Math 4330 Prelim 1

Due Monday, October 7, 2019

Your work on this exam is to be done in accordance with the following:

1. You may not obtain aid nor discuss the exam with any other person.
2. If you have any questions, please send e-mail, call, or come by my office.
3. Please return the exam to me by 4:00 pm, Monday, October 7. You may instead submit your solutions as a pdf file by e-mail.

Please write your answers very carefully, explaining exactly what you are doing and showing all the computations.

Problem 1. Let V be a vector space and $\{W_i \mid i \in I\}$ be a collection of subspaces. The projections $p_i : V \to V/W_i$ can be combined in a linear mapping

$$p : V \to \prod_i V/W_i.$$

a) Find necessary and sufficient condition that p is one-to-one.

b) Find necessary and sufficient condition that p is isomorphism, provided that the index set I is finite.

c) If I is finite and p is an isomorphism construct subspaces $\{U_i \mid i \in I\}$ such that U_i is isomorphic to V/W_i and V is the internal direct sum of U_i.

d) Is there analog of part c) if the index set I is infinite?

Problem 2. Use only the universal mapping property characterising the direct sum of two vector spaces to prove that

a) $V \oplus W \approx W \oplus V$.

b) $(V_1 \oplus V_2) \oplus V_3 \approx V_1 \oplus (V_2 \oplus V_3)$.

Problem 3. Let F be a field of characteristic 0 and let $F[x]$ denotes the space of polynomials over F, and $F_k[x]$ the subspace consisting of polynomials of degree at most k. For any $\lambda \in F$ and any natural number $k \geq 1$ define a linear transformation

$$\phi_{\lambda,k} : F[x] \to F^k \quad \text{by} \quad \phi_{\lambda,k}(f) = \left(f(\lambda), f'(\lambda), \ldots, f^{(k-1)}(\lambda)\right),$$

where $f', f'', \ldots, f^{(k-1)}$ are the formal derivatives of the polynomial f; and $f'(\lambda)$, $f''(\lambda)$, $\ldots$, $f^{(k-1)}(\lambda)$ are their values at the point λ.

If $\{\lambda_i, i \leq l\}$ are (pairwise) distinct elements in F and $\{k_i, i \leq l\}$ are natural numbers, we can define

$$\phi : F[x] \to F^{\Sigma\, k_i}$$

by taking the direct sum of the maps ϕ_{λ_i,k_i} and identifying $\bigoplus F^{k_i} = \prod F^{k_i}$ with $F^{\Sigma\, k_i}$

a) Show that the restriction of $\phi_{\lambda,k}$ induces an isomorphism between $F_{k-1}[x]$ and F^k.

b) Show that the restriction of ϕ induces an isomorphism between $F_{\sum k_i - 1}[x]$ and $F^{\Sigma\, k_i}$.

c) Using part b) and picking a suitable basis of $F_{\sum k_i - 1}[x]$ show that the following matrices are inverible

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{k-1} \\ \vdots & & \ddots & & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^{k-1} \end{pmatrix} \quad \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \ldots & \lambda_1^{2k-1} \\ 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & \ldots & (2k-1)\lambda_1^{2k-2} \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \ldots & \lambda_2^{k-1} \\ 0 & 1 & 2\lambda_2 & 3\lambda_2^2 & \ldots & (2k-1)\lambda_2^{2k-2} \\ & \vdots & & \ddots & \vdots & \\ 1 & \lambda_k & \lambda_k^2 & \lambda_k^3 & \ldots & \lambda_k^{2k-1} \\ 0 & 1 & 2\lambda_k & 3\lambda_k^2 & \ldots & (2k-1)\lambda_k^{2k-2} \end{pmatrix}$$

d) Can you describe the inverses of these matrices?

e) Is the condition $charF = 0$ necessary in parts a) and b)? Can you relax it? What about parts c) and d)?

Problem 4. Let

$$V_0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_n \subset \ldots$$

be a filtration of a vector space $V = \bigcup V_n$ consisting of finite dimensional vector spaces over a field F of characteristic 0. Let D_1 and D_2 are two operators which satisfy $D_i(V_k) \subset V_{k-1}$ for all k-es.

a) Show that D_1 and D_2 are locally nilpotent transformations, i.e., for every $v \in V$ there exists n (depending on v) such that $D_1^n(v) = D_2^n(v) = 0$.

b) Prove that any finite linear combination of compositions of D_1 and D_2 is locally nilpotent transformation.

c) Prove that for any $t_1, t_2 \in F$ one has

$$\exp(t_1 D_1) \circ \exp(t_2 D_1) = \exp\left((t_1 + t_2) D_1\right)$$

d) Let $[A, B] = A \circ B - B \circ A$ denotes the commutator of the operators A and B. Assume that $D = [D_1, D_2]$ satisfies $[D, D_1] = [D, D_2] = 0$. Prove that for any $t_1, t_2 \in F$ the composition

$$\exp(t_1 D_1) \circ \exp(t_2 D_2) \circ \exp(-t_1 D_1) \circ \exp(-t_2 D_2).$$

can be expressed as a formal power series of D. How?

e) Construct an example of a vector space $V = \bigcup V_n$ and two transformations D_1 and D_2 satisfying the conditions is part d) and that D is not nilpotent.

f) Is it necessary that all V_i are finite dimensional?

Here $\exp(tD)$ is the operator which sends any $v \in V$ to $\sum_{i=0}^{\infty} \frac{t^i}{i!} D^i(v)$ – this is well defined since for any $v \in V$ the sum is finite (why?).

Problem 5. For any natural number n and integer i define $\phi_i(n) := \sum_{d|n} d^i$, where the sum is taken over all divisors of n. For every N and every i the $N \times N$ define the symmetric matrix $A^{(i,N)} = \{a_{p,q}^{(i)}\}$ with entries

$$a_{p,q}^{(i)} = \phi_i\left(\gcd(p,q)\right) \qquad 1 \le p, q \le N.$$

a) Prove that the matrix $A^{(0,N)}$ is invertible for all N.

b) Show that the inverse matrix $B^{(0,N)} = (A^{(0,N)})^{-1}$ has integer entries. Can you find formulas for entries of this matrix?

c) Prove that the matrix $A^{(i,N)}$ is invertible for all i, N.

d) Show that the inverse matrix $B^{(i,N)} = (A^{(i,N)})^{-1}$ has integer entries when $i \le 0$.

Here, $\gcd(p,q)$ denotes *the greatest common divisor* of the integers p, q.

Hint: Express the matrix $A^{(i,N)}$ as a product of two matrices.

Problem 6. There is a general theorem about the existence of solutions of ordinary linear differential equation: For any open interval $U \subset \mathbb{R}$ and any smooth functions $a_i : U \to \mathbb{R}$ the n-th order differential equation

$$a_n(x) f^{(n)}(x) + a_{n-1} f^{n-1}(c) + \cdots + a_2(x) f''(x) + a_1(x) f'(x) + a_0(x) f(x) = 0$$

has n dimensional space of solutions V (provided that $a_n(x) \neq 0$ for all $x \in U$). More over the maps $\phi_\lambda : V \to \mathbb{R}^n$ defined by

$$\phi_\lambda(f) = \left(f(\lambda), f'(\lambda), \ldots, f^{(n-1)}(\lambda)\right)$$

are isomorphisms for all $\lambda \in U$.

Consider the second order differential equation

$$x^2 f''(x) + x f'(x) + 4 f(x) = 0$$

on the interval $U = (0, \infty)$. This equation has a 2 dimensional space of solutions V. Let θ_u denote the linear transformation from V to $\mathbb{R}^2$ given by

$$\theta_u(f) = (f(1), f(u)).$$

Let $\{T_\alpha \mid \alpha \in \mathbb{R}\}$ denote the linear operators on the space of all smooth functions defined on U given by

$$(T_\alpha(f))(x) := f(e^\alpha x)$$

a) Show that $T_\alpha \circ T_\beta = T_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{R}$.

b) Show that the space V is invariant under the operators T_α.

c) Let S denote the set of all values of u such that θ_u is not isomorphism, i.e.,

$$S = \{u \mid \theta_u \text{ is not isomorphism}\}.$$

Show that S is discrete subset of U.

d) Use part b) to show that S is closed under multiplication.

e) Describe the set S and find a basis of V.

Hint: Part d) can be done only using a) and b) and the properties of the operators T_α.