

Math 4330 Final Exam

Due Saturday, December 14, 2019

Your work on this exam is to be done in accordance with the following:

1. You may not obtain aid nor discuss the exam with any other person.
2. If you have any questions, please send e-mail, call, or come by my office.
3. Please return the exam to me by noon on Saturday, December 14. You may instead submit your solutions as a pdf file by e-mail.

Please write your answers very carefully, explaining exactly what you are doing and showing all the computations.

Problem 1. a) Let F be a field. Describe all pairs of finite dimensional spaces U and V over F such that $U \oplus V$ is isomorphic to $U \otimes_F V$.

b) Does the answer in part a) changes if the finite dimensionality is removed?

Problem 2. Let R be a principal ideal domain and let M and N are finitely generated modules over R .

a) Show that if $M \neq 0$ then $M \otimes_R M$ is also non-zero module.

b) Use part a) show that if $M \otimes_R N$ is non-zero then any tensor product (over R) involving only M and N (like $M \otimes N \otimes M \otimes M$) is non-zero.

c) Does part a) holds if the module M is not finite dimensional?

Problem 3. Let $T : V \rightarrow V$ be a diagonalizable linear endomorphism acting on a n -dimensional vector space V over a field F , with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Show that the following linear transformations are diagonalizable and find their eigenvalues

a) $T^{\otimes 2} : V \otimes V \rightarrow V \otimes V$,

b) $\wedge^3 T : \wedge^3 V \rightarrow \wedge^3 V$.

Problem 4. Let U and V be vector spaces over a field F . If both U and V are finite dimensional $\text{Hom}(U \rightarrow V)$ is canonically isomorphic to $U^* \otimes V$ (here U^* is the dual spaces of U).

- a) In general there is only an injective map between these two spaces. Define this map and prove that it is injective.
- b) Give an example where this map is not an isomorphism.
- c) Give an example where the map is isomorphism but at least one of the spaces is not finite dimensional.

Problem 5. Let F be a field of characteristic 0 and let A and B are $n \times n$ matrices which are a single Jordan block (of size n) with eigenvalues a and b respectively. Let $C = A \otimes B$ denotes the Kronecker product of A and B (this is a $n^2 \otimes n^2$ matrix).

- a) Find the eigenvalues of C and their algebraic multiplicities.
- b) If $n = 2$ or $n = 3$, find the sizes of the blocks in the Jordan normal form of C if $a = b = 0$.
- c) If $n = 2$ or $n = 3$, find the sizes of the blocks in the Jordan normal form of C if $a = 0$ but $b \neq 0$.
- d) If $n = 2$ or $n = 3$, find the sizes of the blocks in the Jordan normal form of C if $a \neq 0$ and $b \neq 0$.
- e) Based on the previous 3 parts can you guess what happens for general n ? You might workout some more complicated cases like $n = 4$ and $n = 5$ to see the pattern. Can you proof some your guesses? For example what is the size of the largest Jordan block in the 3 cases.