Problem 5.3. This is the direct corollary of the axioms of scalar product. Indeed, using the fact that for any $k \in \mathbb{N}$, u_k is ortonormal to the subspace of V generated by $\{u_j\}$, $j \in \mathbb{N} - \{k\}$, we have $< a_k u_k, \sum_{k=1}^{\infty} b_k u_k >=< a_k u_k, b_k u_k > + < u_k, \sum_{j \in \mathbb{N}-\{k\}} b_j u_j >= a_k \overline{b_k} + 0 = a_k \overline{b_k}$. Taking a sum over k, we get the desired expression.

Problem 5.4. (a) By definition of p_k, $\phi(x) = \sum_{k \in \mathbb{Z}} p_k \phi(2x - k)$. This implies $\phi(x - l) = \sum_{k \in \mathbb{Z}} p_{k-2l} \phi(2x - k)$. Combining these two equalities with Parseval's equation and the fact that $\{\phi(2x - k)\}$ is an orthonormal basis, we get the equality $< \phi(x - l), \phi(x) >= \sum_{k \in \mathbb{Z}} p_{k-2l} \overline{p_k}$. Remembering that $< \phi(x - l), \phi(x) >= 2\delta_{l0}$, we conclude that $\sum_{k \in \mathbb{Z}} p_{k-2l} \overline{p_k} = 2\delta_{l0}$.

(b) We have $\psi_{0m} = \sum_{k \in \mathbb{Z}} (-1)^k 2^{-1/2} \overline{p_{1-k}} \phi(2x - 2m - k) = \sum_{k' \in \mathbb{Z}} (-1)^{k'-2m} 2^{-1/2} \overline{p_{1-k'+2m}} \phi(2x - k')$, where we set $k' = k + 2m$. Also $\psi_{0m} = \sum_{j \in \mathbb{Z}} (-1)^{j-2l} 2^{-1/2} \overline{p_{1-j+2l}} \phi(2x - j)$. Using Parseval's equation and the fact that $\{\phi(2x - k)\}$ is an orthonormal basis, we get the desired result.

(c) This part can be solved combining the expressions for ϕ_{0l} and ψ_{0m} from parts 1 and 2 with Parseval's equation.

Problem 5.6. (a) We have $u_j = \sum_{k \in \mathbb{Z}} < u, \phi_{jk} > \phi_{jk}$, so we have only to determine what $< u, \phi_{jk} >$ equals to. By definition, $< u, \phi_{jk} >= \int_{-\infty}^{+infty} = \int_0^1 u(x) \phi(2^j x - k) dx$. Using change of variable $y = 2^j - k$, we get $< u, \phi_{jk} >= 2^{-j/2} \int_{-k}^{2^j - k} \phi(y) dy$.

(b) As u_j is an orthogonal projection of u to some subspace, we have $< u_j, u >= < u_j, u_j >$, whence $||u - u_j||^2 =< u - u_j, u - u_j >=< u, u > -2 < u_j, u_j > + < u_j, u_j >= 1 - ||u_j||^2 = 1 - 2^{-j} \sum_{k \in \mathbb{Z}} (\int_{-k}^{2^j - k} \phi(y) dy)^2$. So we are left to show $2^{-j} \sum_{k \in \mathbb{Z}} (\int_{-k}^{2^j - k} \phi(y) dy)^2 < 1/2$ for j large enough. One has to be accurate here (actually no one had the completely correct solution in the assignments I graded). An accurate solution may go as follows. Let $f(j, k) = (\int_{-k}^{2^j - k} \phi(y) dy)^2$; assume that the support of $\phi(x)$ lies in $[-N; N]$, and $max|\phi(x)| = L$; L is finite, as $\phi(x)$ is continuous and compactly supported. Then $f(j, k)$ may be non-zero only if $k \in (-N; N)$ of $2^j - k \in (-N; N)$. So for any fixed j, there is at most $4N$ values of k such that $f(j, k) \neq 0$. Further, $|f(j, k)| \leq \int_{-N}^{N} |phi(x)|^2 dx \leq 2NL^2$. Hence, $|\sum_{k \in \mathbb{Z}} f(j, k)| \leq 8N^2 L^2$ for any $j \in \mathbb{N}$, whence $2^{-j} \sum_{k \in \mathbb{Z}} f(j, k) \to 0$ when $j \to \infty$.

(c) As $\{V_j\}$ is an MRA, we should have $u_j \to u$ as $j \to \infty$. Comparing this with the result of part (b), we see that $\int \phi(x) dx \neq 0$.

Problem 5.7. (a) Axioms 1 and 4 are trivially satisfied. To show axiom 2 is satisfied, let $g = F(f)$, and take $g_n = g\chi_{[-2^n \pi, 2^n \pi]}$. Then $g_n \to g$, and $f_n = F^{-1}(g_n) \in V_n$, $f_n \to f$, as Fourier transform is an isometry. To see axiom 3 is satisfied, note that if *supp* $g \in [-2^n \pi, 2^n \pi]$ for any $n \in \mathbb{Z}$ implies that *supp* $g = \{0\}$, so g is equal to 0 in L^2, and so is f.

(b) If we take $\Omega = \pi$ in the Sampling Theorem, we will get the desired result.

(c), (d). You just have to compute the corresponding coefficients here, using common integral representation for them.