Examples of Vector Spaces

Let F be an arbitrary field.

1. F^n for $n > 0$ an integer, where F^n is the set of n-tuples of elements of F. Here the vector space structure is the usual one:

   a. $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n)$, for $a_i \in F$, $b_i \in F$.

   b. $c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n)$ for $c \in F$. $a_i \in F$.

2. $F^{m\times n}$ for $m, n > 0$ integers. Again the vector space structure is the standard one:

   a.
$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} + \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{pmatrix} =$$
$$\begin{pmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1.2} & \cdots & a_{1,n}+b_{1,n} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2.2} & \cdots & a_{2,n}+b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1}+b_{m,1} & a_{m,2}+b_{m.2} & \cdots & a_{m,n}+b_{m,n} \end{pmatrix}$$
   for $a_{i,j} \in F$, $b_{i,j} \in F$.

   b.
$$c\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = \begin{pmatrix} ca_{1,1} & ca_{1,2} & \cdots & ca_{1,n} \\ ca_{2,1} & ca_{2,2} & \cdots & ca_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m,1} & ca_{m,2} & \cdots & ca_{m,n} \end{pmatrix}$$
   for $c \in F$, $a_{i,j} \in F$.

3. Let $M \in F^{m\times n}$, $m, n > 0$ be a fixed matrix. $W = \{\, C \in F^{n\times 1} \,|\, MC = 0 \,\}$, the solution space for the matrix equation, is a vector space over F as is easily checked.

4. Let S be a non-empty set and F^S the set of all functions from S to F. Addition and scalar multiplication for F^S is given as in calculus:

   a. $(f+g)(s) = f(s) + g(s)$ for $f, g \in F^S$, $s \in S$.

   b. $(af)(s) = af(s)$ for $f \in F^S$, $s \in S$, $a \in F$.

The notation F^S may appear a bit unusual if you haven't seen it before. For X, Y sets we denote by X^Y the set of all functions from Y to X. For finite sets it is easy to check the sizes of the sets satisfy $|X^Y| = |X|^{|Y|}$; hence the notation. For arbitrary sets, this yields the definition of powers for cardinal numbers.

In addition, one can define a multiplication of two such functions by

c. $(f \cdot g)(s) = f(s) \cdot g(s)$ for $f, g \in F^S$, $s \in S$.

5. Let S be a non-empty set and let $F^{(S)}$ be the subset of F^S consisting of those functions with *finite support*. For a function $f : S \longrightarrow F$ its *support* is the set

$$\operatorname{Supp}(f) = \{ s \in S \mid f(s) \neq 0 \}$$

where it takes on non-zero values. Finite support just means that $\operatorname{Supp}(f)$ is a finite set. Addition and scalar multiplication for $F^{(S)}$ is the same as in F^S. One can also define multiplication of two such functions as before.

Note that this only gives a vector space different from F^S in case the set S is infinite.

6. Let S be a non-empty set and let V be a vector space over the field F. We denote by V^S the set of all functions from S to V. Addition and scalar multiplication for V^S is given as earlier:

a. $(f + g)(s) = f(s) + g(s)$ for $f, g \in V^S$, $s \in S$.

b. $(af)(s) = af(s)$ for $f \in V^S$, $s \in S$, $a \in F$.

7. Let S be a non-empty set and let V be a vector space over the field F. We denote by $V^{(S)}$ the subset of V^S consisting of those functions with finite support. Addition and scalar multiplication are the same as in V^S.

Again this gives a vector space different from V^S only in case the set S is infinite.

8. $F[x]$ denotes the set of all polynomials with coefficients in F, treated formally. Addition and scalar multiplication are given by the usual formulas:

a. $\sum_{i=0}^{i=n} a_i x^i + \sum_{i=0}^{i=n} b_i x^i = \sum_{i=0}^{i=n} (a_i + b_i) x^i$ for $a_i \in F$, $b_i \in F$.

b. $c \sum_{i=0}^{i=n} a_i x^i = \sum_{i=0}^{i=n} c a_i x^i$

The phrase "treated formally" means that two polynomials in $F[x]$ are equal if and only if their corresponding coefficients are equal. They are NOT functions. However, we will later show that a function is determined by a polynomial, but it is not true that different coefficients on f and g mean one obtains different functions - it depends on the field. A more detailed treatment of this can be found in the lectures and the exercises.

In addition, one can define multiplication in the usual way:

c. $(\sum_{i=0}^{i=n} a_i x^i) \cdot (\sum_{i=0}^{i=m} b_i x^i) = \sum_{i=0}^{i=n+m} c_i x^i$
with $c_i \in F$ given by $c_i = \sum a_k b_l$ where the sum is taken over all k, l such that $i = k+l$ (that is, the terms which have the same x^i are added together).

9. $F[[x]]$ denotes the set of all formal power series with coefficients in F. Addition and scalar multiplication are given by the usual formulas:

   a. $\sum_{i=0}^{i=\infty} a_i x^i + \sum_{i=0}^{i=\infty} b_i x^i = \sum_{i=0}^{i=\infty} (a_i + b_i) x^i$ for $a_i \in F$, $b_i \in F$.
   b. $c \sum_{i=0}^{i=\infty} a_i x^i = \sum_{i=0}^{i=\infty} c a_i x^i$

   As in the previous example, "formal" means that two power series in $F[[x]]$ are equal if and only if all of their corresponding coefficients are equal. Another way to think of this is that there is a one-to-one correspondence between $F[[x]]$ and $F^{\mathbb{N}}$ for $\mathbb{N}$ the set of non-negative integers. This correspondence does not preserve the multiplication.

   $F[x]$ is the subring of $F[[x]]$ consisting of those power series whose coefficients a_i are all 0 for sufficiently large i.

   Elements of $F[[x]]$ are NOT functions. For an arbitrary field, they don't even determine functions. For very special fields, those with a topology, perhaps given by a metric, one can define a meaning using convergence (which may or may not happen for a given power series). However, we will normally not consider such here.

   In addition, one can define multiplication in the usual way:

   c. $(\sum_{i=0}^{i=\infty} a_i x^i) \cdot (\sum_{i=0}^{i=\infty} b_i x^i) = \sum_{i=0}^{i=\infty} c_i x^i$
   with $c_i \in F$ given by $c_i = \sum a_k b_l$ where the sum is taken over all k, l such that $i = k + l$ just as for polynomials. The c_i is given by a finite sum (with $i + 1$ terms) since $k, l \geq 0$.

   It should again be noted that this is NOT the same multiplication (not the same ring) as example 4.

10. Let $F \subseteq K$ be fields where the addition and multiplication of elements in F is the same as that when they are considered elements of K. It is easy to check that K is a vector space over F since the required axioms are just a subset of the statements that are valid for the field K. We thus obtain many examples this way:

   (1) $\mathbb{R} \subseteq \mathbb{C}$, (i.e., $\mathbb{C}$ is a vector space over $\mathbb{R}$)

   (2) $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

   (3) $\mathbb{Q} \subseteq \mathbb{Q}[i]$,

   (4) $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{3}]$.

   (5) $\mathbb{F}_2 \subseteq \mathbb{F}_4$

11. Let $D \subseteq \mathbb{R}$ where, for example, $D = (0, 1)$ is the open interval from 0 to 1. In all cases addition and scalar multiplication are defined as in calculus (see example 4. above). The following are all vector spaces over $\mathbb{R}$:

    (1) The set of all functions from D to $\mathbb{R}$,

    (2) The set of all continuous functions from D to $\mathbb{R}$,

    (3) The set of all differentiable functions from D to $\mathbb{R}$,

    (4) The set. $\mathcal{C}^m(D)$, of all functions from D to $\mathbb{R}$, which are differentiable $m > 1$ times (including the case $m = \infty$).

    (5) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. f is called *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$; f is called *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$. The set of all odd functions is a vector space over $\mathbb{R}$; the set of all even functions is a vector space over $\mathbb{R}$..

12. Let F be a field, V a vector space over F, and $L : V \longrightarrow V$ a linear transformation. If $f \in F[x]$ is a polynomial of degree n, say $f = a_0 + a_1 x + \cdots + a_n x^n$, we will write $f(L) = a_0 I + a_1 L + \cdots + a_n L^n$ to mean the linear transformation given by the formula
$$f(L)(v) = a_0 v + a_1 L(v) + \cdots + a_n L^n(v)$$
for any vector $v \in V$. Here L^i means L composed with itself i times (and $L^0 = I$, the identity function on V).

    a. Let $F = \mathbb{R}$ be the field of real numbers, V the vector space of all infinitely differentiable real valued functions defined on $\mathbb{R}$. Take $L = D$ where $D : V \longrightarrow V$ denotes ordinary differentiation, $Dg(x) = g'(x)$ for any function $g \in V$. Now let $f \in \mathbb{R}[x]$ be a polynomial of degree n (as above). Then $f(D) : V \longrightarrow V$ is a linear transformation. The set
$$\ker(f(D)) = \{\, v \in V \mid f(D)(v) = 0 \,\}$$
is a subspace of V. This is the set of solutions to the differential equation $f(D)(v) = 0$. For example, if $f(x) = x^2 + 1$, the corresponding differential equation is $D^2 g(x) + g(x) = 0$ or $g''(x) + g(x) = 0$ (sometimes written as $y'' + y = 0$). It is easy to check that the functions $\sin x$ and $\cos x$ are both solutions to this equation.

    b. Let F be a field and $F[[x]]$ the ring of formal power series over F. Define $D : F[[x]] \longrightarrow F[[x]]$ by $D(\sum_{i=0}^{i=\infty} a_i x^i) = \sum_{i=0}^{i=\infty} i a_i x^{i-1}$. It is easy to check that D is a linear transformation which has the usual properties of the derivative, including $D(gh) = hD(g) + gD(h)$ for any $g, h \in F[[x]]$. For a polynomial $f = b_0 + b_1 x + \cdots + b_n x^n$ of degree n, we write $f(D) = b_0 I + b_1 D + \cdots + b_n D^n$, which is also a linear transformation defined on $F[[x]]$. Here D^i is the composition of D with itself i times, $D^0 = I$, the identity transformation. $\ker f(D) = \{\, g \in F[[x]] \mid f(D)(g) = 0 \,\}$ is called the set of solutions to the formal homogeneous linear differential equation with constant coefficients.

13. Let X be an arbitrary set and let $V = \mathcal{P}(X)$ be the set of all subsets of X. Let $\mathbb{F}_2 = \{0, 1\}$ be the field with two elements. the integers mod 2. We define an addition on V as follows: for subsets $S, T \subseteq X$.

$$S + T = S \triangle T$$

where $S \triangle T$ denotes the *symmetric difference*: the elements that are either in S or in T but not in both (i.e., $S \cup T$ with $S \cap T$ removed). Note that with this definition of addition, V is an abelian group:

a. The operation is associative. (Draw the Venn diagram!)
b. The empty set is the zero 0.
c. Every element is its own inverse: $S + S = 0$.
d. The addition is commutative:
$S + T = S \triangle T = T \triangle S = T + S$

Finally, V is a vector space over the field $\mathbb{F}_2$ via the only possible definition of multiplication by scalars:

$$\begin{aligned} 0 \cdot S &= 0 \\ 1 \cdot S &= S . \end{aligned}$$

Note that V even has a multiplication defined on it by intersection:

$$S \cdot T = S \cap T .$$

Multiplication is associative, commutative and there is an identity element, namely X. The only element with a multiplicative inverse is the identity X.

This example is actually an *algebra* over $\mathbb{F}_2$ (see the following remark) and is called the *boolean algebra* of all subsets of X. Some generalizations of such algebras are also called boolean algebras.

It is relatively easy to show that there is an isomorphism of vector spaces of

$$\chi : V \longrightarrow \mathbb{F}_2^X$$

using the *characteristic function* χ.

$$\chi(S) : X \longrightarrow \mathbb{F}_2$$

is given by

$$\chi(S)(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S . \end{cases}$$

In fact, this function gives an isomorphism of $\mathbb{F}_2$-algebras.

Note that sometimes this vector space $\mathcal{P}(X)$ is denoted by 2^X.

A number of assertions were given without proof in this example. You should verify them.

Remark 1. In the preceding Examples 2 (when $m = n$), 4, 8, 9, 10, 11, and 13 there is also a multiplication defined. In each case we get what is called an F-algebra. That is, a set R which is a vector space over F, is a ring, and for which the scalar multiplication and multiplication are compatible, that is, satisfy:

d. $c(fg) = (cf)g = f(cg)$ for all $c \in F$ and $f, g \in R$.

In all examples except for matrices (with $n > 1$), these are commutative F-algebras, that is, the multiplication in the ring is commutative. By using an F-algebra A (instead of V) in Example 6 one could also define a multiplication as in 6, which would yield another example of an F-algebra. See the section "Some Useful Definitions".

Exercises

VSExamples 1. Let V be a vector space and let I, J, K be set sets

a. Construct isomorphisms between $\left(V^I\right)^J$, $\left(V^J\right)^I$ and $V^{I\times J}$.

b. Using the above isomorphisms both $\left(V^{(I)}\right)^J$ and $\left(V^J\right)^{(I)}$ can be viewed as subspaces of $V^{I\times J}$. Which one is larger? Why?

c. Assume that V is an algebra. Define (an interesting) multiplication

$$\left(V^{(I)}\right)^J \times \left(V^{(J)}\right)^K \to \left(V^{(I)}\right)^K.$$

Can you do the same for the full spaces?

d. Can you interpret the above multiplication as matrix multiplication?

VSExamples 2. a. Describe the set of the formal solution of the differential equation $D(g) = 0$ in $F[[x]]$.

b. Does there exists a non trivial polynomial $f(x)$ such that the set of formal solutions of $f(D)(g) = 0$ is the whole $F[[x]]$. Can you describe all such f-es?

c. Can you construct a subspaces of $F[[x]]$ which is not equal to the space of formal solutions of $f(D)(g) = 0$ for any polynomial f?

Hint: The cases $\operatorname{char} F = 0$ and $\operatorname{char} F = p > 0$ are very different.

The Notes for the course *Math 4330, Honors Linear Algebra* at Cornell University have been developed over the last ten years or so mainly by the following (in chronological order):

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Most sections have been revised so many times the original author may no longer recognize it. The intent is to provide a modern treatement of linear algebra using consistent terminology and notation. Some sections are written simply to provide a central source of information such as those on "Useful Definitions", "Subobjects", and "Universal Mapping Properties" rather than as a chapter as one might find in a traditional textbook. Additionally there are sections whose intent is to provide proofs of some results which are not given in the lectures, but rather provide them as part of a more thorough development of a tangential topic (e.g., Zorn's Lemma to develop cardinal numbers and the existence of bases and dimension in the general case).

A large number of challenging exercises from many different sources have been included. Although most should be readily solvable by students who have mastered the material, a few even more challenging ones still remain.

Much still remains to be done. Corrections and suggestions for additional exercises, topics and supplements are always welcome.