MODULATION SPACES AND A CLASS OF BOUNDED MULTILINEAR PSEUDODIFFERENTIAL OPERATORS

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Abstract. We show that multilinear pseudodifferential operators with symbols in the modulation space $\mathcal{M}^{\infty,1}$ are bounded on products of modulation spaces. In particular, $\mathcal{M}^{\infty,1}$ includes non-smooth symbols. Several multilinear Calderón–Vaillancourt-type theorems are then obtained by using certain embeddings of classical function spaces into modulation spaces.

1. Introduction

The study of multilinear operators has been actively pursued in recent years due to their many applications in linear and nonlinear partial differential equations. For example, it it known that the formal solutions to certain evolution equations reduce to infinite sums of multilinear pseudodifferential operators; see [7] and the references therein. The simplest example of a multilinear operator is the pointwise product of n functions, and in this case Hölder's inequality regulates the boundedness properties on Lebesgue spaces. In this paper we address the question of how much of Hölder's inequality carries over to the much more complicated class of general multilinear pseudodifferential operators.

An m-linear pseudodifferential operator is defined à priori through its (distributional) symbol σ to be the mapping T_σ from the m-fold product of Schwartz spaces $\mathcal{S}(\mathbb{R}^d) \times \cdots \times \mathcal{S}(\mathbb{R}^d)$ into the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions given by the formula

$$T_\sigma(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^d)^m} \sigma(x, \xi_1, \ldots, \xi_m)\, \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m)\, e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\, d\xi_1 \cdots d\xi_m, \qquad (1)$$

for $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^d)$. The pointwise product $f_1 \cdots f_m$ corresponds to the case $\sigma \equiv 1$.

Various authors have searched for sufficient (nontrivial) conditions on σ that guarantee the boundedness of T_σ on products of appropriately chosen Banach spaces. For instance, by using wavelet decompositions and a multilinear version of Schur's test, Grafakos and Torres [13] have obtained results on Besov-Triebel-Lizorkin spaces. For other results, including the boundedness of multilinear Hörmander-Mihlin and

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Marcinkiewicz multipliers, that use classical harmonic analysis techniques, see, e.g., [6], [12], [14]. Another line of investigation uses the class of modulation spaces both as symbols and as the underlying Banach spaces on which a multilinear pseudodifferential operator acts. The modulation spaces figure implicitly in the analysis of linear pseudodifferential operators presented in [3], [19], [24]. The paper [17] explicitly recognized the space $M^{\infty,1}(\mathbb{R}^{2d})$ as the appropriate symbol class to establish the boundedness of $T_\sigma = \sigma(X,D)$ acting on $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$, including $M^2 = L^2$ as a special case. Further developments using modulation spaces have since been obtained in [5], [15], [18], [21], [22], [25]. The analogous investigation of multilinear pseudodifferential operators on modulation spaces was initiated in [2] and is certainly only in its infancy.

We will investigate the boundedness of multilinear pseudodifferential operators on products of modulation spaces. As our symbol class we use the modulation space $\mathcal{M}^{\infty,1}(\mathbb{R}^{(m+1)d})$. This modulation space can be seen as a useful and conceptually simple extension of the standard symbol class $S^0_{0,0}$. In particular, $\mathcal{M}^{\infty,1}$ includes nonsmooth symbols. Our main result shows that an m-linear pseudodifferential operator T_σ with symbol $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{(m+1)d})$ is bounded on modulation spaces with indices that obey a relation similar to Hölder's inequality. In contrast to pure analysis results which would use decomposition techniques, Schur's test, or Cotlar's Lemma, we will use tools developed in time-frequency analysis, especially techniques developed in [15, Ch. 14] and [18]. Further, by using some recent embeddings theorems from [23], we can state new boundedness results on products of certain Besov spaces.

While concrete boundedness problems are rarely easy to deal with, the bilinear or multilinear case offers additional difficulties. To give an example of these new problems, consider the classical symbol class $S^0_{0,0}$ consisting of those σ which satisfy estimates of the form

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta}, \qquad \forall\, \alpha, \beta \ge 0. \tag{2}$$

A classical result of Calderón and Vaillancourt [4] asserts that the corresponding *linear* pseudodifferential operator T_σ is bounded on $L^2(\mathbb{R}^d)$. In the *bilinear* case, however, the analogous class of symbols which satisfy the conditions

$$|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\alpha,\beta,\gamma}, \qquad \forall\, \alpha, \beta, \gamma \ge 0, \tag{3}$$

does not necessarily yield bounded operators from $L^2 \times L^2$ into L^1, unless additional size conditions are imposed on the symbols; see [1]. However, as a consequence of our main result we will show that the Calderón–Vaillancourt-like condition (3) does yield boundedness from $L^2 \times L^2$ into a modulation space that contains L^1.

Our conditions should also be compared to a typical hard analysis result of Coifman and Meyer [6, Thm. 12]: If the symbol σ of a bilinear pseudodifferential operator satisfies the conditions

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\beta,\gamma} \tag{4}$$

and

$$|\partial_\xi^\beta \partial_\eta^\gamma \sigma(x',\xi,\eta) - \partial_\xi^\beta \partial_\eta^\gamma \sigma(x,\xi,\eta)| \le C_{\beta,\gamma}\, |x'-x|^\delta \tag{5}$$

for all $\beta,\ \gamma \ge 0$ and some $\delta > 0$, then the corresponding operator is bounded on products of certain Lebesgue spaces. It turns out that conditions (4) and (5) are not comparable to the condition $\sigma \in M^{\infty,1}$; neither set of conditions implies the other.

Our paper is organized as follows. In Section 2 we set the notation, define the modulation spaces and collect some of their basic properties and the embeddings that will be needed later on. The main results are then stated and proved in Section 3, and some applications of these results are obtained in Section 4.

2. Notation and Preliminaries

2.1. **General notation.** Translation and modulation of a function f with domain $\mathbb{R}^d$ are, respectively, $T_x f(t) = f(t-x)$ and $M_y f(t) = e^{2\pi i y\cdot t} f(t)$. The inner product $f,\ g \in L^2(\mathbb{R}^d)$ is $\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t)}\,dt$, and the same notation is used for the extension of the inner product to $\mathcal{S}' \times \mathcal{S}$. The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)\, e^{-2\pi i t\cdot\omega}\,dt$.

The Short-Time Fourier Transform (STFT) of a function f with respect to a window g is

$$V_g f(x,y) = \langle f, M_y T_x g\rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot t}\,\overline{g(t-x)}\, f(t)\,dt, \qquad (x,y) \in \mathbb{R}^{2d},$$

whenever the integral makes sense. If $g \in \mathcal{S}$ and $f \in \mathcal{S}'$ then $V_g f$ is a uniformly continuous function on $\mathbb{R}^{2d}$. One important technical tool is the extended isometry property of the STFT [15, (14.31)]: If $\phi \in \mathcal{S}(\mathbb{R}^d)$, $\|\phi\|_{L^2} = 1$, then

$$\langle f, h\rangle = \langle V_\phi f, V_\phi h\rangle \qquad \forall\, f \in \mathcal{S}',\ h \in \mathcal{S}. \tag{6}$$

A second important tool is the fundamental identity $V_g f(x,y) = e^{-2\pi i x\cdot y}\, V_{\hat{g}} \hat{f}(y,-x)$.

We let $L^{p,q}(\mathbb{R}^{2d})$ be the mixed-norm Lebesgue space defined by the norm

$$\|f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x,y)|^p\,dx\right)^{q/p} dy\right)^{1/q},$$

with the usual adjustment if p or q is infinite, and we use a similar notation for the mixed-norm sequence spaces $\ell^{p,q}$.

2.2. **Modulation spaces.** Given $1 \le p, q \le \infty$, and given a fixed, nonzero window function $g \in \mathcal{S}(\mathbb{R}^d)$, the modulation space $\mathcal{M}^{p,q}(\mathbb{R}^d)$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ for which the following norm is finite:

$$\|f\|_{\mathcal{M}^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,y)|^p\,dx\right)^{q/p} dy\right)^{1/q} = \|V_g f\|_{L^{p,q}}, \tag{7}$$

with the usual modifications if p or q are infinite. Note that $\mathcal{M}^{2,2} = L^2$.

We refer to [15] for a detailed description of the theory of modulation spaces and their weighted counterparts. In particular, $\mathcal{M}^{p,q}$ is a Banach space, and any nonzero function $\phi \in \mathcal{M}^{1,1}$ can be substituted for g in (7) to define an equivalent norm for $\mathcal{M}^{p,q}$. The Schwartz class is dense in $\mathcal{M}^{p,q}$ for all $p,\ q < \infty$.

For $1 \le p, q < \infty$, the dual of $\mathcal{M}^{p,q}$ is $\mathcal{M}^{p',q'}$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. To deal with duality properly in the cases $p = \infty$ or $q = \infty$, we introduce the following new related modulation spaces.

Definition 2.1. Let $L^0(\mathbb{R}^{2d})$ denote the space of bounded, measurable functions on $\mathbb{R}^{2d}$ which vanish at infinity. We define

$$\begin{aligned}
\mathcal{M}^{0,q}(\mathbb{R}^d) &= \{f \in \mathcal{M}^{\infty,q}(\mathbb{R}^d) : V_g f \in L^0(\mathbb{R}^{2d})\}, \quad 1 \le q < \infty,\\
\mathcal{M}^{p,0}(\mathbb{R}^d) &= \{f \in \mathcal{M}^{p,\infty}(\mathbb{R}^d) : V_g f \in L^0(\mathbb{R}^{2d})\}, \quad 1 \le p < \infty,\\
\mathcal{M}^{0,0}(\mathbb{R}^d) &= \{f \in \mathcal{M}^{\infty,\infty}(\mathbb{R}^d) : V_g f \in L^0(\mathbb{R}^{2d})\},
\end{aligned}$$

equipped with the norms of $\mathcal{M}^{\infty,q}$, $\mathcal{M}^{p,\infty}$, and $\mathcal{M}^{\infty,\infty}$, respectively.

Though not yet explicitly mentioned in the literature, we will see that these spaces are useful for the treatment of end-point results and in the study of compactness properties of pseudodifferential operators. The following properties are easily established.

Lemma 2.2.

(a) $\mathcal{M}^{0,q}$ is the $\mathcal{M}^{\infty,q}$-closure of $\mathcal{S}$ in $\mathcal{M}^{\infty,q}$, hence is a closed subspace of $\mathcal{M}^{\infty,q}$. Likewise, $\mathcal{M}^{p,0}$ is the $\mathcal{M}^{p,\infty}$-closure of $\mathcal{S}$ in $\mathcal{M}^{p,\infty}$, and $\mathcal{M}^{0,0}$ is the closure of $\mathcal{S}$ in the $\mathcal{M}^{\infty,\infty}$-norm.

(b) The following duality results hold for $1 \le p, q < \infty$: $(\mathcal{M}^{0,q})' = \mathcal{M}^{1,q'}$, $(\mathcal{M}^{p,0})' = \mathcal{M}^{p',1}$, and $(\mathcal{M}^{0,0})' = \mathcal{M}^{1,1}$.

Proof. Statement (a) is proved exactly as [15, Prop. 11.3.4], and (b) can be obtained by a modification of [15, Thm. 11.3.6]. Both statements can also be seen as special cases of the coorbit space theory developed in [8]. □

Using these spaces, we can prove that the following compactness result for linear pseudodifferential operators is a corollary of the boundedness result for the symbol class $\mathcal{M}^{\infty,1}$. Other compactness results have been obtained by Labate in [22].

Proposition 2.3. If $\sigma \in \mathcal{M}^{0,1}(\mathbb{R}^{2d})$, then T_σ is a compact mapping of $\mathcal{M}^{p,q}$ into itself for all $1 \le p, q \le \infty$.

Proof. Assume first that $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$. Then we can write T_σ as an integral operator with kernel $k \in \mathcal{S}(\mathbb{R}^{2d})$. Let $\phi \in M^{1,1}(\mathbb{R}^d)$ and α, $\beta > 0$ be such that $\{\phi_{kn}\}_{k,n\in\mathbb{Z}^d}$ is a Parseval Gabor frame for $L^2(\mathbb{R}^d)$, where $\phi_{kn} = M_{\beta n} T_{\alpha k}\phi$. Then $\{\Phi_{k\ell mn}\}_{k,\ell,m,n\in\mathbb{Z}^d}$ is a Parseval Gabor frame for $L^2(\mathbb{R}^{2d})$, where $\Phi_{k\ell mn}(x,y) = \phi_{kn}(x)\,\overline{\phi_{\ell m}(y)}$. Since $k \in M^{1,1}$, we therefore have

$$k = \sum_{k,\ell,m,n} \langle k, \Phi_{k\ell mn}\rangle\, \Phi_{k\ell mn}, \quad \text{with} \quad \sum_{k,\ell,m,n} |\langle k, \Phi_{k\ell mn}\rangle| < \infty,$$

and hence

$$T_\sigma f = \sum_{k,\ell,m,n} \langle k, \Phi_{k\ell mn}\rangle\, \langle f, \phi_{\ell m}\rangle\, \phi_{kn}.$$

Since the ϕ_{kn} are uniformly bounded in $M^{p,q}$-norm, it follows easily that T_σ is a compact mapping of $M^{p,q}$ into itself; in fact, T_σ is nuclear.

For the general case, if $\sigma \in \mathcal{M}^{0,1}(\mathbb{R}^{2d})$ then by Lemma 2.2 there exists a sequence $\sigma_n \in \mathcal{S}(\mathbb{R}^{2d})$ such that $\|\sigma - \sigma_n\|_{\mathcal{M}^{\infty,1}} \to 0$. By the boundedness theorem for linear pseudodifferential operators with symbols $\sigma \in \mathcal{M}^{\infty,1}$ [15, Thm. 14.5.2], the operator norm can be estimated as $\|T_\sigma - T_{\sigma_n}\|_{\mathcal{M}^{p,q}\to\mathcal{M}^{p,q}} \le C\|\sigma - \sigma_n\|_{\mathcal{M}^{\infty,1}}$. Since the ideal of compact operators is closed in the operator norm, this implies that T_σ is compact on $\mathcal{M}^{p,q}$. □

2.3. **Embeddings.** We conclude this section by listing a few embeddings proved in [23] between Lebesgue or Besov spaces and modulation spaces. Further embeddings and comparisons of modulation space with standard spaces can be found in [16], [11], [20], [25].

(a) $\mathcal{B}^s_{p,q} \subseteq L^p \subseteq \mathcal{M}^{p,p'}$ for $s > 0$, $1 \le p \le 2$ and $1 \le q \le \infty$;
(b) $\mathcal{B}^s_{p,q} \subseteq L^p \subseteq \mathcal{M}^{p,p}$ for $s > 0$, $2 \le p \le \infty$ and $1 \le q \le \infty$;
(c) $\mathcal{B}^s_{p,p} \subseteq \mathcal{M}^{p,p'}$ for $s > d/p'$, $1 \le p \le \infty$.

3. Boundedness of Multilinear Pseudodifferential Operators

Our main result is the following.

Theorem 3.1. If $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{(m+1)d})$, then the m-linear pseudodifferential operator T_σ defined by (1) extends to a bounded operator from $\mathcal{M}^{p_1,q_1} \times \cdots \times \mathcal{M}^{p_m,q_m}$ into $\mathcal{M}^{p_0,q_0}$ when $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}$, $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0}$, and $1 \le p_i, q_i \le \infty$ for $0 \le i \le m$.

Theorem 3.1 has the following intuitive explanation. Though not literally correct, it is instructive to think of $f \in \mathcal{M}^{p,q}$ as being represented by the statement "$f \in L^p$ and $\hat{f} \in L^q$" (for a rigorous comparison of modulation spaces and Fourier-Lebesgue spaces see the embeddings in [11]). Under this analogy, the first condition $\sum p_j^{-1} = p_0^{-1}$ is the condition required to estimate the pointwise product $f_1 \cdots f_m$ by Hölder's inequality, and the second condition $\sum q_j^{-1} = m - 1 + q_0^{-1}$ is the condition needed to apply Young's inequality to the convolution product $\hat{f}_1 * \cdots * \hat{f}_m$. Thus, loosely speaking, Theorem 3.1 asserts that the symbol class $\mathcal{M}^{\infty,1}$ yields multilinear operators T_σ that behave like pointwise multiplication with respect to *both* time and frequency.

The proof of Theorem 3.1 requires some preparation. To compactify the notation, let us write $\vec{\xi} = (\xi_1, \ldots, \xi_m)$, $d\vec{\xi} = d\xi_1 \cdots d\xi_m$, etc. Then for $f_1, \ldots, f_m, g \in \mathcal{S}(\mathbb{R}^d)$, the action of T_σ can be expressed by the formula

$$\begin{aligned}\langle T_\sigma \vec{f},\, g\rangle &= \langle T_\sigma(f_1, \ldots, f_m),\, g\rangle \\ &= \int_{\mathbb{R}^{(m+1)d}} \sigma(x, \xi_1, \ldots, \xi_m)\, \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m)\, e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\, \overline{g(x)}\, d\xi_1 \cdots d\xi_m\, dx \\ &= \langle \sigma,\, W_m(g, f_1, \ldots, f_m)\rangle = \langle \sigma,\, W_m(g, \vec{f})\rangle,\end{aligned}$$

where

$$W_m(g, f_1, \ldots, f_m)(x, \xi_1, \ldots, \xi_m) = g(x)\, \overline{\hat{f}_1(\xi_1)} \cdots \overline{\hat{f}_m(\xi_m)}\, e^{-2\pi i x\cdot(\xi_1+\cdots+\xi_m)}.$$

Remark 3.2. For $m = 1$, the Kohn-Nirenberg correspondence can be written as $\langle T_\sigma f, g\rangle = \langle \sigma, W_1(g,f)\rangle$ where $W_1(g,f) = e^{-2\pi i x\cdot\xi}\, g(x)\,\hat{f}(\xi)$ is the so-called cross-Ryhaczek distribution of f and g. Thus, one may think of W_m as a multilinear version of the Ryhaczek distribution.

The following multilinear "magic formula" will be an important tool.

Lemma 3.3. Let $(\phi_0, \vec{\phi}) = (\phi_0, \phi_1, \dots, \phi_m) \in (\mathcal{S}(\mathbb{R}^d))^{m+1}$ be given. Then for $(\vec{f}, g) \in (\mathcal{M}^{\infty,\infty}(\mathbb{R}^d))^{m+1}$ and $(u_0, \vec{u}) = (u_0, u_1, \dots, u_m)$, $(v_0, \vec{v}) = (v_0, v_1, \dots, v_m) \in \mathbb{R}^{(m+1)d}$ we have

$$V_{W_m(\phi_0,\vec{\phi})} W_m(g, \vec{f})((u_0, \vec{u}), (v_0, \vec{v})) = e^{2\pi i u_0\cdot(u_1+\cdots+u_m)}\, V_{\phi_0} g(u_0, v_0 + u_1 + \cdots + u_m)\, \overline{\prod_{i=1}^m V_{\phi_i} f_i(u_0 + v_i, u_i)}. \tag{8}$$

Proof. Note first that $W_m(\phi_0, \vec{\phi}) \in \mathcal{S}(\mathbb{R}^{(m+1)d})$. Assume that we also had $(\vec{f}, g) \in (\mathcal{S}(\mathbb{R}^d))^{m+1}$. Then the integral defining the STFT $V_{W_m(\phi_0,\vec{\phi})} W_m(g, \vec{f})$ converges absolutely, and hence the following manipulations are justified:

$$\begin{aligned}
&V_{W_m(\phi_0,\vec{\phi})} W_m(g, \vec{f})((u_0, \vec{u}), (v_0, \vec{v})) \\
&\quad = \int_{\mathbb{R}^{(m+1)d}} W_m(g, \vec{f})(x, \vec{\xi})\, e^{-2\pi i (x,\vec{\xi})\cdot(v_0,\vec{v})}\, \overline{W_m(\phi_0, \vec{\phi})((x, \vec{\xi}) - (u_0, \vec{u}))}\, dx\, d\vec{\xi} \\
&\quad = \int_{\mathbb{R}^{(m+1)d}} g(x)\, \overline{\prod_{i=1}^m \hat{f}_i(\xi_i)}\, e^{-2\pi i x\cdot \sum_{i=1}^m \xi_i}\, e^{-2\pi i (x\cdot v_0 + \sum_{i=1}^m \xi_i\cdot v_i)}\, \times \\
&\qquad\qquad \overline{\phi_0(x - u_0)}\, \prod_{i=1}^m \hat{\phi}_i(\xi_i - u_i)\, e^{2\pi i (x-u_0)\cdot\sum_{i=1}^m (\xi_i - u_i)}\, dx\, d\vec{\xi} \\
&\quad = e^{2\pi i u_0\cdot\sum_{i=1}^m u_i} \int_{\mathbb{R}^{(m+1)d}} g(x)\, \prod_{i=1}^m \overline{\hat{f}_i(\xi_i)}\, \overline{\phi_0(x-u_0)}\, \prod_{i=1}^m \hat{\phi}_i(\xi_i - u_i)\, \times \\
&\qquad\qquad e^{-2\pi i x\cdot(v_0 + \sum_{i=1}^m u_i)}\, \prod_{i=1}^m e^{-2\pi i \xi_i\cdot(u_0+v_i)}\, dx\, d\vec{\xi} \\
&\quad = e^{2\pi i u_0\cdot\sum_{i=1}^m u_i}\, V_{\phi_0} g\Big(u_0, v_0 + \sum_{i=1}^m u_i\Big)\, \overline{\prod_{i=1}^m V_{\hat{\phi}_i} \hat{f}_i(u_i, -u_0 - v_i)} \\
&\quad = e^{2\pi i u_0\cdot\sum_{i=1}^m u_i}\, V_{\phi_0} g\Big(u_0, v_0 + \sum_{i=1}^m u_i\Big)\, \overline{\prod_{i=1}^m V_{\phi_i} f_i(u_0 + v_i, u_i)},
\end{aligned}$$

and the result follows in this case.

Now assume that $(g, \vec{f}) \in (\mathcal{M}^{\infty,\infty}(\mathbb{R}^d))^{m+1}$. Then we have $\bar{g} \otimes \hat{f}_1 \otimes \cdots \otimes \hat{f}_m \in \mathcal{M}^{\infty,\infty}(\mathbb{R}^{(m+1)d})$. Since pointwise multiplication by the "chirp" $e^{-2\pi i x\cdot(\xi_1+\cdots+\xi_m)}$ leaves $\mathcal{M}^{\infty,\infty}$ invariant ([9] or [15, Thm. 12.1.3]), we find that $W_m(g, \vec{f}) \in \mathcal{M}^{\infty,\infty}(\mathbb{R}^{(m+1)d})$

as well. Consequently $V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})$ is a well-defined, bounded, uniformly continuous function on $\mathbb{R}^{(m+1)d}$.

We prove the validity of the identity (8) by approximation. Since $\mathcal{S}$ is weak*-dense in $\mathcal{M}^{\infty,\infty}$, we can choose sequences $g_n \in \mathcal{S}(\mathbb{R}^d)$ and $\vec{f_n} \in (\mathcal{S}(\mathbb{R}^d))^m$ such that $(g_n,\vec{f_n}) \xrightarrow{w^*} (g,\vec{f})$ in $\mathcal{M}^{\infty,\infty}$. By continuity of tensor products and multiplication by chirps, we obtain that $W_m(g_n,\vec{f_n}) \xrightarrow{w^*} W_m(g,\vec{f})$ in $\mathcal{M}^{\infty,\infty}$. Since weak*-convergence of distributions is equivalent to uniform convergence of the STFT on compact sets [8], we find that $V_{W_m(\phi_0,\vec{\phi})}W_m(g_n,\vec{f_n})$ converges uniformly on compact sets to $V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})$.

Similarly, for the right-hand side of (8) we obtain that $V_{\phi_0}g_n \to V_{\phi_0}g$ and $V_{\phi_i}(f_n)_i \to V_{\phi_i}f_i$ uniformly on compact sets. Consequently the right-hand side converges uniformly to $e^{2\pi i u_0\cdot\sum_{i=1}^m u_i}\, V_{\phi_0}g(u_0, v_0+\sum_{i=1}^m u_i)\, \overline{\prod_{i=1}^m V_{\phi_i}f_i(u_0+v_i,u_i)}$. This proves the identity in the general case. $\square$

Lemma 3.4. Let $(\phi_0,\vec{\phi}) \in (\mathcal{S}(\mathbb{R}^d))^{m+1}$ be given. Assume that $\frac{1}{p_1}+\cdots+\frac{1}{p_m} = \frac{1}{p_0}$ and $\frac{1}{q_1}+\cdots+\frac{1}{q_m} = m-1+\frac{1}{q_0}$, with $1 \le p_0, q_0 \le \infty$ for $0 \le i \le m$. Then

$$\|V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})\|_{L^{1,\infty}} \le C\,\|f_1\|_{\mathcal{M}^{p_1,q_1}} \cdots \|f_m\|_{\mathcal{M}^{p_m,q_m}}\, \|g\|_{\mathcal{M}^{p_0',q_0'}},$$

whenever the right-hand side is defined.

Proof. Lemma 3.3 implies that for all $(v_0,\vec{v}) \in \mathbb{R}^{(m+1)d}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^{(m+1)d}} |V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})((u_0,\vec{u}),(v_0,\vec{v}))|\, du_0\, d\vec{u} \\
&\quad = \int_{\mathbb{R}^{(m+1)d}} \Big|V_{\phi_0}g(u_0, v_0+\sum_{i=1}^m u_i)\Big|\, \Big|\prod_{i=1}^m V_{\phi_i}f_i(u_0+v_i,u_i)\Big|\, du_0\, d\vec{u} \\
&\quad \le \int_{\mathbb{R}^{md}} \|V_{\phi_0}g(\cdot, v_0+\sum_{i=1}^m u_i)\|_{L^{p_0'}} \prod_{i=1}^m \|V_{\phi_i}f_i(\cdot,u_i)\|_{L^{p_i}}\, d\vec{u} \;=\; (*),
\end{aligned}$$

the last line following by applying Hölder's inequality in the first variable, since $\frac{1}{p_1}+\cdots+\frac{1}{p_m}+\frac{1}{p_0'} = 1$. Now write $G(v) = \|V_{\phi_0}g(\cdot,v)\|_{L^{p_0'}}$ and $F_i(u_i) = \|V_{\phi_i}f_i(\cdot,-u_i)\|_{L^{p_i}}$. With this notation, $\|G\|_{L^{q_0'}} = \|g\|_{\mathcal{M}^{p_0',q_0'}}$ and $\|F_i\|_{L^{q_i}} = \|f_i\|_{\mathcal{M}^{p_i,q_i}}$ (more precisely, a different equivalent norm for $\mathcal{M}^{p_i,q_i}$ is used for each i because of the different choice of window functions), and we may rewrite the term $(*)$ above as

$$(*) = \int_{\mathbb{R}^{md}} G(v_0+\sum_{i=1}^m u_i) \prod_{i=1}^m F_i(-u_i)\, d\vec{u} = (G * F_1 * \cdots * F_m)(v_0).$$

Note that this expression is independent of $\vec{v}$. Applying now Young's inequality for convolutions, since $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=m-1+\frac{1}{q_0}$ we obtain

$$\begin{aligned}
&\|V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})\|_{L^{1,\infty}} \\
&\qquad = \sup_{(v_0,\vec{v})\in\mathbb{R}^{(m+1)d}} \int_{\mathbb{R}^{(m+1)d}} |V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})((u_0,\vec{u}),(v_0,\vec{v}))|\,du_0\,d\vec{u} \\
&\qquad \le \|G * F_1 * \cdots * F_m\|_{L^\infty} \\
&\qquad \le \|G\|_{q_0'} \prod_{i=1}^m \|F_i\|_{L^{q_i}} \\
&\qquad \le C\,\|g\|_{\mathcal{M}^{p_0',q_0'}}\,\|f_1\|_{\mathcal{M}^{p_1,q_1}}\,\cdots\,\|f_m\|_{\mathcal{M}^{p_m,q_m}},
\end{aligned}$$

the constant C arising from the use of different windows to measure the modulation space norms. □

We can now prove our main result.

Proof of Theorem 3.1. Let $f_i\in\mathcal{M}^{p_i,q_i}$ be given, and let $\phi_0,\phi_1,\dots,\phi_m\in\mathcal{S}(\mathbb{R}^d)$ be fixed so that $\|\phi_i\|_{L^2}=1$ for each i. Then, using the extended isometry property of the STFT, Hölder's inequality, and Lemma 3.4, for any $g\in\mathcal{M}^{p_0',q_0'}$ we may estimate that

$$\begin{aligned}
|\langle T_\sigma\vec{f},g\rangle| &= \left|\left\langle \sigma,\,W_m(g,\vec{f})\right\rangle\right| \\
&= \left|\left\langle V_{W_m(\phi_0,\vec{\phi})}\sigma,\,V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})\right\rangle\right| \\
&\le \|V_{W_m(\phi_0,\vec{\phi})}\sigma\|_{L^{\infty,1}}\,\|V_{W_m(\phi_0,\vec{\phi})}W_m(g,\vec{f})\|_{L^{1,\infty}} \\
&\le C\,\|\sigma\|_{\mathcal{M}^{\infty,1}} \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i,q_i}}\,\|g\|_{\mathcal{M}^{p_0',q_0'}}.
\end{aligned}$$

If $p_0',q_0'<\infty$, then the duality properties of the modulation spaces implies that $T_\sigma\vec{f}\in\mathcal{M}^{p_0,q_0}$ with the norm estimate

$$\|T_\sigma\vec{f}\|_{\mathcal{M}^{p_0,q_0}} \le C\,\|\sigma\|_{\mathcal{M}^{\infty,1}} \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i,q_i}}.$$

If either $p_0'=\infty$ or $q_0'=\infty$ or both, then we take $g\in\mathcal{M}^{0,q_0'}$, $\mathcal{M}^{p_0',0}$, or $\mathcal{M}^{0,0}$ instead. Again the duality stated in Lemma 2.2 then implies that $T_\sigma\vec{f}\in\mathcal{M}^{p_0,q_0}$ with the correct norm estimate, which completes the proof. □

4. Applications

In this final section we give some applications of Theorem 3.1.

First we consider that boundedness of T_σ from $\mathcal{M}^{p_1,p_1}\times\cdots\times\mathcal{M}^{p_m,p_m}$ into $\mathcal{M}^{p_0,p_0}$. The required conditions on the exponents p_i and q_i then imply that we must necessarily have $m=1$, since $p_i=q_i$. Thus we recover the following boundedness condition for linear pseudodifferential operators, which, as explained in the introduction, extends the classical result of Calderón and Vaillancourt.

Corollary 4.1. ([17, Theorem 1.1]) If $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{2d})$, then T_σ extends to a bounded operator from $\mathcal{M}^{p,p}$ into $\mathcal{M}^{p,p}$ for $1 \le p \le \infty$.

If instead we choose $q_i = p_i'$ for $1 \le i \le m$, then the conditions of Theorem 3.1 yield $q_0 = p_0'$. Hence we have the following.

Corollary 4.2. If $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{(m+1)d})$ and $1 \le p_0, p_1, \dots, p_m \le \infty$ satisfy $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}$, then T_σ extends to a bounded operator from $\mathcal{M}^{p_1,p_1'} \times \cdots \times \mathcal{M}^{p_m,p_m'}$ into $\mathcal{M}^{p_0,p_0'}$.

Using the embedding (c) from Section 2.2 of Besov spaces into the modulation spaces, we obtain the following.

Corollary 4.3. Let $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{(m+1)d})$, and let $1 < p_0, p_1, \dots, p_m < \infty$ be given so that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}$. If $s_i > \frac{d}{p_i'}$ for $1 \le i \le m$, then T_σ extends to a bounded operator from $\mathcal{B}^{s_1}_{p_1,p_1} \times \cdots \times \mathcal{B}^{s_m}_{p_m,p_m}$ into $\mathcal{M}^{p_0,p_0'}$.

It is tempting to seek a similar result for Lebesgue spaces by using the embedding (a) from Section 2.2. However, in this case the embeddings and the conditions of Theorem 3.1 do not seem to lead to interesting results.

Next we consider the multilinear Calderón–Vaillancourt class of symbols σ defined by the inequalities

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} \sigma(x, \xi_1, \dots, \xi_m)| \le C_{\alpha_0,\alpha_1,\dots,\alpha_m}, \tag{9}$$

for all multiindices α_i, $0 \le i \le m$ up to a certain order. It was shown in [1] that condition (9) does not necessarily yield an operator T_σ that is bounded from $L^2 \times L^2$ into L^1, or more generally from $L^p \times L^q$ into L^r for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Here the use of the modulation spaces clarifies the situation. In particular, by applying Theorem 3.1 with $p_1 = p_2 = q_1 = q_2$ and $\mathcal{M}^{2,2} = L^2$ we see by how much $L^2 \times L^2$ fails to be mapped into L^1.

Corollary 4.4. If $\sigma \in \mathcal{M}^{\infty,1}(\mathbb{R}^{3d})$, then T_σ maps $L^2 \times L^2$ into $\mathcal{M}^{1,\infty}$ (in fact, into $\mathcal{M}^{1,0}$).

The relationship between $\mathcal{M}^{\infty,1}$ and the Calderón–Vaillancourt class (9) is illuminated by the following embeddings.

Corollary 4.5. A symbol σ belongs to $\mathcal{M}^{\infty,1}$ under each of the following conditions:

(a) Equation (9) is satisfied for all α_j such that $\sum_{j=1}^m |\alpha_j| \le m(d+1) + 1$.
(b) Equation (9) is satisfied for all α_j such that $|\alpha_j| \le d + 1$ for $j = 0, \dots, m$.
(c) Equation (9) is satisfied for all α_j such that $\alpha_j \in \{0, 1, 2\}^d$.
(d) $\sigma \in C^s(\mathbb{R}^{(m+1)d})$ with $s > (m+1)d$.

In each of these cases, T_σ extends to a bounded operator from $\mathcal{M}^{p_1,q_1} \times \cdots \times \mathcal{M}^{p_m,q_m}$ into $\mathcal{M}^{p_0,q_0}$ when $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}$, $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0}$, and $1 \le p_i, q_i \le \infty$ for $0 \le i \le m$.

Proof. The embeddings (a) and (d) are well-known, see, e.g., [15], [19], [23]. The embeddings (b) and (c) are new, but their proofs are almost identical to the proof of [15, Thm. 14.5.3]. □

Remark 4.6. Finally, we compare membership of the symbol in $\mathcal{M}^{\infty,1}(\mathbb{R}^{3d})$ with the requirement that σ satisfy (4) and (5). These two conditions are distinct, in the sense that neither implies the other. The condition presented in this paper is more general in the variables ξ and η, but too strong in the x-variable. We can easily construct examples satisfying one but not the other condition. For instance, consider a symbol of the form

$$\sigma(x,\xi,\eta) = \sum_{k,l\in\mathbb{Z}^d} a_{k,l}(x)\, e^{2\pi i(k\cdot\xi + l\cdot\eta)}$$

with $\sum_{k,l} |a_{kl}(x)| < \infty$ for all x. Choosing the coefficients suitably, we can make $\sigma \in \mathcal{M}^{\infty,1}$, but σ obviously does not satisfy the Coifman–Meyer conditions.

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