upper left-hand corner that is universal with respect to an incoming arrow that makes the appropriate diagram commute. Verify that the the pullback exists for vector spaces and is unique up to isomorphism. Give an analogue to the last part of the previous problem.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

Gerhard O. Michler

R. Keith Dennis

Martin Kassabov

W. Frank Moore

and

Yuri Berest.

Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on “Useful Definitions”, “Subobjects”, and “Universal Mapping Properties” rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn’s Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.