BILINEAR PSEUDODIFFERENTIAL OPERATORS ON MODULATION SPACES

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Abstract. We use the theory of Gabor frames to prove the boundedness of bilinear pseudodifferential operators on products of modulation spaces. In particular, we show that bilinear pseudodifferential operators corresponding to non-smooth symbols in the Feichtinger algebra are bounded on products of modulation spaces.

1. Introduction

A bilinear pseudodifferential operator T_σ is á priori defined through its (distributional) symbol σ as a mapping from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by:

$$T_\sigma(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(x,\xi,\eta)\,\hat{f}(\xi)\,\hat{g}(\eta)\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta, \tag{1}$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$. A natural question then is to find sufficient (nontrivial) conditions on the symbol that ensure the boundedness of the operator on products of certain Banach spaces such as Lebesgue, Sobolev, or Besov spaces; see the works of Coifman and Meyer [4], [5], [6], Gilbert and Nahmod [14], [15], Muscalu, Tao and Thiele [25], Grafakos and Torres [16], [17], Bényi and Torres [2], [3], and Bényi [1] and the references therein for more details. For instance, it is known that the condition

$$|\partial_x^\alpha\,\partial_\xi^\beta\,\partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}\,(1+|\xi|+|\eta|)^{-|\beta|-|\gamma|} \tag{2}$$

for $(x,\xi,\eta) \in \mathbb{R}^{3d}$ and all multi-indices α, β, γ is enough to prove the boundedness of the operator defined by (1) from $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $p, q > 1$. This result was first obtained by Coifman and Meyer [4], [5], [6] who noticed that, in general, if the symbol is smooth and has certain decay, then the boundedness of the corresponding operator can be studied through its decomposition into elementary operators via techniques related to Littlewood-Paley theory. In this case, the smoothness and decay conditions play an important role and cannot be removed from the proof. In [14] and [16], for example, the authors used a different decomposition approach where the functions on which the operator acts are instead decomposed and the boundedness of the bilinear operator reduces to the boundedness of an infinite matrix acting on appropriate sequence spaces. In particular, Grafakos and Torres [16] used wavelet expansions of Triebel-Lizorkin spaces due to Frazier and

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Jawerth [12] (see also the books by Frazier, Jawerth and Weiss [13], and by Meyer [24]) to impose convenient decay conditions on the entries of the corresponding matrix that yield boundedness results on the operator side. Here again, the symbols of the operators are assumed to be sufficiently smooth and have decay at infinity.

In this paper, we obtain certain boundedness results for operators with symbols which are not necessarily smooth. We also employ decomposition techniques of functions spaces, but the novelty is the use of Gabor expansions of tempered distributions in the so-called modulation spaces, which were introduced by Feichtinger and Gröchenig [9], [10]. The modulation spaces play a crucial role in the theory of Gabor frames. Moreover, modulation spaces were recently used to formulate and to prove boundedness results and Schatten-class properties of linear pseudodifferential operators; see e.g., the works by Gröchenig and Heil [19], Heil, Ramanathan and Topiwala [20], Labate [22], [23], and Toft [26]. This is yet another motivation to study the boundedness properties of bilinear pseudodifferential operators in terms of modulation spaces. Most of the techniques used in studying the linear pseudodifferential operators on modulation spaces are based on the relationship between the Weyl and/or the Kohn-Nirenberg correspondences, and some time-frequency representations of distributions such as the Wigner transform. The approach we use here is fundamentally different from the ones previously employed in dealing with the linear case, namely, we decompose the functions in the modulation spaces into their Gabor expansions and thereby transform the boundedness of the bilinear operator into that of an infinite matrix acting on sequence spaces associated to the modulation spaces. The conditions we impose on the infinite matrix to prove our results turn out to be equivalent to membership of the corresponding symbol to a particular modulation space. However, these conditions do not imply any smoothness nor decay of the symbols. In particular, the modulation space $\mathcal{M}^1$, also known as the Feichtinger algebra, turns out to be an important class of symbols that guarantees the boundedness of the operator defined by (1) on products of certain modulation spaces.

Our paper is organized as follows. In Section 2 we set the notations and definitions that will be used throughout this paper. In Section 3 we define the modulation spaces and Gabor frames, and collect some of their properties that will be needed later on. In Section 4 we present a brief review of the bilinear integral operators, of which the bilinear pseudodifferential operators are a special case. This sets the stage for our main results which, we state and prove in Section 5.

It is worth noting that our main results can be stated in the more general setting of multilinear pseudodifferential operators. To ease the flow of reading and for notational convenience we restrict ourselves to the bilinear case. The interested reader could easily adapt the proofs to the multilinear case.

2. Preliminaries

2.1. **General notation.** We will be working on the d-dimensional space $\mathbb{R}^d$. We let $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ be the subspace of $C^\infty(\mathbb{R}^d)$ of Schwartz rapidly decreasing functions, with its usual topology. Its dual is $\mathbb{S}' = \mathbb{S}'(\mathbb{R}^d)$, the set of all tempered distributions on $\mathbb{R}^d$. Translation and modulation of a function f with domain $\mathbb{R}^d$ are defined, respectively, by

$$T_x f(t) = f(t-x) \qquad \text{and} \qquad M_y f(t) = e^{2\pi i y\cdot t} f(t).$$

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i t\cdot\omega}\, dt$, $\omega \in \mathbb{R}^d$. The Fourier transform is an isomorphism of the Schwartz space $\mathbb{S}$ onto itself, and extends to the space $\mathbb{S}'(\mathbb{R}^d)$ of tempered distributions by duality. The inverse Fourier transform is $\check{f}(x) = \hat{f}(-x)$.

The inner product of two functions $f, g \in L^2$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}\, dt$, and its extension to $\mathbb{S}' \times \mathbb{S}$ will be also denoted by $\langle \cdot, \cdot \rangle$.

The Short-Time Fourier Transform (STFT) of a function f with respect to a window g is

$$V_g f(x, y) = \langle f, M_y T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot t}\, \overline{g(t-x)}\, f(t)\, dt,$$

whenever the integral makes sense. Analogously to the Fourier transform, the STFT extends in a distributional sense to f, g in the space of tempered distributions $\mathbb{S}'$, cf. [11, Prop. 1.42].

Given a strictly positive function ν on $\mathbb{R}^{2d}$, we let $L^{p,q}_\nu = L^{p,q}_\nu(\mathbb{R}^{2d})$ be the spaces of measurable functions $f(x, y)$ for which the weighted mixed norms

$$\|f\|_{L^{p,q}_\nu} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,y)|^p\, \nu(x,y)^p\, dx\right)^{q/p} dy\right)^{1/q}$$

are finite. If $p = q$, we have $L^{p,p}_\nu(\mathbb{R}^{2d}) = L^p_\nu(\mathbb{R}^{2d})$, a weighted Lebesgue space. By $l^{p,q}_{\tilde{\nu}}(\mathbb{Z}^{2d})$ we denote the spaces of sequences $a = (a_{kl})_{k,l\in\mathbb{Z}^d}$ for which the mixed norms

$$\|a\|_{l^{p,q}_{\tilde{\nu}}} = \left(\sum_{k\in\mathbb{Z}^d} \left(\sum_{l\in\mathbb{Z}^d} |a_{kl}|^p\, \tilde{\nu}(k,l)^p\right)^{q/p}\right)^{1/q}$$

are finite, where $\tilde{\nu}(k, l) = \nu(\alpha k, \beta l)$ for some given $\alpha, \beta > 0$. If $p = q$, we recover the weighted sequence spaces $l^p_{\tilde{\nu}}(\mathbb{Z}^{2d})$.

2.2. **Weight functions.** Given $s \geq 0$, a positive, continuous, and symmetric function ν is called an s-moderate weight if there exists a constant $C > 0$ such that

$$\forall\, x, y \in \mathbb{R}^d, \quad \nu(x+y) \leq C\,(1+|x|^2)^{s/2}\, \nu(y). \tag{3}$$

For example, $\nu(x) = (1+|x|^2)^{t/2}$ is s-moderate exactly for $|t| \leq s$. If ν is s-moderate, then by manipulating (3) we see that

$$\frac{1}{\nu(x+y)} \leq C\,(1+|x|^2)^{s/2}\, \frac{1}{\nu(y)},$$

hence, $1/\nu$ is also s-moderate (with the same constant).

In the sequel we let $\omega_s(x) = (1 + |x|^2)^{s/2}$ for $s \geq 0$. Notice that the definition of s-moderate weight functions can be generalized in an obvious manner to higher dimensions. If ω_s defined above is a function over $\mathbb{R}^{2d}$, we let $\Omega_s = \omega_s \otimes \omega_s \otimes \omega_s$, i.e.,

$$\Omega_s(x_1, x_2, y_1, y_2, z_1, z_2) = \omega_s(x_1, x_2)\, \omega_s(y_1, y_2)\, \omega_s(z_1, z_2),$$

be a weight function defined on $\mathbb{R}^{6d}$. Moreover, if A is an invertible transformation on $\mathbb{R}^{6d}$, i.e., $A \in GL(\mathbb{R}, 6d)$, we denote by Ω_s^A the weight function defined on $\mathbb{R}^{6d}$ by $\Omega_s^A(X) = \Omega_s(A(X))$ for $X \in \mathbb{R}^{6d}$.

3. Modulation spaces and Gabor frames

3.1. Modulation spaces.

Definition 1. Given $1 \leq p, q \leq \infty$, and given a window function $g \in \mathcal{S}$, and an s-moderate weight ν defined on $\mathbb{R}^{2d}$, the modulation space $\mathcal{M}_\nu^{p,q} = \mathcal{M}_\nu^{p,q}(\mathbb{R}^d)$ is the space of all distributions $f \in \mathbb{S}'$ for which the following norm is finite:

$$\|f\|_{\mathcal{M}_\nu^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,y)|^p\,\nu(x,y)^p\,dx\right)^{q/p}dy\right)^{1/q} = \|V_g f\|_{L_\nu^{p,q}}, \tag{4}$$

with the usual modifications if p and/or q are infinite. If $\nu = 1$, we will simply write $\mathcal{M}^{p,q}$ for the modulation space $\mathcal{M}_\nu^{p,q}$. Moreover, when $p = q$, we will write $\mathcal{M}_\nu^p$ for the modulation space $\mathcal{M}_\nu^{p,p}$

Remark 1. The definition is independent of the choice of the window g in the sense of equivalent norms. If $1 \leq p, q < \infty$, and ν is an s-moderate weight, then $\mathcal{M}_{\omega_s}^1$ is densely embedded into $\mathcal{M}_\nu^{p,q}$. In fact, the Schwartz class $\mathbb{S}$ is dense in $\mathcal{M}_\nu^{p,q}$ for $1 \leq p, q < \infty$ and for all s-moderate weights ν. One can also show that the dual of $\mathcal{M}_\nu^{p,q}$ is $\mathcal{M}_{1/\nu}^{p',q'}$, where $1 \leq p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. We refer to [18] and the references therein for more details about modulation spaces.

Remark 2. For $p = q = 2$, if $\nu \equiv 1$ then $\mathcal{M}_\nu^2 = L^2$; if $\nu(x,y) = (1+|x|^2)^{s/2}$ then $\mathcal{M}_\nu^2 = L_s^2$, a weighted L^2-space; if $\nu(x,y) = (1+|y|^2)^{s/2}$ then $\mathcal{M}_\nu^2 = H^s$, the standard Sobolev space, and if $\nu(x,y) = (1+|x|^2+|y|^2)^{s/2}$ then $\mathcal{M}_\nu^2 = L_s^2 \cap H^s$.

Remark 3. The modulation space $\mathcal{M}_{\omega_s}^1$ is a Banach algebra under both pointwise multiplication and convolution and is invariant under Fourier transform. It plays also an important role in the theory of Gabor frames where it serves as a convenient class of windows that generate Gabor frames for the whole class of the modulation spaces. In particular, if $s = 0$ (equivalently if $\nu = \omega_s \equiv 1$), then $\mathcal{M}^1$ is the Feichtinger algebra. We point out that functions in $\mathcal{M}_{\omega_s}^1$ are, in general, not smooth. Notice also that $\mathcal{M}_{\omega_s}^1$ is continuously embedded into $L_{\omega_s^1}^1$ where ω_s^1 is the restriction of ω_s to $\mathbb{R}^d \times \{0\}$; see e.g. [18] for the proof of this embedding.

3.2. Gabor Frames.

Definition 2. Given a window function $\phi \in L^2(\mathbb{R}^d)$ and constants $\alpha, \beta > 0$, we say that $\{M_{\beta n}T_{\alpha k}\phi\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ if there exist constants $A, B > 0$ (called frame bounds) such that

$$A\,\|f\|_{L^2(\mathbb{R}^d)} \leq \sum_{k,n\in\mathbb{Z}^d}|\langle f, M_{\beta n}T_{\alpha k}\phi\rangle|^2 \leq B\,\|f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d).$$

We refer to [7], [18], and [21] for extensive treatments of frames and Gabor frames.

Let ϕ be a well-localized window in the time-frequency plane. One can generalize the theory of Gabor frames from the pure L^2 theory to obtain a characterization of the whole class of modulation spaces [9], [10], [18]. We summarize in the next theorem certain facts about Gabor frames in modulation spaces that will be needed in the sequel of this paper.

Theorem 1. *Let $\phi \in \mathbb{S}(\mathbb{R}^d)$ be such that $\{M_{\beta n} T_{\alpha k}\phi\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame for L^2. Let $1 \le p, q \le \infty$, and let ν be an s-moderate weight. Then the following hold.*
a. *The frame operator S_ϕ is a continuously invertible operator from $\mathcal{M}^{p,q}_\nu$ onto itself, and the (canonical) dual $\gamma = S_\phi^{-1}\phi$ belongs to $\mathbb{S}(\mathbb{R}^d)$.*
b. *Every tempered distribution in $\mathcal{M}^{p,q}_\nu$ has a Gabor expansion that converges unconditionally (or weak* unconditionally if $p = \infty$ or $q = \infty$), namely*

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha k}\gamma\rangle\, M_{\beta n} T_{\alpha k}\phi, \qquad \forall f \in \mathcal{M}^{p,q}_\nu(\mathbb{R}^d);$$

moreover, we have the following norm equivalences

$$\|f\|_{\mathcal{M}^{p,q}_\nu} \asymp \|\langle f, M_{\beta n} T_{\alpha k}\phi\rangle\|_{l^{p,q}_{\tilde{\nu}}} \asymp \|\langle f, M_{\beta n} T_{\alpha k}\gamma\rangle\|_{l^{p,q}_{\tilde{\nu}}}.$$

To summarize, a tempered distribution f belongs to the modulation space $\mathcal{M}^{p,q}_\nu(\mathbb{R}^d)$ if and only if the sequence of its Gabor coefficients $(\langle f, M_{\beta n} T_{\alpha k}\phi\rangle)_{k,n\in\mathbb{Z}^d}$ belongs to the sequence space $l^{p,q}_{\tilde{\nu}}(\mathbb{Z}^{2d})$. Moreover, the norm of f is equivalent to the norm of its Gabor coefficients.

The next proposition gives a characterization of $\mathcal{M}^1_{\Omega_s}(\mathbb{R}^{3d})$ in terms of Gabor frames using standard tensor product arguments; see [18, p. 272] for further details.

Proposition 1. *Let $\phi \in \mathcal{M}^1_{\omega_s}(\mathbb{R}^d)$ be such that $\{M_{\beta n} T_{\alpha k}\phi\}_{k,n\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with (canonical) dual $\gamma \in \mathcal{M}^1_{\omega_s}(\mathbb{R}^d)$. Then $\mathcal{K} \in \mathcal{M}^1_{\Omega_s}$ if and only if*

$$\mathcal{K} = \sum_{k,m,i,l,n,j\in\mathbb{Z}^d} \langle \mathcal{K}, M_{\beta n} T_{\alpha m}\gamma \otimes M_{\beta l} T_{\alpha k}\overline{\gamma} \otimes M_{\beta j} T_{\alpha i}\overline{\gamma}\rangle\, M_{\beta n} T_{\alpha m}\phi \otimes M_{\beta l} T_{\alpha k}\overline{\phi} \otimes M_{\beta j} T_{\alpha i}\overline{\phi}$$

with unconditional convergence of the series in $\mathcal{M}^1_{\Omega_s}(\mathbb{R}^{3d})$. Moreover, the norm of $\mathcal{K}$ in $\mathcal{M}^1_{\Omega_s}$ is equivalent to the norm of its Gabor coefficients $\big(\langle \mathcal{K}, M_{\beta n} T_{\alpha m}\gamma \otimes M_{\beta l} T_{\alpha k}\overline{\gamma} \otimes M_{\beta j} T_{\alpha i}\overline{\gamma}\rangle\big)_{k,m,i,l,n,j\in\mathbb{Z}^d}$ in $l^1_{\tilde{\Omega}_s}(\mathbb{Z}^{6d})$.

4. Bilinear operators and a discrete model

4.1. Bilinear operators.

Definition 3. A bilinear operator associated with a kernel $K \in \mathbb{S}'(\mathbb{R}^{3d})$, is a mapping B_K defined á priori from $\mathbb{S}(\mathbb{R}^d) \times \mathbb{S}(\mathbb{R}^d)$ into $\mathbb{S}'(\mathbb{R}^d)$ by

$$B_K(f,g)(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y,z)\, f(y)\, g(z)\, dy\, dz, \tag{5}$$

for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

The next proposition establishes the relationship between a bilinear integral operator and a bilinear pseudodifferential operator defined by (1). Notice also the similarity with the relationship between the Weyl and Kohn-Nirenberg correspondences in the linear case.

Proposition 2. *Let T_σ be a bilinear pseudodifferential operator associated to a symbol $\sigma \in \mathcal{S}'(\mathbb{R}^{3d})$. Then T_σ is a bilinear integral operator B_K with kernel $K(x,y,z) = \mathcal{F}_1^{-1}\hat{\sigma}(N(x,y,z))$, where $N(x,y,z) = (x, y-x, z-x)$ is a change of coordinates on $\mathbb{R}^{3d}$ and $\mathcal{F}_1^{-1}$ denotes the inverse Fourier transform in the first variable.*

Proof. For $f, g \in \mathcal{S}$ we have:

$$\begin{aligned} T_\sigma(f,g)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(x,\xi,\eta)\,\hat{f}(\xi)\,\hat{g}(\eta)\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta \\ &= \iiiint \sigma(x,\xi,\eta)\,f(y)\,g(z)\,e^{-2\pi i\xi\cdot y}\,e^{-2\pi i\eta\cdot z}\,e^{2\pi i x\cdot(\xi+\eta)}\,d\xi\,d\eta\,dy\,dz \\ &= \iint K(x,y,z)\,f(y)\,g(z)\,dy\,dz = B_K(f,g)(x), \end{aligned}$$

where

$$\begin{aligned} K(x,y,z) &= \int\int \sigma(x,\xi,\eta)e^{-2\pi i\xi\cdot(y-x)}\,e^{-2\pi i\eta\cdot(z-x)}\,d\xi\,d\eta \\ &= \mathcal{F}_2\mathcal{F}_3\sigma(x, y-x, z-x) = \mathcal{F}_1^{-1}\hat{\sigma}(N(x,y,z)). \end{aligned}$$

Here, $\mathcal{F}_j$ denotes the Fourier transform in the j^{th} variable. □

Let $X = (x_1, x_2, x_3), Y = (y_1, y_2, y_3) \in \mathbb{R}^{3d}$. We define an invertible linear transformation on $\mathbb{R}^{6d}$ by $A(X,Y) = ((N^T)^{-1}(x_1, -y_2, -y_3), N(y_1, x_2, x_3))$, where N is the change of variables defined in Proposition 2. We show in the next proposition that the symbol of the bilinear pseudodifferential operator is in $\mathcal{M}^1_{\Omega_s^B}$ if and only if the corresponding integral kernel as defined in Proposition 2 is in $\mathcal{M}^1_{\Omega_s}$, where $B = A^{-1}$ is the inverse of A.

Proposition 3. *$\sigma \in \mathcal{M}^1_{\Omega_s^B}(\mathbb{R}^{3d})$ if and only if $K = \mathcal{F}_1^{-1}\hat{\sigma} \circ N \in \mathcal{M}^1_{\Omega_s}(\mathbb{R}^{3d})$.*

Proof. Let $G \in \mathbb{S}(\mathbb{R}^{3d})$. For $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$, and $t = (t_1, t_2, t_3) \in \mathbb{R}^{3d}$ we have

$$\begin{aligned} V_G K(u,v) &= \int_{\mathbb{R}^{3d}} K(t)\,e^{-2\pi i t\cdot v}\overline{G}(t-u)\,dt \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{\sigma}(x, t_2 - x, t_3 - x)\,e^{-2\pi i(v_2\cdot t_2 + v_3\cdot t_3)}\,e^{2\pi i t_1\cdot(x - v_1)}\times \\ &\qquad \overline{G(t_1 - u_1, t_2 - u_2, t_3 - u_3)}\,dx\,dt_1\,dt_2\,dt_3 \\ &= e^{-2\pi i u_1\cdot v_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \hat{\sigma}(x, t_2, t_3)\,e^{-2\pi i(x,t_2,t_3)\cdot(-u_1+v_2+v_3, v_2, v_3)}\times \\ &\qquad \overline{\mathcal{F}_1 G((x, t_2 + x, t_3 + x) - (v_1, u_2, u_3))}\,dx\,dt_2\,dt_3 \\ &= e^{-2\pi i u_1\cdot v_1}\int_{\mathbb{R}^{3d}} \hat{\sigma}(Z)\,e^{-2\pi i Z\cdot(N^{-1})^T(-u_1, v_2, v_3)}\times \end{aligned}$$

$$\overline{(\mathcal{F}_1 G)\circ N^{-1}(Z - N(v_1,u_2,u_3))}\,dZ$$

$$= e^{-2\pi i u_1\cdot v_1}\, V_{\left(\mathcal{F}_1 G\right)\circ N^{-1}}\hat{\sigma}(N(v_1,u_2,u_3),(N^{-1})^T(-u_1,v_2,v_3))$$

$$= e^{-2\pi i u_1\cdot v_1}\, V_H\hat{\sigma}(N(v_1,u_2,u_3),(N^T)^{-1}(-u_1,v_2,v_3))$$

where $H = \left(\mathcal{F}_1 G\right)\circ N^{-1}$. Since $|V_g f(x,y)| = |V_{\hat{g}}\hat{f}(-y,x)|$, whenever the STFT is defined, we have

$$\begin{aligned}|V_G K(u,v)| &= |V_H\hat{\sigma}(N(v_1,u_2,u_3),(N^T)^{-1}(-u_1,v_2,v_3))|\\ &= |V_{\check{H}}\sigma((N^T)^{-1}(u_1,-v_2,-v_3),N(v_1,u_2,u_3))|\\ &= |V_{\check{H}}\sigma(A(u,v))|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_G K(u,v)|\Omega_s(u,v)\,du\,dv &= \int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_{\check{H}}\sigma(A(u,v))|\Omega_s(u,v)\,du\,dv\\ &= \int_{\mathbb{R}^{3d}}\int_{\mathbb{R}^{3d}} |V_{\check{H}}\sigma(u,v))|\Omega_s^B(u,v)\,du\,dv < \infty,\end{aligned}$$

and the proof is complete. □

4.2. **A discrete model.** Consider $\phi \in \mathbb{S}(\mathbb{R}^d)$ that generates a Gabor frame for L^2 with (canonical) dual $\gamma \in \mathbb{S}(\mathbb{R}^d)$. We can then expand f, g and h in $\mathcal{S}(\mathbb{R}^d)$ as in Theorem 1, where the series converge unconditionally in every modulation space norm as long as $p, q \neq \infty$. Then using (5), we obtain:

$$\begin{aligned}\langle B_K(f,g),h\rangle &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y,z) \sum_{k,l\in\mathbb{Z}^d}\langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\, M_{\beta l}T_{\alpha k}\phi(y)\times\\ &\qquad \sum_{m,n\in\mathbb{Z}^d}\langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\, M_{\beta n}T_{\alpha m}\phi(z)\,\overline{\sum_{i,j\in\mathbb{Z}^d}\langle h, M_{\beta j}T_{\alpha i}\gamma\rangle\, M_{\beta j}T_{\alpha i}\phi(x)}\,dx\,dy\,dz\\ &= \sum_{i,j}\sum_{k,l}\sum_{m,n}\langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\,\langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\,\overline{\langle h, M_{\beta j}T_{\alpha i}\gamma\rangle}\times\\ &\qquad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y,z)\,\overline{M_{\beta j}T_{\alpha i}\phi(x)}M_{\beta l}T_{\alpha k}\phi(y)\,M_{\beta n}T_{\alpha m}\phi(z)\,dx\,dy\,dz\\ &= \sum_{i,j}\sum_{k,n}\sum_{l,m}\langle f, M_{\beta l}T_{\alpha k}\gamma\rangle\,\langle g, M_{\beta n}T_{\alpha m}\gamma\rangle\,\overline{\langle h, M_{\beta j}T_{\alpha i}\gamma\rangle}\times\end{aligned}$$

$$\langle B_K(M_{\beta l}T_{\alpha k}\phi, M_{\beta n}T_{\alpha m}\phi), M_{\beta j}T_{\alpha i}\phi\rangle. \tag{6}$$

The exchange of the integrals and summations above is justified, since, $f, g, h \in \mathbb{S}$ have absolutely summable Gabor coefficients. Moreover, $K \in \mathbb{S}'(\mathbb{R}^{3d}) = \bigcup_{s\geq 0}\mathcal{M}^{\infty}_{1/\omega_s}$

(cf. [18, Prop. 11.3.1]) and $\phi \in \mathbb{S}$ imply that the triple integral in the second equality is uniformly bounded with respect to $i, j, k, l, m, n \in \mathbb{Z}^d$. Therefore, to study the boundedness of B_K on products of modulation spaces, it suffices to analyze the boundedness of the matrix $B = (b_{ij,kl,mn})$ defined by

$$b_{ij,kl,mn} = \langle B_K(M_{\beta l} T_{\alpha k}\phi, M_{\beta n} T_{\alpha m}\phi), M_{\beta j} T_{\alpha i}\phi\rangle \tag{7}$$

on products of appropriate sequence spaces.

The next theorem will be of special importance in proving our main results. In particular, it shows that, under some mild condition on its entries, an infinite matrix yields a bounded operator on products of sequence spaces associated with the modulation spaces as in Theorem 1. For an infinite matrix $(a_{mn,ij,kl})$, let $\mathcal{O}$ denote the bilinear operator associated to it, i.e., $(\mathcal{O}(f_{ij}), (g_{kl}))_{mn} = \sum_{ij,kl} a_{mn,ij,kl}\, f_{ij}\, g_{kl}$, where (f_{ij}) and (g_{kl}) are sequences defined on $\mathbb{Z}^{2d}$.

Theorem 2. *Let ν be an s-moderate weight, and let $1 \le p_i, q_i, r_i < \infty$, for $i = 1, 2$ be such that $\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}$. If $(a_{mn,ij,kl}) \in l^1_{\tilde{\Omega}_s}(\mathbb{Z}^{6d})$, then $\mathcal{O}$ is a bounded operator from $l^{p_1,p_2}_{\tilde{\nu}}(\mathbb{Z}^{2d}) \times l^{q_1,q_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$ into $l^{r_1,r_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$. In particular, if $(a_{mn,ij,kl}) \in l^1(\mathbb{Z}^{6d})$ then $\mathcal{O}$ is a bounded operator from $l^{p_1,p_2}(\mathbb{Z}^{2d}) \times l^{q_1,q_2}(\mathbb{Z}^{2d})$ into $l^{r_1,r_2}(\mathbb{Z}^{2d})$.*

Proof. Let $(f_{ij}) \in l^{p_1,p_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$, $(g_{kl}) \in l^{q_1,q_2}_{\tilde{\nu}}(\mathbb{Z}^{2d})$ and $(h_{mn}) \in l^{r_1',r_2'}_{1/\tilde{\nu}}(\mathbb{Z}^{2d})$ where r_1', r_2' are the dual indices of r_1, respectively r_2. We have

$$\begin{aligned}
|\langle \mathcal{O}((f_{ij}), (g_{kl})), (h_{mn})\rangle| \le{}& \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|\,|f_{ij}|\,|g_{kl}|\,|h_{mn}| \\
={}& \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|^{\frac{1}{p_1}} |f_{ij}| \frac{\tilde{\nu}(i,j)}{\tilde{\nu}(i,j)}\, |a_{mn,kl,ij}|^{\frac{1}{q_1}} |g_{kl}| \frac{\tilde{\nu}(k,l)}{\tilde{\nu}(k,l)} \times \\
& |a_{mn,kl,ij}|^{\frac{1}{r_1'}} |h_{mn}| \frac{\tilde{\nu}(m,n)}{\tilde{\nu}(m,n)} \\
\le{}& C^3 \sum_{m,n,i,j,k,l} |a_{mn,kl,ij}|^{\frac{1}{p_1}} \tilde{\nu}(i,j) |f_{ij}| \tilde{\omega_s}(i,j)\, |a_{mn,ij,kl}|^{\frac{1}{q_1}} \tilde{\nu}(k,l) \times \\
& |g_{kl}| \tilde{\omega_s}(k,l)\, |a_{mn,ij,kl}|^{\frac{1}{r_1'}} \frac{1}{\tilde{\nu}(m,n)} |h_{mn}| \tilde{\omega_s}(m,n) \\
={}& C^3 \sum_{m,n,i,j,k,l} \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/p_1} |f_{ij}| \tilde{\nu}(i,j) \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/q_1} |g_{kl}| \times \\
& \tilde{\nu}(k,l) \big(|\tilde{a}_{mn,kl,ij}|\big)^{1/r_1'} |h_{mn}| \frac{1}{\tilde{\nu}(m,n)}
\end{aligned}$$

where $\tilde{a}_{mn,kl,ij} = a_{mn,kl,ij}\, \tilde{\Omega}_s(m,n,k,l,i,j) = a_{mn,kl,ij}\, \tilde{\omega_s}(m,n)\, \tilde{\omega_s}(k,l)\, \tilde{\omega_s}(i,j)$. We have used the fact that $\tilde{\nu}$, and $\frac{1}{\tilde{\nu}}$ are s-moderate with the same constant C. Since

$\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{r_1}$ we can apply Hölder's inequality to obtain the following:

$$
\begin{aligned}
|\langle \mathcal{O}((f_{ij}),(g_{kl})),(h_{mn})\rangle| &\le C^3\Big(\sum_{m,n,i,j,k,l}|\tilde{a}_{mn,ij,kl}|\,|f_{ij}|^{p_1}\tilde{\nu}(i,j)^{p_1}\Big)^{\frac{1}{p_1}}\times\\
&\quad\Big(\sum_{m,n,i,j,k,l}|\tilde{a}_{mn,ij,kl}|\,|g_{kl}|^{q_1}\tilde{\nu}(k,l)^{q_1}\Big)^{\frac{1}{q_1}}\times\\
&\quad\Big(\sum_{m,n,i,j,k,l}|\tilde{a}_{mn,ij,kl}|\,|h_{mn}|^{r_1'}\frac{1}{\tilde{\nu}(m,n)^{r_1'}}\Big)^{\frac{1}{r_1'}}\\
&\le C^3\,\sup_i\big(\sup_j|f_{ij}|\tilde{\nu}(i,j)\big)\,\sup_k\big(\sup_l|g_{kl}|\tilde{\nu}(k,l)\big)\times\\
&\quad\sup_m\big(\sup_n|h_{mn}|\frac{1}{\tilde{\nu}(m,n)}\big)\,\Big(\sum_{m,i,k}\sum_{n,j,l}|\tilde{a}_{mn,ij,kl}|\Big)\\
&\le C^3\,\|a_{mn,ij,kl}\|_{l^1_{\tilde{\Omega}_s}}\Big(\sum_i\Big(\sum_j|f_{ij}|^{p_1}\tilde{\nu}(i,j)^{p_1}\Big)^{\frac{p_2}{p_1}}\Big)^{\frac{1}{p_2}}\times\\
&\quad\Big(\sum_k\Big(\sum_l|g_{kl}|^{q_1}\tilde{\nu}(k,l)^{q_1}\Big)^{\frac{q_2}{q_1}}\Big)^{\frac{1}{q_2}}\times\\
&\quad\Big(\sum_m\Big(\sum_n|h_{mn}|^{r_1'}\frac{1}{\tilde{\nu}(m,n)^{r_1'}}\Big)^{\frac{r_2'}{r_1'}}\Big)^{\frac{1}{r_2'}}
\end{aligned}
$$

where we have used the fact that $l^{p,q}(\mathbb{Z}^{2d})\subset l^{\infty}(\mathbb{Z}^{2d})$. By duality we get that

$$\|\mathcal{O}((f_{ij}),(g_{kl}))\|_{l^{r_1,r_2}_{\tilde{\nu}}}\le C^3\,\|a_{mn,ij,kl}\|_{l^1_{\tilde{\Omega}_s}}\,\|(f_{ij})\|_{l^{p_1,p_2}_{\tilde{\nu}}}\,\|(g_{kl})\|_{l^{q_1,q_2}_{\tilde{\nu}}}.$$

The second part of the theorem follows by choosing $\nu=\omega_0\equiv 1$. □

5. Boundedness of bilinear pseudodifferential operators

Our first main result shows that a bilinear integral operator with kernel in the modulation space $\mathcal{M}^1_{\Omega_s}$ —in particular, in the Feichtinger algebra— gives rise to a bounded operator.

Theorem 3. *Let ν be an s-moderate weight, and let $1\le p_i,q_i,r_i<\infty$ for $i=1,2$ be such that $\frac{1}{p_1}+\frac{1}{q_1}=\frac{1}{r_1}$. If $K\in\mathcal{M}^1_{\Omega_s}(\mathbb{R}^{3d})$, then the bilinear integral operator B_K defined by (5) can be extended as a bounded operator from $\mathcal{M}^{p_1,p_2}_{\nu}(\mathbb{R}^d)\times\mathcal{M}^{q_1,q_2}_{\nu}(\mathbb{R}^d)$ into $\mathcal{M}^{r_1,r_2}_{\nu}(\mathbb{R}^d)$.*

Proof. Let $f,g,h\in\mathbb{S}(\mathbb{R}^d)$ and expand each of these functions into their Gabor series, i.e., $f=\sum_{i,j}\langle f,M_{\beta j}T_{\alpha i}\phi\rangle\,M_{\beta j}T_{\alpha i}\gamma$, $g=\sum_{k,l}\langle g,M_{\beta l}T_{\alpha k}\phi\rangle\,M_{\beta l}T_{\alpha k}\gamma$, and $h=$

$\sum_{m,n}\langle h, M_{\beta n}T_{\alpha m}\phi\rangle\, M_{\beta n}T_{\alpha m}\gamma$, where ϕ and γ are dual Gabor frames as in Theorem 1. By Proposition 1, the matrix defined by (7) belongs to $l^1_{\tilde{\Omega}_s}$ since $K \in \mathcal{M}^1_{\Omega_s}$. Therefore, using Theorem 2 we have the following estimates:

$$\begin{aligned}
|\langle B_K(f,g), h\rangle| &= |\sum_{mn}\sum_{ij}\sum_{kl} a_{mn,ij,kl}\,\langle f, M_{\beta j}T_{\alpha i}\phi\rangle\,\langle g, M_{\beta l}T_{\alpha k}\phi\rangle\,\overline{\langle h, M_{\beta n}T_{\alpha m}\phi\rangle}| \\
&\le C\,\|a_{mn,ij,kl}\|_{l^1_{\tilde{\Omega}_s}}\,\|\langle f, M_{\beta j}T_{\alpha i}\phi\rangle\|_{l^{p_1,p_2}_{\tilde{\nu}}}\times \\
&\qquad \|\langle g, M_{\beta l}T_{\alpha k}\phi\rangle\|_{l^{q_1,q_2}_{\tilde{\nu}}}\,\|\langle h, M_{\beta n}T_{\alpha m}\phi\rangle\|_{l^{r_1',r_2'}_{1/\tilde{\nu}}} \\
&= C\,\|K\|_{\mathcal{M}^1_{\Omega_s}}\,\|f\|_{\mathcal{M}^{p_1,p_2}_{\nu}}\,\|g\|_{\mathcal{M}^{q_1,q_2}_{\nu}}\,\|h\|_{\mathcal{M}^{r_1',r_2'}_{1/\nu}} \qquad (8)
\end{aligned}$$

and by duality we obtain,

$$\|B_K(f,g)\|_{\mathcal{M}^{r_1,r_2}_{\nu}} \le C\,\|K\|_{\mathcal{M}^1_{\Omega_s}}\,\|f\|_{\mathcal{M}^{p_1,p_2}_{\nu}}\,\|g\|_{\mathcal{M}^{q_1,q_2}_{\nu}}.$$

The result then follows by standard density arguments, using the fact that $\mathbb{S}(\mathbb{R}^d)$ is dense in $\mathcal{M}^{p,q}_{\nu}$ for $1 \le s,t < \infty$. □

The previous theorem together with Propositions 2 and 3 yield our second main result, which provides a sufficient condition on the symbol so that the operator (1) is bounded on products of modulation spaces. Recall that the invertible transformation A was defined on $\mathbb{R}^{6d}$ by $A(X,Y) = ((N^T)^{-1}(x_1,-y_2,-y_3), N(y_1,x_2,x_3))$, where N is the change of variable defined in Proposition 2; we also let $B = A^{-1}$.

Theorem 4. *Let ν be an s-moderate weight, and let $1 \le p_i, q_i, r_i < \infty$ for $i = 1,2$ be such that $\frac{1}{p_1}+\frac{1}{q_1} = \frac{1}{r_1}$. If $\sigma \in \mathcal{M}^1_{\Omega_s^B}(\mathbb{R}^{3d})$, then the bilinear pseudodifferential operator T_σ defined by (1) can be extended to a bounded operator from $\mathcal{M}^{p_1,p_2}_{\nu}(\mathbb{R}^d)\times\mathcal{M}^{q_1,q_2}_{\nu}(\mathbb{R}^d)$ into $\mathcal{M}^{r_1,r_2}_{\nu}(\mathbb{R}^d)$.*

Proof. By Proposition 3, $\sigma \in \mathcal{M}^1_{\Omega_s^B}$ if and only if $K \in \mathcal{M}^1_{\Omega_s}$ where K is the kernel of the corresponding integral operator, and the result follows from Theorem 3. □

If we assume that $\nu = \omega_0 \equiv 1$, and that $p_1 = p_2 = p$ and $q_1 = q_2 = q$ (hence $r_1 = r_2 = r$), we obtain the following

Corollary 1. *Let $2 \le p,q < \infty$ and $1 \le r \le 2$ be such that $\frac{1}{p}+\frac{1}{q} = \frac{1}{r}$. If $\sigma \in \mathcal{M}^1(\mathbb{R}^{3d})$, then T_σ can be extended to a bounded operator from $L^p(\mathbb{R}^d)\times L^q(\mathbb{R}^d)$ into $L^r(\mathbb{R}^d)$. In particular, if $\sigma \in \mathcal{M}^1(\mathbb{R}^{3d})$, then T_σ has a bounded extension from $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ into $L^1(\mathbb{R}^d)$.*

Proof. For the range of p,q being considered we have the following continuous embeddings: $L^p \subset \mathcal{M}^p$, $L^q \subset \mathcal{M}^q$ and so $L^p\times L^q \subset \mathcal{M}^p\times\mathcal{M}^q$. Moreover, since $1 \le r \le 2$, we have that $\mathcal{M}^r \subset L^r$ (see [18]). These continuous embeddings combined with Theorem 4 imply then the result. □

Remark 4. It is remarkable that the condition $\sigma \in \mathcal{M}^1(\mathbb{R}^{3d})$, does not imply any smoothness nor decay on the symbol. In particular, Coifman-Meyer-type conditions (2) are not satisfied by the symbols we consider. Note also that while the symbols considered in [15] or [25] are very singular along the anti-diagonal in the frequency plane, they are independent of the space variable x. Furthermore, the techniques used there to prove the boundedness of the corresponding operators fit the one dimensional situation, but they are yet to be developed in a multidimensional setting. Our result has the advantage of dealing with symbols in both a non-smooth and higher dimensional framework.

Assume that $\nu(x,y) = \omega_s(x,y) = (1+|x|^2+|y|^2)^{s/2}$ for some $s > 0$, and that $p_i = q_i = 2$. Let ω_s^1 be the restriction of ω_s to $\mathbb{R}^d \times \{0\}$. Then the following holds.

Corollary 2. *If $\sigma \in \mathcal{M}^1_{\Omega_s^B}$ then T_σ can be extended as a bounded bilinear pseudodifferential operator from $\mathcal{M}^2_{\omega_s} \times \mathcal{M}^2_{\omega_s}$ into $L^1_{\omega_s^1}$.*

Proof. Notice that $\mathcal{M}^1_{\omega_s}$ is continuously embedded in $L^1_{\omega_s^1}$, cf. [18, Prop. 12.1.4]. So we only need to prove that under the hypotheses of the corollary, the bilinear pseudodifferential operator can be extended to a bounded operator from $\mathcal{M}^2_{\omega_s} \times \mathcal{M}^2_{\omega_s}$ into $\mathcal{M}^1_{\omega_s}$. But this follows from Theorem 4 by taking $\nu = \omega_s$. $\square$

Remark 5. If the symbol σ satisfies the estimates

$$\sup_x \|\partial^{\alpha_j}_{\xi_j}\partial^{\beta_k}_{\eta_k}\sigma(x,\cdot,\cdot)\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \leq C \tag{9}$$

for all $j,k = 1,\ldots,n$, and $\alpha_j,\beta_k = 0$ or 1, it was shown in [2, Theorem 2] that the corresponding bilinear pseudodifferential operator is bounded from $L^2 \times L^2$ into L^1. We wish to point out that, in general, neither that result nor Corollary 1 in this paper imply each other. On one hand, if $g \in \mathbb{S}(\mathbb{R}^{2d})$ then $\sigma_1(x,\xi,\eta) = \chi_{[0,1[^d}(x)g(\xi,\eta)$ where $\chi_{[0,1[^d}$ is the characteristic function of the unit cube in $\mathbb{R}^d$, satisfies (9) and hence it yields a bounded operator from $L^2 \times L^2$ into L^1. However, because σ_1 is not a continuous function, it is not in $\mathcal{M}^1(\mathbb{R}^{3d})$. Therefore, our corollary does not apply. On the other hand, functions in $\mathcal{M}^1$ must be continuous, but there are non-differentiable functions in $\mathcal{M}^1$, hence they do not satisfy (9), thus [2, Theorem 2] does not apply here.

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