ASYMPTOTICS OF EIGENVALUE CLUSTERS FOR SCHRÖDINGER OPERATORS ON THE SIERPIŃSKI GASKET

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Abstract. In this note we investigate the asymptotic behavior of spectra of Schrödinger operators with continuous potential on the Sierpiński gasket SG. In particular, using the existence of localized eigenfunctions for the Laplacian on SG we show that the eigenvalues of the Schrödinger operator break into clusters around certain eigenvalues of the Laplacian. Moreover, we prove that the characteristic measure of these clusters converges to a measure. Results similar to ours were first observed by A. Weinstein and V. Guillemin for Schrödinger operators on compact riemannian manifolds.

1. Introduction and Motivations

The construction by Kigami [9] of a Laplacian on the post critically finite (pcf) fractals, spurs new researches on analysis on fractals. In particular, numbers of results in this area are reminiscent of results from analysis on manifolds, while other results are only true in this fractal setting. We refer to [1, 10, 13, 14] and the references therein for more background on analysis on fractals. In this note we consider the typical example of pcf fractal, namely the Sierpiński gasket denoted by $K = SG$. This is the attractor of the iterated function systems (ifs) consisting of the functions F_1, F_2 and F_3 defined on $\mathbb{R}^2$ respectively by $F_1(x) = \frac{1}{2}x$, $F_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0)$ and $F_3(x) = \frac{1}{2}x + (\frac{1}{4}, \frac{\sqrt{3}}{4})$. In other words, the Sierpiński gasket is the unique nonempty compact subset of $\mathbb{R}^2$ such that $K = \cup_{i=1}^{3} F_i K$. Alternatively, SG can be defined as a limit of graphs. For a word $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ of length m, the set

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$F_\omega K = F_{\omega_1} \circ \cdots \circ F_{\omega_{m-1}} \circ F_{\omega_m} K$ with $\omega_i \in \{1, 2, 3\}$, is called an m-cell. Let $V_0 = \{(0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$, i.e., V_0 is the set of fixed points of the contractions F_i. Let $V_n = \cup_{i=1}^{3} F_i V_{n-1}$, $n \geq 1$, and define a sequence of graphs Γ_m with vertices in V_m and edge relation $x \sim_m y$ given inductively by: Γ_0 is the complete graph with vertices in V_0, and $x \sim_m y$ if and only if x and y belong to the same m-cell $F_\omega K$. Notice also that K is the closure of V_* in the Euclidean metric, where $V_* = \cup_{n=0}^{\infty} V_n$.
In all that follows, we assume that K is equipped with the probability measure μ that assigns the measure 3^{-m} to each m cell.

A Laplacian on SG can be defined as a renormalized limit of graph Laplacian in the following way. First, a graph Laplacian Δ_m is defined on the graph Γ_m by

$$\Delta_m f(x) = \sum_{y \sim_m x} f(y) - 4f(x) \tag{1}$$

for $x \in V_m \setminus V_0$. A Laplacian on SG can now be defined by

$$\Delta = \tfrac{3}{2} \lim_{m\to\infty} 5^m \Delta_m. \tag{2}$$

Using the spectral decimation method of Rammal and Toulouse [11], Fushikima and Shima gave a complete characterization of the spectrum of the Laplacian on SG [5]. Further description of this spectrum has been investigated in [4, 15]. The tenet of the spectral decimation method is based on the fact the eigenvalues and the eigenfunctions of Δ on SG are completely determined by the eigenvalues and eigenfunctions of the graph Laplacians Δ_m. More specifically, for every Dirichlet eigenvalue λ of Δ on SG, there exits an integer $m \geq 1$ ("the generation of birth"), such that every λ-eigenfunction f, when restricted to V_m is an eigenfunction of Δ_m with eigenvalue $2, 5$ or 6. Additionally, 2 occurs only for $m = 1$, while 6 occurs only for $m > 1$. Moreover, the restriction of f to V_k for $k > m$ is an eigenfunction of Δ_k with eigenvalue λ_k given by $\lambda_{k-1} = \lambda_k(5 - \lambda_k)$. Furthermore, the eigenvalue λ is related to the graph eigenvalues by

$$\lambda = \tfrac{3}{2} \lim_{k\to\infty} 5^k \lambda_k.$$

The eigenvalues obtained from the graph eigenvalue 2 i.e., the generation of birth is $m = 1$ are called the 2-series eigenvalues and all have multiplicity 1. However, the 5-series and the 6-series eigenvalues have very high multiplicities. A similar description for the Neumann eigenvalues exists and can be found in [14] and the references therein. A unique feature of the spectrum of the Laplacian on SG (and other pcf fractals) is the existence of localized eigenfunctions [2, 5]. These are eigenfunctions which satisfy both Dirichlet and Neumann boundary conditions. In particular, every 6-series eigenspace starting with a large m is spanned by localized eigenfunctions, and a similar statement is true for most of the 5-series eigenfunctions. Moreover, it is known that the proportion of localized eigenvalues of Δ is close to 1 [14]. Another feature of the Laplacian on SG, which we heavily use below, is the existence of gaps in its spectrum.

Our goal in this note is to study the asymptotics of the spectrum of the Schrödinger operator $H = -\Delta + \chi$ on SG, where the potential χ is a real-valued continuous function defined on SG. However, we first reduce the problem to the case where χ is a step function on SG. In this case the existence of localized eigenfunctions for Δ implies that the spectrum of the Schrödinger operator breaks into clusters whose asymptotic distribution may be described precisely. These clusters appear very naturally due to the existence of gaps in the spectrum of Δ. By then approximating every continuous function on SG by such step functions we establish the same result for general Schrödinger operator with continuous potential. Notice that some experimental results pertaining the Schrödinger equation on the Sierpiński gasket were first introduced in [3]. Finally, we wish to point out that a result similar to ours was obtained by Weinstein [16] for the Schrödinger operator corresponding to the Laplace-Beltrami operator on compact riemannian manifold, with a smooth potential; see also [6, 7, 8]. In particular, the eigenvalues of the Laplace-Beltrami operator $-\Delta$ on the unit n-sphere S^n are of the form $k(k+n-1)$, for $k = 0, 1, 2, \ldots$, and where the multiplicity of each eigenvalue grows as a polynomial of order $n-1$ in k. It was shown in [16] that the eigenvalues of the Schrödinger operator on S^n break into clusters whose distribution converges to a certain measure when k tends to infinity. However, the techniques used in the above paper are completely different from ours and rely on the well establish theory of pseudodifferential operators on riemannian manifolds, which as yet has no analog in this fractal setting.

We begin by proving the following lemma, which uses a mini-max argument ([12]) to compare the eigenvalues of two Schrödinger operators with different potential. More precisely we have the following result.

Lemma 1. *For $i = 1, 2$, let χ_i be real-valued bounded measurable functions on SG. Let $H_i = -\Delta + \chi_i$ denote the corresponding Schrödinger operators. For $n \geq 1$, the nth eigenvalues ν_n^i of H_i, $i = 1, 2$, satisfy the following inequality*

$$|\nu_n^1 - \nu_n^2| \leq \|\chi_1 - \chi_2\|_{L^\infty}.$$

Proof. Note that for $i = 1, 2$, H_i is a self-adjoint operator. Moreover, since χ_i is bounded from below, it is readily seen that H_i is bounded from below. More specifically we have

$$\langle H_i f, f\rangle \geq (\lambda_1 + \min_K \chi_i)\|f\|_{L^2}^2$$

where λ_1 is the smallest (positive) eigenvalue of $-\Delta$ on K. Therefore, by [12, Theorem XIII.2], it follows that the eigenvalues of the operators are given by

$$\nu_n^i = \sup_{\phi_1,\phi_2,\cdots,\phi_{n-1}} \inf_{\psi\in[\phi_1,\cdots,\phi_{n-1}]^\perp;\ \|\psi\|_{L^2}=1} \langle H_i\psi, \psi\rangle. \tag{3}$$

Moreover, we have the following estimate

$$\begin{aligned}\langle H_1\psi, \psi\rangle &= \langle H_2\psi, \psi\rangle + \langle(\chi_1 - \chi_2)\psi, \psi\rangle \\ &\leq \langle H_2\psi, \psi\rangle + \|\chi_1 - \chi_2\|_{L^\infty}\, \|\psi\|_{L^2}^2. \end{aligned} \tag{4}$$

Using (4) together with (3) completes the proof. □

2. Schrödinger operators with step function potential

Let $N > 1$ be an integer and consider a partition of the Sierpiński gasket into

$$K = \cup_{|\omega|=N} F_\omega K, \tag{5}$$

where for each word ω of length N, $F_\omega K$ is an N cell. Note that there are exactly 3^N such cells, each of which has a measure 3^{-N}.
Consider a real-valued function associated to this partition: $\chi_N = \sum_{j=1}^{3^N} a_j\, \chi_{C_j}$ where the constants $a_j \in \mathbb{R}$ are not necessarily all distinct, and where $C_j = F_\omega K$ for some word ω of length N.

We wish in this section to study the asymptotics of the spectrum of the Schrödinger operator $H_N = -\Delta + \chi_N$.

We adopt the notation of [4], and let $\lambda_k := 5^{k-1}\lambda_n^{(6)}$ be one of the eigenvalues corresponding to the 6-series as described above. Note that we ignore the second index n as it is irrelevant for our investigation. Additionally, because we are looking at the limiting behavior of the spectrum of H_N as $k \to \infty$, we may assume without loss of generality that $k >> N$. Let E_k be the eigenspace associated to λ_k, and set $E_k = span\{u_l^k, l = 1, 2, \ldots, d_k\}$ where $d_k = dim E_k = \frac{3^k-3}{2}$ is the multiplicity of λ_k. We may choose u_l^k so that the first $\tilde{d}_k$ of them are entirely supported in a single N cell, where $\tilde{d}_k = 3^N m_N^k$, with m_N^k representing the multiplicity of eigenfunctions supported in a given N cell. Denote by $\tilde{E}_k$ the span of these localized eigenfunctions. It follows from the description of the spectrum of Δ in [4] that $m_N^k = \frac{3^{k-N}-3}{2}$. Each of the m_N^k eigenfunctions u_l^k supported in the cell C_j is trivially seen to be an eigenfunction of H_N. Indeed, it follows from the form of the potential χ_N that $H_N u_l^k = (\lambda_k + a_j)u_l^k$, for all $l = 1, 2, \ldots, m_N^k$. Thus, $\lambda_j^k = \lambda_k + a_j$ is an eigenvalue of H_N with multiplicity m_N^k. Note that these eigenvalues all belong to the interval $[\lambda_k + \min \chi_N, \lambda_k + \max \chi_N]$. Denote by $\tilde{\Lambda}_N^k$ the portion of the spectrum of H_N lying in this interval, and denote the eigenvalues in $\tilde{\Lambda}_N^k$ by $\{\tilde{\lambda}_l^k\}$ (counting multiplicity). We claim that, for k large enough $\tilde{\Lambda}_N^k$ contains exactly d_k eigenvalues. This follows from Lemma 1, since the separation between λ_k and the next higher and lower eigenvalues of Δ grows exponentially in k, and so eventually exceeds $\|\chi_N\|_{L^\infty}$. So we have identified $\tilde{d}_k$ eigenvalues in $\tilde{\Lambda}_N^k$, and the remaining number of eigenvalues is $\alpha_N^k = d_k - \tilde{d}_k = \frac{3^{N+1}-3}{2}$, which is independent of k, so $\lim_{k\to\infty} \frac{\alpha_N^k}{d_k} = 0$.

Define the characteristic measure of the subset $\tilde{\Lambda}_N$ of the spectrum of H_N by

$$\begin{aligned}
\phi_k^N(\lambda) &= \frac{1}{d_k}\sum_{l=1}^{d_k} \delta[\lambda - (\tilde{\lambda}_l^k - \lambda_k)] \\
&= \frac{m_N^k}{d_k}\sum_{j=1}^{3^N} \delta[\lambda - (\lambda_j^k - \lambda_k)] + \frac{1}{d_k}\sum_{l\in A_N^k} \delta[\lambda - (\tilde{\lambda}_l^k - \lambda_k)] \\
&= \frac{m_N^k}{d_k}\sum_{j=1}^{3^N} \delta(\lambda - a_j) + \frac{1}{d_k}\sum_{l\in A_N^k} \delta[\lambda - (\tilde{\lambda}_l^k - \lambda_k)], \qquad (6)
\end{aligned}$$

where $\# A_N^k = \alpha_N^k$. Observe that the factor m_N^k in the definition of ϕ_k^N is due to the fact that the eigenvalue λ_j^k has multiplicity m_N^k. We prove below that the measures ϕ_k^N converge to a measure ϕ_0^N, defined to be the pullback of μ under χ_N. More specifically, ϕ_0^N is defined by

$$\phi_0^N(\lambda) = \sum_{j=1}^{3^N} \mu(C_j)\,\delta(\lambda - a_j) = 3^{-N}\sum_{j=1}^{3^N} \delta(\lambda - a_j), \qquad (7)$$

or equivalently, for any continuous bounded function f,

$$\langle \phi_0^N, f\rangle = \int_K f(\chi_N(x))\,d\mu(x).$$

Lemma 2. *$\phi_k^N \to \phi_0^N$ weakly as $k \to \infty$; in other words, for any continuous and bounded function $f \in \mathcal{C}_b(\mathbb{R})$,*

$$\lim_{k\to\infty} \langle \phi_k^N, f\rangle = \langle \phi_0^N, f\rangle.$$

Proof. Let $f \in \mathcal{C}_b(\mathbb{R})$,

$$\begin{aligned} |\langle \phi_k^N - \phi_0^N, f\rangle| &\leq \sum_{j=1}^{3^N} \left| \Big(\frac{m_N^k}{d_k} - 3^{-N}\Big) f(a_j) \right| + \frac{1}{d_k} \sum_{l \in A_N^k} |f(\tilde{\lambda}_l^k - \lambda_k)| \\ &\leq \|f\|_{L^\infty} \tfrac{3^{N+1}-3}{3^k-3} + \|f\|_{L^\infty} \tfrac{\alpha_N^k}{d_k} \\ &= 2\|f\|_{L^\infty} \tfrac{3^{N+1}-3}{3^k-3}. \qquad (8) \end{aligned}$$

Letting $k \to \infty$ in (8) concludes the proof of the lemma. □

3. Schrödinger operator with continuous potential

In this section we would like to investigate the asymptotics of the eigenvalues of the Schrödinger operator $H = -\Delta + \chi$. For each $N \geq 1$ construct a partition of K as in (5). Any continuous function $\chi \in \mathcal{C}(K)$ can be approximated uniformly by a sequence of simple functions $\chi_N = \sum_{j=1}^{3^N} a_j \chi_{C_j}$. Then Lemma 1 says that the spectrum of H is approximated by the spectrum of H_N. In particular, if $\tilde{\Lambda}^k$ denotes the portion of the spectrum of H contained in the interval $[\lambda_k + \min\chi, \lambda_k + \max\chi]$, then $\tilde{\Lambda}^k$ is approximated by $\tilde{\Lambda}_N^k$ (we can choose the approximation so that $\min\chi \leq \min\chi_N$ and $\max\chi_N \leq \max\chi$). As before, for k large enough, $\tilde{\Lambda}^k$ contains d_k eigenvalues which we will denote $\{\nu_l^k\}_{l=1}^{d_k}$. We call this the λ_k cluster of the eigenvalues of H.

Define the characteristic measure of the λ_k cluster of H by

$$\psi_k = \frac{1}{d_k} \sum_{l=1}^{d_k} \delta[\lambda - (\nu_l^k - \lambda_k)]. \qquad (9)$$

We show below that these measures converge to the pullback of the measure μ on SG under χ defined by

$$\langle \psi_0, f\rangle = \int_K f(\chi(x))\, d\mu(x) \qquad (10)$$

for all test function f.

As mentioned in Section 2, for each k large enough, most of the eigenvalues of H_N are just $\lambda_j^k = \lambda_k + a_j$, which are just shifts of λ_k by $-a_j$. Moreover, according to Lemma 1, for all k and l

$$|\nu_l^k - \tilde{\lambda}_l^k| \leq \|\chi - \chi_N\|_{L^\infty}.$$

Theorem 1. *$\psi_k \to \psi_0$ weakly as $k \to \infty$, i.e., for all continuous bounded function $f \in \mathcal{C}_b(\mathbb{R})$,*

$$\lim_{k\to\infty} \langle \psi_k, f\rangle = \langle \psi_0, f\rangle.$$

Proof. Given any $f \in \mathcal{C}_b(\mathbb{R})$, let $\epsilon, \eta > 0$.

$$\begin{aligned}\langle \psi_k - \psi_0, f\rangle &= \langle \psi_k - \phi_k^N, f\rangle + \langle \phi_k^N - \phi_0^N, f\rangle + \langle \phi_0^N - \psi_0, f\rangle \\ &= I + II + III,\end{aligned}$$

where ϕ_k^N and ϕ_0^N where previously defined in section 2.

Choose $N_0 \geq 1$ such that for all $N \geq N_0$,

$$\|\chi - \chi_N\|_{L^\infty} \leq \eta. \tag{11}$$

Now let $N \geq N_0$ be fixed but arbitrary. Notice that f is uniformly continuous on the compact set

$$\chi(K) \cup \chi_N(K) = \chi(K) \cup_{j=1}^{3^N} \{a_j\}.$$

Therefore, there exists $\alpha > 0$ such that for all $z_1, z_2 \in \chi(K) \cup \chi_N(K)$ with $|z_1 - z_2| \leq \alpha$, we have

$$|f(z_1) - f(z_2)| \leq \epsilon/3.$$

By choosing $\eta = \alpha$ in (11), it follows that

$$|III| \leq \int_K |f(\chi(x)) - f(\chi_N(x))|\, dx \leq \epsilon/3. \tag{12}$$

Moreover, by Lemma 2, it is readily seen that

$$|II| \leq 2\, \|f\|_{L^\infty} \left(\tfrac{3^{N+1}-3}{3^k-3}\right).$$

Consequently, we can choose $l_0 >> N$ such that for all $k \geq l_0$ we have

$$|II| \leq \epsilon/3. \tag{13}$$

Note that ν_l^k and $\tilde{\lambda}_l^k$ belong to the interval $[\lambda_k + \min \chi, \lambda_k + \max \chi]$, and f is uniformly continuous on this interval and on $\chi(K) \cup \chi_N(K)$. Using Lemma 1 and the observations made above, one can choose $0 < \alpha' \leq \alpha$ such that for all a, b in the above set, with $|a - b| \leq \alpha'$, we have $|f(a) - f(b)| \leq \epsilon/3$. Therefore,

$$\begin{aligned}|I| &\leq \tfrac{1}{d_k} \sum_{l=1}^{d_k} |f(\nu_l^k) - f(\tilde{\lambda}_l^k)| \\ &\leq \tfrac{1}{d_k} \sum_{l=1}^{d_k} \epsilon/3 \\ &= \epsilon/3. \end{aligned} \tag{14}$$

Combining (12), (13) and (14) we conclude the proof of the theorem. □

Remark 1. Similar results can be proved for the Neumann spectrum. Since the proof is quite similar to the one above, we omit it.

One could do the same analysis starting from the 5-series eigenvalues of $-\Delta$. However, in this case the analysis is a little different. In particular, the proof of Theorem 1 relies on the existence of large gaps between the (6-series) eigenvalues of $-\Delta$. But the key argument when dealing with the 5-series eigenvalues is that while other eigenvalues of H can be very close, it is true that for a given cluster, the number of (other) eigenvalues, counting multiplicity, that are nearby is bounded by a constant that only depends on the seize of the neighborhood.

Our results are also likely to hold on all pcf fractals with localized eigenfunctions as described in [2].

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