# TIME-FREQUENCY ESTIMATES FOR PSEUDODIFFERENTIAL OPERATORS

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Abstract. We present several boundedness results for linear and multilinear pseudodifferential operators on modulation spaces.

## 1. Introduction

The recent years have seen an increasing interest in the theory of modulation spaces. The family of modulation spaces appear naturally in the study of certain functions and operators, especially when one is interested in both the time and frequency description of such objects. They constitute a family of Banach spaces of distributions that behave very much like the Besov spaces: the dilation in the definition of Besov spaces is essentially replaced by frequency shifts.

In this paper, we present certain results and techniques regarding the boundednes of linear and multilinear operators acting on these spaces. Our goal is to provide a story of estimates, known and unknown, on modulation spaces. We would like to convey to the reader that such estimates are natural substitutes when other classical function space estimates fail. Moreover, general estimates on modulation spaces translate-via embeddings- into estimates on Lebesgue, Besov, or Sobolev spaces. In telling our story of estimates, we choose a non-uniform approach by jumping from linear to multilinear estimates. Several topics presented here might seem unrelated when read separately. The unifying theme, however, is the general framework of modulation spaces that led us to ask certain questions in the first place once previous questions have been answered in a satisfactory manner.

Our paper is organized as follows. In Section 2 we set the notations and definitions that will be used throughout this paper. We also define the modulation spaces and collect some of their properties that will be needed later on. In Section 3 we present an overview of estimates for various operators. To the best of our knowledge, some of these estimates or techniques do not appear elsewhere in the literature.

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## 2. Preliminaries

2.1. **General notation.** Translation and modulation of a function $f$ with domain $\mathbb{R}^d$ are defined, respectively, by $T_x f(t) = f(t-x)$ and $M_y f(t) = e^{2\pi i y\cdot t} f(t)$. The Fourier transform of $f \in L^1$ is $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i t\cdot\omega}\, dt$, $\omega \in \mathbb{R}^d$. The Fourier transform is an isomorphism of the Schwartz space $\mathbb{S}$ onto itself, and extends to the space $\mathbb{S}'$ of tempered distributions by duality. The inverse Fourier transform is $\check{f}(x) = \hat{f}(-x)$.

The inner product of two functions $f, g \in L^2$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\, dt$, and its extension to $\mathbb{S}' \times \mathbb{S}$ will be also denoted by $\langle\cdot,\cdot\rangle$.

The Short-Time Fourier Transform (STFT) of a function $f$ with respect to a window $g$ is

$$V_g f(x,y) = \langle f, M_y T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot t}\, \overline{g(t-x)}\, f(t)\, dt,$$

whenever the integral makes sense. If $g \in \mathbb{S}$ and $f \in \mathbb{S}'$ then $V_g f$ is a uniformly continuous function on $\mathbb{R}^{2d}$. One important technical tool is the extended isometry property of the STFT [11, (14.31)]: If $\phi \in \mathbb{S}$, $\|\phi\|_{L^2} = 1$, then

$$\langle f, h\rangle = \langle V_\phi f, V_\phi h\rangle, \qquad \forall\, f \in \mathbb{S}',\, h \in \mathbb{S}. \tag{1}$$

Given a strictly positive function $\nu$ on $\mathbb{R}^{2d}$, we let $L^{p,q}_\nu$ be the spaces of measurable functions $f(x,y)$ for which the weighted mixed norms

$$\|f\|_{L^{p,q}_\nu} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x,y)|^p\, \nu(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q}$$

are finite. If $p = q$, we have $L^{p,p}_\nu(\mathbb{R}^{2d}) = L^p_\nu(\mathbb{R}^{2d})$, a weighted Lebesgue space.

2.2. **Weight functions.** Given $s \geq 0$, a positive, continuous, and symmetric function $\nu$ is called an $s$-moderate weight if there exists a constant $C > 0$ such that

$$\forall\, x, y \in \mathbb{R}^d, \quad \nu(x+y) \leq C\,(1+|x|^2)^{s/2}\,\nu(y). \tag{2}$$

For example, $\nu(x) = (1+|x|^2)^{t/2}$ is $s$-moderate exactly for $|t| \leq s$. If $\nu$ is $s$-moderate, then by manipulating (2) we see that

$$\frac{1}{\nu(x+y)} \leq C\,(1+|x|^2)^{s/2}\,\frac{1}{\nu(y)},$$

hence, $1/\nu$ is also $s$-moderate (with the same constant).

In the sequel we let $\omega_s(x) = (1+|x|^2)^{s/2}$ for $s \geq 0$.

2.3. **Modulation spaces.**

**Definition 1.** Given $1 \leq p, q \leq \infty$, and given a window function $g \in \mathcal{S}$, and an $s$-moderate weight $\nu$ defined on $\mathbb{R}^{2d}$, the modulation space $\mathcal{M}^{p,q}_\nu = \mathcal{M}^{p,q}_\nu(\mathbb{R}^d)$ is the space of all distributions $f \in \mathbb{S}'$ for which the following norm is finite:

$$\|f\|_{\mathcal{M}^{p,q}_\nu} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,y)|^p\, \nu(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q} = \|V_g f\|_{L^{p,q}_\nu}, \tag{3}$$

with the usual modifications if $p$ and/or $q$ are infinite. If $\nu = 1$, we will simply write $\mathcal{M}^{p,q}$ for the modulation space $\mathcal{M}^{p,q}_\nu$. If $\nu(x,y) = \omega_s(x)$, we will simply write $\mathcal{M}^{p,q}_s$. Moreover, when $p = q$, we will write $\mathcal{M}^p_\nu$ for the modulation space $\mathcal{M}^{p,p}_\nu$.

The definition is independent of the choice of the window $g$ in the sense of equivalent norms. We refer to [7, 11] and the references therein for more details about modulation spaces.

The definition above quantifies both the time and frequency contents of a function or distribution. Although not completely correct, one can think of $f \in \mathcal{M}^{p,q}$ as being represented by the statement "$f \in L^p$ and $\hat{f} \in L^q$"; for a rigorous comparison of modulation spaces and Fourier-Lebesgue spaces see [10]. Fore more embeddings between modulation spaces and other function spaces, see [10, 12, 14, 17, 21].

*Remark* 1. For $p = q = 2$, if $\nu \equiv 1$ then $\mathcal{M}^2_\nu = L^2$; if $\nu(x,y) = (1+|x|^2)^{s/2}$ then $\mathcal{M}^2_\nu = L^2_s$, a weighted $L^2$-space; if $\nu(x,y) = (1+|y|^2)^{s/2}$ then $\mathcal{M}^2_\nu = H^s$, the standard Sobolev space, and if $\nu(x,y) = (1+|x|^2+|y|^2)^{s/2}$ then $\mathcal{M}^2_\nu = L^2_s \cap H^s$.

## 3. Estimates

A $k$-linear pseudodifferential operator is defined à priori through its (distributional) symbol $\sigma$ to be the mapping $T_\sigma$ from the $k$-fold product of Schwartz spaces $\mathbb{S} \times \cdots \times \mathbb{S}$ into the space $\mathbb{S}'$ of tempered distributions given by the formula

$$\begin{aligned} &T_\sigma(f_1,\ldots,f_k)(x) \\ &= \int_{\mathbb{R}^{kd}} \sigma(x,\xi_1,\ldots,\xi_k)\, \hat{f}_1(\xi_1) \cdots \hat{f}_k(\xi_k)\, e^{2\pi i x\cdot(\xi_1+\cdots+\xi_k)}\, d\xi_1 \cdots d\xi_k, \end{aligned} \tag{4}$$

for $f_1,\ldots,f_k \in \mathbb{S}$.

1-linear operators are simply called *linear*, 2-linear operators are called *bilinear*, and so on. The pointwise product $f_1 \cdots f_k$ corresponds to the case $\sigma \equiv 1$.

We start our story of estimates with a classical class of symbols.

3.1. **The Calderón-Vaillancourt class.** Assume that $\sigma \in L^\infty$ and consider the linear operator $L_1(f) = \sigma f$. Clearly, this operator is bounded on all $L^p$ spaces, $1 \le p \le \infty$. Hölder's inequality tells us that the bilinear operator $L_2(f,g) = \sigma f g$ is bounded from $L^p \times L^q$ into $L^r$ if the exponents are larger than 1 and satisfy the relation $1/p + 1/q = 1/r$. The norms of operators $L_1, L_2$ coincide with $\|\sigma\|_{L^\infty}$. These cases correspond to pseudodifferential operators with symbol $\sigma(x,\xi) = \sigma(x)$ independent of the frequency variable $\xi$.

A much more interesting class of operators is defined on the frequency side by $\widehat{T(f)} = \sigma \hat{f}$. In this case, $T$ coincides with a Fourier multiplier with symbol $\sigma(x,\xi) = \sigma(\xi)$ independent of the space variable $x$. Using Plancherel's identity, we see that $T$ is bounded from $L^2$ into $L^2$ if $\sigma \in L^\infty$. It is not clear, however, how one would approach the boundedness of $T$ on $L^p$ for $p \neq 2$. If one could obtain another $L^{p_0}$ estimate, then interpolation would give boundedness for all exponents in the range $(2,p_0)$ or $(p_0,2)$. A naive approach would be, for example, the following:

$$\|T(f)\|_{L^\infty} = \|(m\hat{f})\check{}\|_{L^\infty} \le \|m\hat{f}\|_{L^1} \le \|m\|_{L^\infty}\|\hat{f}\|_{L^1}.$$

But this is perhaps the most that we can accomplish this way, since, in general, we do not have any good control on the $L^1$ norm of the Fourier transform of a function. The same approach would have failed if we would have started with a different exponent $p_0$. Nevertheless, we proved that the operator $T$ is bounded from $L^2 \to L^2$ and $\widehat{L^1} \to L^\infty$. The natural question then is whether there are any "intermediate" spaces on which the operator remains bounded.

The questions above become yet more difficult if we modify the operators and allow the symbols to be both $x$ and $\xi$ dependent. Consider then the classical symbol class $S^0_{0,0}$ consisting of those $\sigma$ which satisfy estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}, \qquad \forall\, \alpha, \beta \ge 0. \tag{5}$$

A classical result of Calderón and Vaillancourt [6] asserts that the corresponding *linear* pseudodifferential operator $T_\sigma$ is bounded on $L^2$. Notice that these operators are generally unbounded on $L^p$ for $p \neq 2$ [1].

In the *bilinear* case, however, the analogous class of symbols which satisfy the conditions

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}, \qquad \forall\, \alpha, \beta, \gamma \ge 0, \tag{6}$$

does not necessarily yield bounded operators from $L^2 \times L^2$ into $L^1$, unless additional size conditions are imposed on the symbols; see [5]. Nevertheless, a Calderón–Vaillancourt-like condition (6) does yield boundedness from $L^2 \times L^2$ into the *modulation space* $\mathcal{M}^{1,\infty}$ that contains $L^1$. Indeed, considering this problem in the framework of modulation spaces we find the desired answer for general $k$-linear pseudodifferential operators; see [3] for the proof and further details.

**Theorem 1.** *If $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{(k+1)d})$, then the $k$-linear pseudodifferential operator $T_\sigma$ defined by* (4) *extends to a bounded operator from $\mathcal{M}^{p_1,q_1} \times \cdots \times \mathcal{M}^{p_k,q_k}$ into $\mathcal{M}^{p_0,q_0}$ when $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{p_0}$, $\frac{1}{q_1} + \cdots + \frac{1}{q_k} = k - 1 + \frac{1}{q_0}$, and $1 \le p_i, q_i \le \infty$ for $0 \le i \le k$.*

All the questions we asked above can be answered from this theorem, by using the embeddings $S^0_{0,0} \subset \mathcal{M}^{\infty,1}$, $L^1 \subset \mathcal{M}^{1,\infty}$, $\mathcal{M}^{2,2} = L^2$, and choosing $k = 1$ respectively $k = 2$.

It is worth pointing out that there is an extensive literature on the continuity properties of (linear) pseudodifferential operators on modulation spaces; see [9, 13, 18, 21].

3.2. **The Hilbert transform and related multipliers.** Closely related to the linear operators with symbols in the class $S^0_{0,0}$ is the Hilbert transform defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|t|>\epsilon} \frac{f(x-t)}{t}\, dt.$$

$H$ is a Fourier multiplier operator

$$\widehat{Hf} = m\widehat{f}, \tag{7}$$

where $m(\xi) = -i \operatorname{sgn} \xi$. The multiplier $m$ is bounded and its derivatives, which exist everywhere except at the origin, are also bounded. Thus, $m$ "almost" belongs to the class $S^0_{0,0}$. Yet, its behavior does not follow the pattern we saw in the previous subsection: the Hilbert transform is bounded on all $L^p$ spaces, $1 < p < \infty$. This is due to the fact that $H$ enjoys additional cancellation properties, and as such it can be treated in the context of Calderón-Zygmund theory; $m$ is known to be a particular example of a Hörmander-Mihlin multiplier as well.

Nevertheless, $m$ does follow the pattern of the Calderón-Vaillancourt class on modulation spaces. In fact, a larger class of multipliers extending the Hilbert transform enjoys this property of boundedness on modulation spaces. Let $b > 0$ and $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$ be a bounded sequence of complex numbers. The operators $H_{b,\mathbf{c}}$ are defined by

$$\widehat{H_{b,\mathbf{c}}f} = m_{b,\mathbf{c}}\widehat{f}, \tag{8}$$

with Fourier multipliers

$$m_{b,\mathbf{c}} = -2i \sum_{n=-\infty}^{+\infty} c_n \chi_{(bn,b(n+1))}; \tag{9}$$

$\chi_{(a,b)}$ denotes the characteristic function of the real interval $(a, b)$. It is easy to see that $H = \frac{1}{2}H_{b,\mathbf{c}}$ for any $b > 0$ and $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$, with $c_n = 1$ for $n \geq 0$ and $c_n = -1$ for $n < 0$.

The following result was proved in [2].

**Theorem 2.** *The operators $H_{b,\mathbf{c}}$ are bounded from $\mathcal{M}^{p,q}(\mathbb{R})$ into $\mathcal{M}^{p,q}(\mathbb{R})$ for $1 < p < \infty$, $1 \leq q \leq \infty$ with a norm estimate*

$$\|H_{b,\mathbf{c}}f\|_{\mathcal{M}^{p,q}} \leq C\,\|\mathbf{c}\|_\infty\,\|f\|_{\mathcal{M}^{p,q}}$$

*for some constant depending only on $b, p$, and $q$. In particular, the Hilbert transform $H$ is bounded on $\mathcal{M}^{p,q}$ for $1 < p < \infty$ and $1 \leq q \leq \infty$*

*Remark* 2. Interestingly enough, the multiplier operators $H_{b,\mathbf{c}}$ are not bounded in general on $L^p$ spaces, except when $p = 2$. As such, they resemble the behavior of the class $S^0_{0,0}$ investigated in the previous subsection: the modulation spaces are the "appropriate" spaces on which to study their boundedness. See [2] for some extensions of Theorem 2.

In [8], a slightly more general class of multipliers as the one exposed in Theorem 2 was shown to be bounded on modulation spaces.

3.3. **Fourier multipliers and evolution PDEs.** A strong motivation for the development of a theory of pseudodifferential operators is provided by the fact that pseudodifferential operators generalize classical partial differential operators with variable coefficients. As such, it is not surprising that they appear in the study of solutions of certain partial differential equations.

An extensive amount of research in the area of partial differential equations has been devoted in recent years to study the well-posedness or solvability of various dispersive equations; see [19], also [20]. In the remaining of this subsection we would like to explore two such classical evolution equations: the Schrödinger and wave ones. We will see how to approach their behavior in the context of modulation spaces, as well as the connections with the operators discussed above.

The Schrödinger equation $iu_t - \Delta u = 0$, for example, with $u$ a complex valued function in $\mathbb{R}^d \times \mathbb{R}$, describes the evolution of a free non-relativistic quantum particle in $d$ spatial dimensions. This equation can be perturbed in many ways, mainly by adding a potential or an obstacle, and the resulting equations arise as models from several areas of physics. Clearly, the solution of the corresponding Cauchy problem with initial data $f$ is given by a Fourier multiplier operator:

$$u(x,t) = T_t f(x) = \int_{\mathbb{R}^d} m_t(\xi) e^{2\pi i x\cdot\xi} \hat{f}(\xi) d\xi, \tag{10}$$

where $m_t(\xi) = e^{it|\xi|^2}$.

It is known that $T_t$ is bounded on $L^p(\mathbb{R}^d)$ only for $p = 2$ [15]. Therefore, it is natural to ask whether this Fourier multiplier operator is of Calderón-Vaillancourt type on modulation spaces.

Similarly, the wave Cauchy problem $\partial_{tt} u - \Delta u = 0$, $u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$ has a solution represented by a sum of two Fourier multiplier operators. In this case $u(x,t) = T_t^1 f(x) + T_t^2 g(x)$, where, like in (10), the multipliers are given by $m_t^1(\xi) = \cos t|\xi|$, and $m_t^2(\xi) = \sin(t|\xi|)/|\xi|$. Again, it is known in this case that $T_t$ is bounded on $L^p(\mathbb{R}^d)$ only when $p = 2$ and $d \geq 1$ or when $1 < p < \infty$ and $d = 1$ [15, 16].

In a recent work [4], combining tools from Littlewood-Paley theory and time-frequency analysis, it has been proved that a large family of Fourier multipliers whose symbols are given by unimodular functions $e^{i|\xi|^\alpha}$ for a specific range of $\alpha$, are bounded on all modulation spaces. It is worth pointing out that these operators are in general unbounded on other classical function spaces such as the Lebesgue spaces.

3.4. **Strichartz-type estimates.** The solution of the linear Schrödinger equation $iu_t - \Delta u = 0$ with initial condition $u(x,0) = f(x)$ is given by a $t$-parameter Fourier multiplier (10) $u(x,t) = T_t f(x)$, typically written as $u(x,t) = e^{it\Delta} f(x)$.

The classical Strichartz estimates for the Schrödinger equation give $L_t^q L_x^r$ bounds on the solution $u(x,t)$ in terms of the initial condition $f$. Here,

$$\|u(x,t)\|_{L_t^q L_x^r} = \|\|u(\cdot,t)\|_{L_x^r}\|_{L_t^q}.$$

The pair of exponents $(q,r)$ is called *Schrödinger admissible* if $q, r \geq 2, (q,r,d) \neq (2,\infty,2)$ and $1/q + d/2r = d/4$. For such a pair we have

$$\|e^{it\Delta} f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2}. \tag{11}$$

In particular, for the admissible pair $(q^*, q^*), q^* = 2(d+2)/d$, (11) gives

$$\|e^{it\Delta} f\|_{L^{q^*}(\mathbb{R}^d \times R)} \lesssim \|f\|_{L^2(\mathbb{R}^d)}. \tag{12}$$

But, since $q^* > 2$, we have $L^{q^*} \subset \mathcal{M}^{q^*,q^*}$, and $L^2 = \mathcal{M}^{2,2}$, therefore from (12) we obtain a Strichartz estimate for a pair of modulation spaces

$$\|e^{it\Delta} f\|_{\mathcal{M}^{q^*,q^*}} \lesssim \|f\|_{\mathcal{M}^{2,2}}. \tag{13}$$

It is natural to ask then if there are other *Schrödinger admissible quadruples* $(p,q,r,s)$ such that

$$\|e^{it\Delta} f\|_{\mathcal{M}^{p,q}(x,t)} \lesssim \|f\|_{\mathcal{M}^{r,s}(x)}. \tag{14}$$

Similarly, it would be interesting to know for which *Schrödinger admissible triples* $(p,q,r)$ we have an estimate like

$$\|e^{it\Delta}f\|_{\mathcal{M}^{p,q}(x,t)} \lesssim \|f\|_{L^r(x)}. \tag{15}$$

Note that from the embeddings $L^p \subseteq \mathcal{M}^{p,p'}, 1 \le p \le 2$ and $L^p \subseteq \mathcal{M}^{p,p}, 2 \le p \le \infty$, we immediately get the following *spatial* estimates for triples $(p,p',p')$ respectively $(p,p,p')$:

$$\|e^{it\Delta}f\|_{\mathcal{M}^{p,p'}(x)} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}(x)}, 1 \le p \le 2$$

and

$$\|e^{it\Delta}f\|_{\mathcal{M}^{p,p}(x)} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}(x)}, 2 \le p \le \infty.$$

However, only from these inequalities we cannot conclude estimates of the form (15). Indeed, unlike the Lebesgue spaces, the modulation spaces are not lattice spaces, that is, it is not true that if $|f| \le |g|$ and $g \in \mathcal{M}^{p,q}$ then $f \in \mathcal{M}^{p,q}$.

Following ideas from the classical Schrödinger estimates on mixed Lebesgue spaces, we would like to make use of the scaling properties of the equation to arrive to a possible definition of *admissibility for quadruples* in (14). A similar condition will then give admissibility for triples in (15). Unfortunately, the scaling of the STFT with respect to a window function $g$ gets corrupted, therefore we need to correct it by introducing a new parameter in the definition of the STFT.

For $\mu > 0$ and a given window $g$, the $d$-dimensional $\mu$-STFT is defined by

$$V_{g;\mu}f(x,y) = \mu^{-d}\int_{\mathbb{R}^d} e^{-2\pi i y\cdot z}\,\overline{g(\frac{z-x}{\mu})}\, f(z)\,dt. \tag{16}$$

Let $f_\lambda(z) = f(\lambda z)$. It is easy to see that

$$V_{g;\mu}f_\lambda(x,y) = V_{g;\mu\lambda}f(\lambda x, \frac{y}{\lambda})$$

and

$$\|V_{g;\mu}f_\lambda\|_{L^{p,q}} = \lambda^{d(\frac{1}{q}-\frac{1}{p})}\|V_{g;\mu\lambda}f\|_{L^{p,q}}.$$

It is natural then to define the spaces $\widetilde{\mathcal{M}^{p,q}}$ through

$$\|f(x)\|_{\widetilde{\mathcal{M}^{p,q}}} = \sup_{\mu>0}\|V_{g;\mu}f\|_{L^{p,q}}. \tag{17}$$

Clearly, by letting $\mu = 1$ in (17), we have $\widetilde{\mathcal{M}^{p,q}} \subset \mathcal{M}^{p,q}$. For the smaller spaces $\widetilde{\mathcal{M}^{p,q}}$ we have a "good" scaling property

$$\|f_\lambda\|_{\widetilde{\mathcal{M}^{p,q}}} = \lambda^{d(\frac{1}{q}-\frac{1}{p})}\|f\|_{\widetilde{\mathcal{M}^{p,q}}}.$$

This takes care of the scaling of the initial condition. We now take a similar route to deal with the scaling of the solution $u(x,t)$. In particular, we would like to investigate the mixed Lebesgue norms of appropriate $(\mu_1,\mu_2)$-STFT of scaled solutions $u_\lambda(x,t) = u(x/\lambda, t/\lambda^2)$. Here, we distinguish between the positive parameters $\mu_1$ and $\mu_2$ that refer to the distinct homogeneities in $x$ respectively $t$. With $g = g_1 \otimes g_2$, a similar computation as above gives

$$\|V_{g;\mu_1,\mu_2}u_\lambda\|_{L^{p,q}} = \lambda^{(d+2)(\frac{1}{p}-\frac{1}{q})}\|V_{g;\mu_1/\lambda,\mu_2/\lambda^2}u\|_{L^{p,q}}.$$

In particular, by denoting

$$\|u(x,t)\|_{\overline{\mathcal{M}^{p,q}}} = \sup_{\mu_1,\mu_2>0} \|V_{g;\mu_1,\mu_2} u\|_{L^{p,q}}, \tag{18}$$

we obtain the scaling property

$$\|u_\lambda\|_{\overline{\mathcal{M}^{p,q}}} = \lambda^{(d+2)(\frac{1}{p}-\frac{1}{q})}\|u\|_{\overline{\mathcal{M}^{p,q}}}.$$

We know that if $u$ is a solution of the linear Schrödinger equation with initial condition $f$, then $u_\lambda$ is also a solution for the problem with initial condition $f_\lambda$. Therefore, in order for an estimate

$$\|e^{it\Delta}f\|_{\overline{\mathcal{M}^{p,q}}(x,t)} \lesssim \|f\|_{\widetilde{\mathcal{M}^{r,s}}(x)} \tag{19}$$

to hold, the quadruple $(p,q,r,s)$ must satisfy the *admissibility condition*

$$d(\frac{1}{s}-\frac{1}{r}) = (d+2)(\frac{1}{p}-\frac{1}{q}).$$

Admissibility for triples $(p,q,r)$ in the analogue of (15) with $\overline{\mathcal{M}^{p,q}}$-norms on the left hand-side is

$$\frac{d}{r} = (d+2)(\frac{1}{q}-\frac{1}{p}).$$

Note, however, that the admissibility condition for quadruples is not sufficient. For example, the quadruple $(2,2,2,2)$ is admissible, yet (19) does not hold. This is a simple consequence of Plancherel's theorem: $\|u(t,\cdot)\|_{L^2_x} = \|f\|_{L^2_x}$.

Unfortunately, as it was pointed out to us by K. Gröchenig, there is a flaw in this approach. While introducing the modified modulation spaces $\widetilde{\mathcal{M}^{p,q}}$ (or $\overline{\mathcal{M}^{p,q}}$) through the parametrized STFT makes sense, our definition might give only the trivial $\{0\}$ space in certain situations. For example, for $p=q=2$, $\|V_{g,\mu}f_\lambda\|_{L^2} = \|V_{g,\mu\lambda}f\|_{L^2}$, and $\|V_{g,\mu}f\|_{L^2} = \|f\|_{L^2}\|g_\mu\|_{L^2} = \|f\|_{L^2}\|g\|_{L^2}\mu^{-d/2}$. This explains why certain quadruples, although admissible, do not yield the expected boundedness.

It is not clear to these authors what modifications, if any, are needed to our arguments, or what additional restrictions to require on the quadruples that would guarantee estimates of the form (14). It may also well be the case that the quadruple $(q^*,q^*,2,2)$ (which is admissible) is in fact the only one for which a modulation-Strichartz type estimate holds.

3.5. **Derivative estimates.** We end our overview of estimates on modulation spaces with the following boundedness results about partial differential operators. The results we present in this subsection are special cases of more general results obtained independently by Feichtinger [7, Remark 6.3] and by Toft [22, Corollary 3.3].

Let

$$\nu(x,y) = \nu_{a,b}(x,y) = (1+|x|^2)^{a/2}(1+|y|^2)^{b/2}.$$

We denote the weighted modulation space $\mathcal{M}^{p,q}_\nu$ by $\mathcal{M}^{p,q}_{a,b}$. The weighted Lebesgue space $L^p_{\nu_{a,0}}$ is denoted by $L^p_a$.

**Theorem 3.** *If* $1 \leq r \leq p, 1 \leq s \leq q$, $a > d(1/r - 1/p)$, *and* $b > d(\frac{1}{s} - \frac{1}{q})$, *then*

$$\|\frac{\partial^\alpha f}{\partial x^\alpha}\|_{\mathcal{M}^{r,s}} \leq C(p,q,r,s,\alpha,d)\|f\|_{\mathcal{M}^{p,q}_{a,b}}$$

*for all multiindices* $\alpha$ *such that* $|\alpha| < b - d(\frac{1}{s} - \frac{1}{q})$.

*Proof.* Note that for a given window $g$, we have

$$V_g(\frac{\partial^\alpha f}{\partial x^\alpha})(u,v) = (-1)^{|\alpha|}\langle f, \frac{\partial^\alpha}{\partial x^\alpha} M_v T_u g\rangle.$$

Using Leibniz's rule to compute the derivative of the product $M_y T_x g$, and using the notation $g_{\alpha-\gamma} = \frac{\partial^{\alpha-\gamma} g}{\partial x^{\alpha-\gamma}}$, we obtain

$$V_g(\frac{\partial^\alpha f}{\partial x^\alpha})(u,v) = (-1)^{|\alpha|}\sum_{|\gamma|\leq|\alpha|} c_{\alpha,\gamma}(2\pi i)^{|\gamma|} v^\gamma V_{g_{\alpha-\gamma}} f(u,v).$$

Therefore,

$$\|V_g(\frac{\partial^\alpha f}{\partial x^\alpha})(\cdot,v)\|_{L^r} \leq C(\alpha)\sum_{|\gamma|\leq|\alpha|} |v|^{|\gamma|}\|V_{g_{\alpha-\gamma}} f(\cdot,y)\|_{L^r}.$$

Using the conditions of the parameters $a, r, p$ and Hölder's inequality, we can write

$$\|V_{g_{\alpha-\gamma}} f(\cdot,v)\|_{L^r}^r = \int |V_{g_{\alpha-\gamma}} f(u,v)|^r (1+|u|^2)^{ra/2}(1+|u|^2)^{-ra/2}\, du$$

(20) $$\leq C(r,p,d)\|V_{g_{\alpha-\gamma}} f(\cdot,v)\|_{L^p_a}^r.$$

(21)

Integrate now with respect to $y$ to obtain

$$\|V_g(\frac{\partial^\alpha f}{\partial x^\alpha})\|_{L^{r,s}} \leq C(p,r,\alpha,d)\sum_{|\gamma|\leq|\alpha|} \|(1+|v|^2)^{|\gamma|/2}\|V_{g_{\alpha-\gamma}} f(\cdot,v)\|_{L^p_a}\|_{L^s}.$$

Using now the conditions on $b, s, q$, a similar argument as above using Hölder's inequality allows us to conclude the inequality we wanted to prove. □

**Corollary 1.** *Let* $P(D)(f) = \sum_{|\alpha|\leq k} a_\alpha \frac{\partial^\alpha f}{\partial x^\alpha}$ *be a partial differential operator with constant coefficients. Then* $P(D) : \mathcal{M}^{p,q}_{a,b} \to \mathcal{M}^{r,s}$ *for* $a, b, p, q, r, s$ *as in Theorem 3, and* $|k| < b - d(\frac{1}{s} - \frac{1}{q})$.

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