Math 4330 Prelim 2

Due Monday, November 18, 2019

Your work on this exam is to be done in accordance with the following:

1. You may not obtain aid nor discuss the exam with any other person.
2. If you have any questions, please send e-mail, call, or come by my office.
3. Please return the exam to me by 4:00 pm, Monday, November 18. You may instead submit your solutions as a pdf file by e-mail.

Please write your answers very carefully, explaining exactly what you are doing and showing all the computations.

Problem 1. Let V be a vector space over a field F and $A : V \to V$ is locally nilpotent linear transformation.

a) Show that both the vector space V and the dual space V^* can be made into $F[[x]]$ modules using the action of A.

b) Desctibe all prime elements in the ring $F[[x]]$ (up to multiplication by a unit).

c) Find necessary and sufficent conditions for V and V^* to be finitely generated $F[[x]]$ modules.

d) Use part c) to show that if V is finitely generated $F[[x]]$ module then V^* is also finitely generated $F[[x]]$ module.

e) Give an example when V^* is fininitely generated but V is not.

Here, $F[[x]]$ denotes the ring of formal power serries, i.e., the elements are all infnite sums $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in F$.

Problem 2. Let V and W be vector spaces over a field F. A map $\langle , \rangle : V \times W \to F$ is called a *paring* if it is linear in both arguments, i.e.,

$$\langle \lambda v, w \rangle = \langle v, \lambda w \rangle = \lambda \langle v, w \rangle \quad \langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle \quad \langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$$

A pairing is called *non-degenerate* if: for any nozero $v \in V$ there exists $w \in W$ such that $\langle v, w \rangle \neq 0$; and for any nozero $w \in W$ there exists $v \in V$ such that $\langle v, w \rangle \neq 0$.

a) Show that any pairing defines two linear transformations $i_V : V \to W^*$ and $i_W : W \to V^*$. Moreover each of i_V and i_W determines the pairing. Can you define a function $T : \mathrm{Hom}(V \to W^*) \to \mathrm{Hom}(W \to V^*)$ which sends i_V to i_W?

b) Show that the pairing is nondegenerate if and only if both i_V and i_W are injective.

c) Two operators $A \in \mathrm{Hom}(V \to V)$ and $B \in \mathrm{Hom}(W \to W)$ are called *adjoint* (with respect to the pairing) is $\langle Av, w \rangle = \langle v, Bw \rangle$ for any $v \in V$ and any $w \in W$. Prove that if $\langle , \rangle$ is non degenerate then each A have at most one B which is adjoint to it. Give an example of a nondegenerate pairing and a liniear transformation A which does not have adjoint.

d) Give an example of a finite dimensional vector space V over algebraically closed field F of characteristic 0 and a symmetric pairing $\langle , \rangle : V \times V \to F$ and a *self-adjoint* linear trnasformation $A \in \mathrm{Hom}(V \to V)$ which is not diagonalizable.

   The pairing is symmetric if and only if $\langle v, w \rangle = \langle w, v \rangle$ for any $v, w \in V$; a transformtation A is self-adjoint is A is adjoint to itself.

Problem 3. Let P be the vector space of polynomials $F[x]$ over a field of charach-teristic 0. We have the finite difference operators $\delta_+, \delta_-, \delta_c : P \to P$ by

$$(\delta_+ f)(x) = f(x+1) - f(x) \quad (\delta_- f)(x) = f(x) - f(x-1)$$

a) Express the degrees of $\delta_+ f, \delta_- f$ in terms of the degree of f.

b) Construct two families of polynomials $p_+^{(i)}(x)$ and $p_-^{(i)}(x)$ such that

$$p_+^{(0)}(x) = 1; \quad p_+^{(i)}(0) = 0 \text{ for } i > 0; \quad (\delta_+(p_+^{(i)}))(x) = p_+^{(i-1)}(x)$$

and show that they form a basis of P (and similarly for δ_-).

c) Shat that the linear transformation $T : P \to F^{(\mathbb{N})}$ given by

$$T(f)(i) = ((\delta_+)^i f)(0)$$

is isomorphsim and find its inverse.

d) Use the previous part the prove the identity

$$p_+^{(n)}(x+y) = \sum_{k=0}^{n} a_{n,k} p_+^{(k)}(x) p_+^{(n-k)}(y)$$

for some coeficients $a_{n,k}$. Have you seen these coefiecients before?

e) Is there analog of part c) for the transformation $U : P \to F^{(\mathbb{N})}$ given by

$$U(f)(2i) = ((\delta_-)^i (\delta_+)^i f)(0) \quad U(f)(2i+1) = ((\delta_-)^i (\delta_+)^{i+1} f)(0)?$$

f) Can you find identity similar to the one is part d) using the isomorphism U?

g) What changes if F has a possitive characteristic?

Problem 4. Compute the Jordan form of a matrix A with coefficents in $\mathbb{Q}$ and find change of basis matrix P. Use the Jordan normal form of A to find the Smith normal form of the matrix $A - xId$ over $\mathbb{Q}[x]$. What is the characteristic polynomial of A? (The last two parts can be done without amnost any computations.)

Hint. You can use that the minimal polynomial of A is $(x-3)^3(x-2)^2$.

Hint. Computing the Jordan normal form involves non-trivial computations. You can use computers to do/verify all computations, but you need to write all steps as if you do the computations by hand – you are only allowed to skip steps in the row reduction process. Also, you do not need to verify that A satisifies the minimal polynomial given above. It is possible to find P such that all coeffients of P and P^{-1} are integers with absoulute value less that 1000.

$$A = \begin{pmatrix}
-141 & -78 & 16 & -3 & 0 & 0 & -6 & -1 & 1 & 83 & -23 & -24 & -6 \\
22 & 14 & -2 & 0 & 0 & 0 & 6 & 1 & -1 & -16 & 4 & 6 & 0 \\
-1452 & -792 & 167 & -33 & 0 & 0 & -30 & -5 & 5 & 825 & -231 & -231 & -66 \\
-1452 & -792 & 164 & -30 & 0 & 0 & -30 & -5 & 5 & 832 & -232 & -236 & -64 \\
-976 & -536 & 112 & -24 & -85 & 232 & 0 & 0 & 0 & 1885 & -464 & -580 & -145 \\
-366 & -201 & 42 & -9 & -33 & 90 & 0 & 0 & 0 & 715 & -176 & -220 & -55 \\
0 & 0 & 0 & 0 & 0 & 0 & 27 & 4 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 50 & 10 & -8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 200 & 32 & -30 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 25 & 4 & -4 & -206 & 57 & 57 & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -275 & 78 & 75 & 25 \\
0 & 0 & 0 & 0 & 0 & 0 & 50 & 8 & -8 & -374 & 102 & 105 & 34 \\
0 & 0 & 0 & 0 & 0 & 0 & 125 & 20 & -20 & -341 & 93 & 93 & 34
\end{pmatrix}$$