WEAK UNCERTAINTY PRINCIPLE FOR FRACTALS, GRAPHS AND METRIC MEASURE SPACES

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ABSTRACT. We develop a new approach to formulate and prove the weak uncertainty inequality which was recently introduced by Okoudjou and Strichartz. We assume either an appropriate measure growth condition with respect to the effective resistance metric, or, in the absence of such a metric, we assume the Poincaré inequality and inverse volume doubling property. Our results can be applied to a wide variety of metric measure spaces, including graphs, fractals and manifolds.

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1. INTRODUCTION

The weak uncertainty inequality recently introduced in [24] for functions defined on p.c.f. fractals in general, and on the Sierpiński gasket in particular, obeys the same philosophy as the classical uncertainty principle: it is impossible for any non zero function to have a small energy and to be highly localized in space. We refer to [11, 25, 26, 34, 35] for more background on uncertainty principles. However, the existence of localized eigenfunctions on some of these fractals (see [9, 10, 30, 36]), is a main obstacle in proving any analogue of the classical Heisenberg inequality. In this paper we introduce a new approach to prove weak uncertainty principles for functions defined on metric measure spaces equipped with a Dirichlet (or energy) form, and

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which include certain fractals and fractal graphs such as the Sierpiński lattice. More precisely, we show that the weak uncertainty principle holds on all spaces equipped with an effective resistance metric and a measure satisfying an appropriate growth condition. Additionally, we show that if instead of the existence of an effective resistance metric on the space, we assume that a Poincaré-type inequality holds along with another appropriate growth condition on the measure, then it is also possible to prove the weak uncertainty principles in this setting. In particular, our results show that the self-similarity of the measure, which was heavily used in [24], can be replaced by weaker conditions.

In order to formulate any uncertainty inequality, one has to define measures of space and frequency concentration. For example, for complex-valued functions on $\mathbb{R}$ the classical Heisenberg Uncertainty Principle is

$$Var(|\hat{f}(\xi)|^2)Var(|f(x)|^2) \geqslant \frac{1}{16\pi^2}$$

for any function of $f \in L^2(\mathbb{R})$ such that $\|f\|_2 = 1$ and where $\hat{f}$ denotes the Fourier transform on $\mathbb{R}$. This inequality can be rewritten in the following form

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^2|f(x)|^2\,|f(y)|^2\,dx\,dy \int_{\mathbb{R}} |f'(x)|^2\,dx \geqslant \frac{1}{8}$$

for any function of L^2 norm one. We refer to the survey article [11] for more information on the uncertainty principle.

In this paper we consider a metric measure space (K, d, μ), that is (K, d) is a metric space equipped with a Borel measure μ. If $\mathcal{E}_K$ is an energy form on this metric measure space, then we will say that a weak uncertainty principle holds on K if the following estimate

$$Var_K(u)\,\mathcal{E}_K(u,u) \geqslant C \tag{1}$$

holds for any function $u \in L^2(K)\bigcap Dom(\mathcal{E})$ such that $\|u\|_{L^2} = 1$. Here C is a constant independent of u, and the spacial variance is defined by

$$Var_K(u) = \iint_{K\times K} d(x,y)^{b+1}|u(x)|^2\,|u(y)|^2\,d\mu(x)\,d\mu(y), \tag{2}$$

where b is an exponent which often plays the role of a dimension.

The central question of our paper is the relation between d, b, μ and $\mathcal{E}_K$ which implies a weak uncertainty principle as long as the measure μ satisfies an appropriate growth condition. We formulate sufficient conditions in two situations. The first one is when d is the so called effective resistance metric on K with certain scaling properties, which is particularly useful in analysis on fractals and fractal graphs; see [3, 16, 17] for more on the effective resistance metric. As a byproduct of our result in this case not only we provide a different and simpler proof of [24, Theorems 1 and 2], but also extend it to all p.c.f. fractals [15, 17] and fractal graphs [8, 12, 13, 14, 18, 20, 21], as well as to modifications of them such as some fractafolds. Additionally, our result recovers the classical Heisenberg Uncertainty Principle in $\mathbb{R}$, although not with the best constant. The second situation is when we deal with spaces on which an effective resistance metric does not exist. In this case we assume that there is a certain scaling in Poincaré's inequality, which allows us to prove our result. This latter result is applicable for a wide variety of metric measure spaces, ranging from graphs, to elliptic operators on manifolds. Note that, in this case, the number b appearing in (2) cannot, in general, be interpreted as a dimension in the usual sense. However, b will often represent the so-called walk dimension that appears frequently in recent works on heat kernel estimates (see [8] and references therein). One of the features of our results is their robustness. For example, since the weak uncertainty principle holds for the Sierpiński graphs, it also holds for the manifolds with similar structures, e.g., the fractal-like manifolds considered in [19].

Our paper is organized as follows. In Section 2 we state and prove our main results. To this end, we first make explicit the main assumptions needed on the measure, and the metric of the underlying spaces under consideration. In Section 3 we modify our main results so that they are applicable to spaces where global and local structure can be significantly different, such as graphs, manifolds and compact spaces. Section 4 describes a few metric measure spaces for which the main results of Sections 2 and 3 can be applied: p.c.f. fractals, uniform finitely ramified graphs, Sierpiński carpets, fractal-like manifolds. We also discuss relation with recent results on the heat kernel estimates on metric measure spaces.

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2. Main results

Let (K, d) be a metric space equipped with a measure μ and an energy (or Dirichlet) form $\mathcal{E}_K$. We denote by $B_r(x)$ the ball with center x and radius r in the metric d. To simplify notation, we presume that the L^2-norm is infinite if a function is not square integrable, and that the energy form is infinite if a function is not in its domain.

We assume that the space (K, d, μ) is in one of the following two categories:

Category I: the metric d is the effective resistance metric on K, defined by (6), and there exist two positive constants C_1, C_2 such that for all x and for all $r > 0$ the following inequalities hold:

$$C_1 r^b < \mu(B_r(x)) < C_2 r^b. \tag{3}$$

For the spaces in this category, the constant b in the last inequality is usually the Hausdorff dimension of (K, d).

Category II: the energy form $\mathcal{E}_K$ on K satisfies the following Poincaré inequality for all locally square integrable functions

$$\int\limits_{B_r(y)} (u(x) - \bar{u}_{B_r(y)})^2 d\mu(x) \le C_2 r^\gamma \mathcal{E}_K(u, u), \tag{4}$$

where $\bar{u}_{B_r(y)}$ is the average of u over $B_r(y)$, and γ and C_2 are some positive constants; additionally, the measure μ is assumed to satisfy the "inverse volume doubling property", that is there exists a positive constant $C_1 > 1$ such that for all $x \in K$ and

$r > 0$ we have

$$C_1\mu(B_r(x)) < \mu(B_{2r}(x)). \tag{5}$$

Note that in general, and as opposed to the spaces in *Category I*, the constant γ in (4) may not represent any sort of dimension of the space (K, d). However, in some of the examples we consider later, γ can be interpreted as the so called walk dimension (see [4, 8] and references therein). Note also that for spaces in this category, we will take $b = \gamma$ in (2).

We point out that, when the effective resistance metric exists, it is related to the energy form $\mathcal{E}_K$ by

$$d(x, y) = \sup \mathcal{E}_K^{-1}(u, u), \tag{6}$$

where the supremum is taken over all functions u such that $u(x) = 1, u(y) = 0$. The existence of the effective resistance metric is a nontrivial problem, see [2, 3, 4, 5, 16, 17]; in particular, it is worth noticing that there are spaces without an effective resistance metric, e.g., $\mathbb{R}^n$ with $n \geqslant 2$. However, on p.c.f. fractals and on some Sierpiński carpets, which are "not far from being one dimensional", it is known that the effective resistance metric does exist (see Subsection 4.4).

Finally, note that in either category, the measure μ is not necessarily a self-similar measure. But if it is self-similar, then the measure weights and the resistance scaling weights are related by a power law.

One can see that, by a simple scaling argument, we have $b = 1$ in *Category II* if $K = \mathbb{R}^n$, $n \geqslant 1$, as well as in *Category I* if $K = \mathbb{R}^1$.

Remark 1. It is implicit from the definitions that spaces in *Category I* or *Category II* are assumed to be unbounded, and that they are equipped with measures having no atoms. Moreover, the global and local properties of these spaces are comparable in a certain sense. However, with small modifications our main result below still applies to bounded spaces, manifolds, and spaces equipped with measures possessing atoms (graphs, for example) etc. To deal properly with these situations, we make small changes in the definitions of *category I* and *Category II* spaces in Section 3.

Remark 2. Note that in *Category II* the so called energy form $\mathcal{E}$ does not have to be a Dirichlet form, but just a non negatively defined quadratic form on L^2_μ.

The main result of our paper is the following theorem.

Theorem 1. *Let K be a space equipped with a measure μ and a metric d. Assume that the metric measure space (K, d, μ) is either in the Category I or in the Category II. Then there exists a positive constant C such that for all $u \in L^2(K)$ with $\|u\|_2 = 1$ one has*

$$Var_K(u)\mathcal{E}_K(u, u) \geqslant C.$$

Proof. We first give the proof when the metric measure space (K, d, μ) is in *Category I*. Denote $a = b + 1$ and $Var_K(u) = v$. Then there exists y such that

$$\int_K d^a(x, y)u^2(x)d\mu(x) \leqslant v.$$

Let r be defined by

$$r = \sup\left\{s : \int_{B_s(y)} u^2(x)d\mu(x) \leqslant \frac{1}{2}\right\}. \tag{7}$$

For each $s > 0$ such that $\int_{B_s(y)} |u(x)|^2 \, d\mu(x) \leqslant \frac{1}{2}$, we have

$$v \geqslant \int_{K-B_s(y)} d(x,y)^a \, |u(x)|^2 \, d\mu(x) \geqslant r^a \int_{K-B_s(y)} |u(x)|^2 \, d\mu(x),$$

and the definition of r implies that

$$r^a \leqslant 2v. \tag{8}$$

Moreover, by (3) there is $c > 1$ such that

$$\mu(B_{cr}(y)) > 8C_2 r^b. \tag{9}$$

Note that the definition of r also implies that

$$\int_{B_r(y)} u^2(x)d\mu(x) \geqslant \frac{1}{2} \tag{10}$$

which together with (3) yields

$$\max_{B_r(y)} u^2(x) \geq \frac{1}{2C_2 r^b}.$$

Additionally, since $\|u\|_2 = 1$ and using (9) we see that

$$\min_{B_{cr}(y)} u^2(x) \leq \frac{1}{8C_2 r^b}.$$

Consequently, there are $x_1, x_2 \in B_{cr}$ such that

$$(u(x_1) - u(x_2))^2 \geq \frac{1}{8C_2 r^b}.$$

Thus, by the definition of the effective resistance metric,

$$\mathcal{E}_K(u,u) \geqslant \frac{(u(x_1) - u(x_2))^2}{d(x_1, x_2)} \geq \frac{1}{16C_2 cr r^b}.$$

Using now (8) we conclude that

$$v\mathcal{E}_K(u,u) \geq \frac{v}{C_4 v^{\frac{b+1}{a}}} = \frac{v}{C_4 v} = \frac{1}{C_4},$$

where we have used the fact $a = b + 1$.

Now assume that the metric measure space (K, d, μ) is in *Category II*. Recall that in this case we take $b = \gamma$ in the definition of the variance (2). The proof is similar to the above proof, with the main difference that there exists no effective resistance metric relating the energy to the distance d. Thus we need to make some changes and use Poincaré's inequality (4) in place of the effective resistance metric. We use the same notations as above. Note that (5) implies the existence of a positive constant $c > 1$, such that

$$T = \frac{\mu(B_{cr}(y))}{\mu(B_r(y))} > 8. \tag{11}$$

Recall also that using (7) we proved the following

$$S = \int_{B_r(y)} u^2(x)d\mu(x) \geqslant \frac{1}{2}.$$

The following estimates hold:

$$\begin{aligned}\left|\int_{B_{cr}(y)} u(x)d\mu(x)\right| &\leq \left|\int_{B_r(y)} u(x)d\mu(x)\right| + \left|\int_{B_{cr}(y)\setminus B_r(y)} u(x)d\mu(x)\right| \\ &\leq \sqrt{S\mu(B_r(y))} + \sqrt{(1-S)(T-1)\mu(B_r(y))}.\end{aligned}$$

Now if we maximize the last inequality over $S \geq \frac{1}{2}$ we get

$$\left|\int_{B_{cr}(y)} u(x)d\mu(x)\right| \leqslant (1+\sqrt{T-1})\sqrt{\frac{1}{2}\mu(B_r(y))},$$

or equivalently

$$\|\bar{u}_{B_{cr}(y)}\| \leqslant \frac{1+\sqrt{T-1}}{T}\sqrt{\frac{1}{2\mu(B_r(y))}}.$$

Hence

$$\|\bar{u}_{B_{cr}(y)}\|_{L^2_{B_{cr}(y)}} \leqslant (1+\sqrt{T-1})\sqrt{\frac{1}{2T}}.$$

Maximizing this last estimate over $T \geq 8$ we obtain

$$\|\bar{u}_{B_{cr}(y)}\|_{L^2_{B_{cr}(y)}} \leqslant \frac{1+\sqrt{7}}{4} < 1.$$

Therefore, by (4) we see that

$$\begin{aligned} C_3 c^a r^a \mathcal{E}_K(u,u) &\geq \int\limits_{B_{cr}(y)} (u(x) - \bar{u}_{B_r(y)})^2 d\mu(x) \\ &= \|u - \bar{u}_{B_{cr}(y)}\|^2_{L^2_{B_{cr}(y)}} \\ &\geqslant \left(\|u\|_{L^2_{B_{cr}(y)}} - \|\bar{u}_{B_{cr}(y)}\|_{L^2_{B_{cr}(y)}} \right)^2 \\ &\geqslant \left(\frac{3-\sqrt{7}}{4} \right)^2 \\ &= \delta > 0. \end{aligned}$$

Thus, by (8) we have

$$v\mathcal{E}_K(u,u) \geq \delta \frac{v}{C_3 c^a r^a} \geq \frac{v}{C_4 v} = \frac{1}{C_4},$$

which concludes the proof. Note that in this case $a = \gamma + 1$. □

3. Graphs, manifolds and compact spaces

As mentioned above, Theorem 1 implicitly assumes that the space K is unbounded. Thus, for bounded metric measure spaces we introduce the following *Category* I_{bdd} and *Category* II_{bdd}, and the weak uncertainty principle takes a different form. Note however that its proof is identical to the previous one, and so we omit it.

When dealing with bounded spaces, we require that they belong to one of the following categories:

Category I_{bdd}: the metric d is the effective resistance metric, and there exist three positive constants C_0, C_1, C_2 such that for all x and for all $0 < r < C_0$ the following inequalities hold:

(12) $$C_1 r^b < \mu(B_r(x)) < C_2 r^b.$$

For the spaces in this category, the constant b in the last inequality is usually the Hausdorff dimension of (K, d).

Category II_{bdd}: the energy form $\mathcal{E}_K$ on K satisfies the following Poincaré inequality:

(13) $$\int\limits_{B_r(y)} (u(x) - \bar{u}_{B_r(y)})^2 d\mu(x) \leq C_2 r^\gamma \mathcal{E}_K(u,u),$$

where $\bar{u}_{B_r(y)}$ is the average of u over $B_r(y)$, γ and C_2 are some positive constants; the measure μ satisfies the "inverse volume doubling property", that is there exist positive constants $C_0 > 0$ and $C_1 > 1$ such that for all $x \in K$ and $0 < r < C_0$

(14) $$C_1 \mu(B_r(x)) < \mu(B_{2r}(x)).$$

Note that the constant γ appearing in (13) has the same meaning as in the definition of *Category II*.

Theorem 2. *Let K be a space equipped with a measure μ and a metric d. Assume that the metric measure space (K,d,μ) is either in the Category I_{bdd} or in the Category II_{bdd}. Then there exist positive constants C_0' and C such that for all $u \in L^2(K)$ with $\|u\|_2 = 1$ one has*

$$Var_K(u)\mathcal{E}_K(u,u) \geqslant C$$

provided $Var_K(u) < C_0'$.

Similarly, when dealing with spaces where the local structure is significantly different from the global one, for instance, manifolds, graphs and spaces equipped with measure having atoms, we require them to belong to one of the following categories:

Category I_{grph}: the metric d is the effective resistance metric, and there exist three positive constants C_0, C_1, C_2 such that for all x and for all $r > C_0$ the following inequalities hold:

$$C_1 r^b < \mu(B_r(x)) < C_2 r^b. \tag{15}$$

Again, for the spaces in this category, the constant b in the last inequality is usually the Hausdorff dimension of (K,d).

Category II_{grph}: the energy form $\mathcal{E}_K$ on K satisfies the following Poincaré inequality:

$$\int\limits_{B_r(y)} (u(x) - \bar{u}_{B_r(y)})^2 d\mu(x) \leq C_2 r^\gamma \mathcal{E}_K(u,u), \tag{16}$$

for all $x \in K$ and $r > C_0$, where $\bar{u}_{B_r(y)}$ is the average of u over $B_r(y)$, γ, C_0 and C_2 are positive constants; the measure μ satisfies the "inverse volume doubling property", that is there exists a $C_1 > 1$ such that for all $x \in K$ and $r > C_0$

$$C_1 \mu(B_r(x)) < \mu(B_{2r}(x)). \tag{17}$$

Note also that the constant γ appearing in (16) has the same meaning as in the definition of *Category II* above.

For spaces (K,d,μ) is in *Category* I_{grph} or in *Category* II_{grph} Theorem 1 takes now the following form.

Theorem 3. *Let K be a space equipped with a measure μ and a metric d. Assume that the metric measure space (K,d,μ) is either in the Category I_{grph} or in the Category II_{grph}. Then there exist positive constants C and C_0' such that for all $u \in L^2(K)$ with $\|u\|_2 = 1$ one has*

$$Var_K(u)\mathcal{E}_K(u,u) \geqslant C$$

provided $Var_K(u) > C_0'$.

4. Applications and examples

4.1. **Sierpiński gasket and p.c.f. fractals.** As mentioned in the Introduction, the weak uncertainty principle for functions defined on the Sierpiński gasket was first introduced in [24]. While the results in that paper were stated for p.c.f. fractals, they were only proved for the Sierpiński gasket. In this subsection, we use the results of Section 2 not only to provide a simpler proof to the main results of [24] , but also to establish weak uncertainty principles on all p.c.f. fractals. We briefly define the Sierpiński gasket which is a typical example of a p.c.f. fractal, and refer to [1, 17, 33] for more background on analysis on p.c.f. fractals.

Consider the contractions maps F_1, F_2 and F_3 defined on $\mathbb{R}^2$ by $F_1(x) = \frac{1}{2}x$, $F_2(x) = \frac{1}{2}x + (\frac{1}{2}, 0)$ and $F_3(x) = \frac{1}{2}x + (\frac{1}{4}, \frac{\sqrt{3}}{4})$, for $x \in \mathbb{R}^2$. The Sierpiński gasket $K = SG$, is the unique nonempty compact subset of $\mathbb{R}^2$ such that

$$K = \cup_{i=1}^{3} F_i K. \tag{18}$$

Alternatively, SG can be defined as a limit of graphs: Let V_0 be the complete graphs with vertices $\{(0,0), (1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ which these are the fixed points of the contractions F_i. Define $V_n = \cup_{i=1}^{3} F_i V_{n-1}$, $n \geqslant 1$, and let $V_* = \cup_{n=0}^{\infty} V_n$. Then $K = \overline{V_*}$, i.e., K is the closure of V_* in the Euclidean metric. For any positive integer m, $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$ where each $\omega_i \in \{1, 2, 3\}$ is called a word of length $|\omega| = m$, and we denote $F_\omega = F_{\omega_m} \circ F_{\omega_{m-1}} \circ \ldots \circ F_{\omega_1}$. Then $F_\omega K$ is called a cell of level m if ω is a word of length m. The (standard) measure on K is the probability measure on K that assigns to each cell of level m the measure 3^{-m}. It follows that SG is equipped with a self-similar measure that satisfies trivially (12). By defining an energy form on SG, it can be shown that this gives rise to a resistance metric on SG, see [1, 15, 17]. Consequently, SG belongs to *Category* I_{bdd}, and thus Theorem 2 applies.

More generally, let $\{F_i\}_{i=1}^{N}$ be a set of contraction maps on $\mathbb{R}^d$, and consider the following two sets of positive real numbers $\{\alpha_i\}_{i=1}^{N} \subset (0,1)$, and $\{\beta_i\}_{i=1}^{N}$ such that $\sum_{i=1}^{N} \beta_i = 1$. Following [15, 17], one can sometimes define a self-similar p.c.f. fractal K equipped with a (self-similar) measure μ, and an energy form $\mathcal{E}(\cdot, \cdot)$ which gives rise to an effective resistance metric. Moreover, K constructed in this way can be shown to belong to *Category* I_{bdd}, and thus Theorem 2 applies.

From the p.c.f. fractal constructed above, one can construct an increasing sequence of sets K_n, and define the blowup of K to be $K_\infty = \cup_{n=0}^{\infty} K_n$ where $K_0 = K$. Then K_∞ is an unbounded self-similar set, called fractal blowup and was first introduced in [32], see also [30, 29, 28, 36] for more about fractal blowups. Choose the measure scaling factors β_i such that $\beta_i = \alpha_i^b$ for each $i = 1 \ldots N$, where b is the Hausdorff dimension of (K, d). Then one can show by a simple scaling argument that Theorem 2 holds on K_n for each $n \geqslant 0$ with a constant independent of n, and hence in the limit Theorem 1 holds for the unbounded self-similar set K_∞. The choice $\beta_i = \alpha_i^b$ is natural, as with this choice the asymptotic behavior of the Weyl function is known [17].

4.2. **Sierpiński graphs.** As another application of the results of Section 2, we prove a weak uncertainty principle of some graphs related to the Sierpiński gasket $K = SG$, and its blowup K_∞. More precisely, for any integer $n \geqslant 0$, let V_n be the *nth* pre-gasket approximation to K, i.e., the *nth* graph approximation of K. We define a (finite) graph V_{-n} by $V_{-n} = F_{\omega_i}^{-1} \circ F_{\omega_2}^{-1} \ldots \circ F_{\omega_n}^{-1}(V_n)$, and an infinite graph V_∞ by: $V_\infty = \cup_{n\geqslant 0} V_{-n}$. V_∞ is an example of an infinite self-similar graph, which is also referred to as the Sierpiński lattice; we refer to [36] and the references therein for more on this type of graphs. Note that for all integer $n \geqslant 0$, V_{-n} is similar to the (finite) graph obtained by taking $F_{\omega_i} = F_1$ for all i, in which case, $V_{-n} = 2^n V_n$. Consequently, we will assume without any loss of generality that $V_{-n} = 2^n V_n$. It is easy to see that V_{-n} and V_∞ belongs to *Category* I_{grph}, and we can apply Theorem 3 in these two cases.

Remark 3. We would like to indicate how to use the results in [24] to obtain a weak uncertainty principle on graphs related to the Sierpiński gasket.

Observe that Laplacians and a corresponding energies can be naturally defined on V_{-n} and on V_∞. We will denote them respectively by Δ_n, $\mathcal{E}_n(,)$ for V_{-n}, and by Δ_∞ and $\mathcal{E}_\infty(,)$ for V_∞. Given a function u defined on V_{-n} such that $\|u\|_2^2 = \sum_{x\in V_{-n}} |u(x)|^2 = 1$, extend it to a function $\tilde{u}$ on the blowup K_n as follows:

- $\tilde{u}$ is harmonic in the interior of each triangle (similar to V_0), which makes up K_n,
- and the restriction of $\tilde{u}$ to V_{-n} coincides with u, i.e., $\tilde{u}|_{V_{-n}} = u$.

By the construction of the extension $\tilde{u}$ of u, it is clear that the energy of $\tilde{u}$ on K_n and that of u on V_{-n} satisfy the following relation:

$$\mathcal{E}_{K_n}(\tilde{u}, \tilde{u}) = \mathcal{E}_n(u, u), \tag{19}$$

where $\mathcal{E}_{K_n}(u, u)$ denotes the energy form on the blowup K_n. Note that in this case the variance (2) takes the following discrete form: For a function u define on V_{-n} such that $\|u\|_2 = 1$ by

$$Var_{dis,n}(u) = \sum_{x,y\in V_{-n}} d_R^{d+1}(x, y)|u(x)|^2\,|u(y)|^2. \tag{20}$$

Since harmonic functions form a three dimensional space, it is clear that the variance of the extended function $\tilde{u}$ can be controlled by the discrete variance of u. Thus one can apply some of the results from [24] to obtain analogous weak uncertainty on the graphs V_{-n}, and passing to the limit establish such inequality for functions defined on V_∞. We leave it to the interested reader to fill in the details of this argument.

4.3. **Uniform finitely ramified graphs.** These are graphs obtained from uniform finitely ramified fractals (u.f.r.). Note that u.f.r. fractals include nested fractals and is contained in the class of p.c.f. self-similar sets, see [14, 17, 15]. It was proved in [14, Section 2] that there exists an effective resistance metric on these class of graphs, and moreover [14, Lemma 3.2] establishes that these graphs belongs to *Category* I_{grph}. Thus Theorem 3 applies in this setting as well.

4.4. **Sierpiński carpets and graphical Sierpiński carpets.** These are examples of non finitely ramified fractals and fractal graphs [2, 3, 4, 5]. In particular, they are non p.c.f. fractals, and it is interesting to notice that most of our results apply in this setting. Hence, we answer affirmatively a question posed in [24] of whether the main results of that paper apply to "genuine" non-p.c.f. fractals. More precisely, on the generalized Sierpiński carpets (GSC) and the unbounded sets that can be constructed based on them, it is known that a two sided heat kernel estimate holds, [2, 4, 6]. Thus, following [2, 4, 6, 8] or [13, Theorem 3.2], one can show that (3) holds on the GSC and all related sets; this in turn implies that (5) holds also in these settings. Consequently, Theorem 1 or Theorem 2 applies for GSC and all related sets.

Notice also that on the graphical Sierpiński carpets [5] and the unbounded sets that can be constructed based on them, one could also prove that Theorem 3 holds.

It is worth noticing that if the Sierpiński carpet is constructed in the 2 dimensional Euclidean space, then it is known that there exists an effective resistance metric d. Moreover, a (double-sided) heat kernel estimate holds also in this context, see [3, 6]. Thus, the two dimensional Sierpiński carpet is also in $Category I_{\text{bdd}}$, so Theorem 2 holds.

4.5. **Metric measure spaces and heat kernel estimates.** Our results of Section 2 are applicable to the general setting of metric measure spaces. For a metric measure space (K, d, μ) the main assumption we make is the existence of a heat kernel $\{p_t\}_{t>0}$, which is the fundamental solution of the heat equation where the self-adjoint operator associated with the energy form $\mathcal{E}$ plays the role of a Laplacian. If the heat kernel, which is a non-negative measurable function $p_t(x, y)$ on $[0, \infty) \times K \times K$, satisfies the following two sided estimate for μ-almost $x, y \in K$ and all $t \in (0, \infty)$ (see [13]):

$$\frac{1}{t^{\alpha/\beta}} \Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \leqslant p_t(x,y) \leqslant \frac{1}{t^{\alpha/\beta}} \Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{21}$$

where α is the Hausdorff dimension of (K, d) and $\beta = \alpha + 1$, and Φ_1, Φ_2 are non-negative monotone decreasing functions on $[0, \infty)$, then under a mild decay condition on Φ_2, it is shown in [13, Theorem 3.2] that (21) implies (3) with $b = \alpha$. This can be used in turn to prove (5). Consequently, once a Poincaré-type estimate is established in this setting, one can conclude that (K, d, μ) belongs to *Category II* and our results can be applied. Fortunately, heat kernel estimates of the type

$$\frac{c_1}{\mu(B(x, t^{1/\gamma}))} \exp\left(-\left(\frac{d(x,y)^\gamma}{c_1 t}\right)^{\frac{1}{\gamma-1}}\right) \leqslant p_t(x,y)$$
$$\leqslant \frac{c_2}{\mu(B(x, t^{1/\gamma}))} \exp\left(-\left(\frac{d(x,y)^\gamma}{c_2 t}\right)^{\frac{1}{\gamma-1}}\right)$$

imply the Poincaré inequality, and these estimates can be established on many fractals and other spaces (see [8] and references therein).

Relevant results of a different kind, related to the scaling in the effective resistance metric and *Category I* spaces, can be found in [22, 23].

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