MULTILINEAR LOCALIZATION OPERATORS

ELENA CORDERO* AND KASSO A. OKOUDJOU*,†

Abstract. In this paper we introduce a notion of multilinear localization operators. By reinterpreting these operators as multilinear Kohn-Nirenberg pseudodifferential operators, we prove that these multilinear localization operators are bounded on products of modulation spaces. In particular, by assuming that the symbols of the localization operators belong to the largest modulation space, i.e., M^∞, we show that the symbols of the corresponding pseudodifferential operators belong to the modulation space $M^{\infty,1}$. The boundedness properties of the localization operators, then follow from known results on boundedness of multilinear pseudodifferential operators on products of modulation spaces. Moreover, we show the sufficient boundedness condition is also necessary. Finally, as application we construct various examples of bounded multilinear pseudodifferential operators with symbols in $M^{\infty,1}$.

1. Introduction

Let the translation and modulation operators respectively defined by

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t), \tag{1}$$

where f is a function on $\mathbb{R}^d$, and $x, \omega, t \in \mathbb{R}^d$. For a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$ the short-time Fourier transform (STFT) of $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt\,. \tag{2}$$

The localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ is the mapping initially defined from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega)\, V_{\varphi_1} f(x,\omega)\, M_\omega T_x \varphi_2(t)\, dx\, d\omega\,. \tag{3}$$

In particular, if $a = \chi_\Omega$ for some compact set $\Omega \subseteq \mathbb{R}^{2d}$ and $\varphi_1 = \varphi_2$, then $A_a^{\varphi_1,\varphi_2}$ is interpreted as the part of f that "lives on the set Ω" in the time-frequency plane. This is why $A_a^{\varphi_1,\varphi_2}$ is called "localization operator". These operators were introduced in 1988 by Daubechies [12] to *localize* a signal in the time-frequency plane and to extract its time-frequency features. Moreover, besides their natural

Date: October 3, 2005.

2000 *Mathematics Subject Classification.* Primary 47G30, 42B35; Secondary 35S99, 46E35.

Key words and phrases. Localization operator, modulation space, Kohn-Nirenberg operators, multilinear pseudodifferential operators, short-time Fourier transform.

* Research partially supported by an Erwin Schrödinger Junior Fellowship.

† Research partially supported by NSF grant DMS-0139261.

use in signal processing, these operators have been studied in numbers of other contexts, see, e.g., [3, 14, 20, 28]. A special case of localization operator occurs when $\varphi_1(t) = \varphi_2(t) = e^{-\pi t^2}$, in which case $A_a = A_a^{\varphi_1,\varphi_2}$ is the classical Anti-Wick operator and the mapping $a \to A_a^{\varphi_1,\varphi_2}$ is interpreted as a quantization rule [3, 24, 28].

A basic question about these operators is to find "minimal" conditions on the symbol a that guarantee the boundedness of $A_a^{\varphi_1,\varphi_2}$ on L^2 and on other Banach spaces such as the Potential spaces or the modulation spaces [4, 6, 7, 26]. It has been recently shown that the optimal results as far as boundedness of the operators is concerned, are obtained in the context of time-frequency analysis [7, 8]. We refer to [6, 7, 13, 22, 23, 26], for a sample of works which use time frequency analysis techniques in the study of localization operators. The boundedness is usually achieved by rewriting the localization operator in terms of a pseudodifferential operator: the Weyl transform. More precisely, recall that for any function $f, g \in \mathcal{S}(\mathbb{R}^d)$, the cross-Wigner distribution of f and g is defined as

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x+\tfrac{t}{2})\, \overline{g(x-\tfrac{t}{2})}\, e^{-2\pi i\omega t}\, dt.$$

Then, $A_a^{\varphi_1,\varphi_2} = L_{a*W(\varphi_2,\varphi_1)}$, where $L_{a*W(\varphi_2,\varphi_1)}$ is the Weyl transform with symbol $\sigma = a * W(\varphi_2, \varphi_1)$. More specifically, for $f, g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\langle L_{a*W(\varphi_2,\varphi_1)} f, g\rangle = \langle a * W(\varphi_2,\varphi_1), W(g,f)\rangle, \tag{4}$$

see [6, 7, 14, 20]. In this setting, it is natural to study the boundedness properties of $A_a^{\varphi_1,\varphi_2}$ in the realm of modulation spaces (to be defined later) which are often "the right setting" for the study of pseudodifferential operators [16]. In particular, assuming that the symbol of the localization operator satisfies some "weak" time-frequency concentration conditions i.e., a belongs to the largest modulation spaces $M^\infty(\mathbb{R}^{2d})$, and provided that the windows φ_1, φ_2 are regular enough, it can be shown that the symbol $a * W(\varphi_2, \varphi_1)$ of the corresponding Weyl transform is in the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$ ([7], see also [26]). Hence boundedness result of $A_a^{\varphi_1,\varphi_2}$ follows from that of $L_{a*W(\varphi_2,\varphi_1)}$ [16]. Moreover, the condition that the symbol a of the time-frequency localization operator belongs to the modulation space $M^\infty(\mathbb{R}^{2d})$ is "optimal" in an appropriate sense, see [8]. We refer to [18, 19, 10, 11, 25] for more on pseudodifferential operators on modulation spaces.

However, to get any "nice" boundedness results for a localization operator, it is not necessary to write it in the Weyl form, but in any *suitable* pseudodifferential form. Here we are interested in the *Kohn-Nirenberg* form of a pseudodifferential operator, informally given by

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega) e^{2\pi i x\omega}\, d\omega, \quad f \in \mathcal{S}(\mathbb{R}^d), \tag{5}$$

where σ is a measurable function, or even a tempered distribution on $\mathbb{R}^{2d}$.

If we define the *rotation* operator $\mathcal{U}$ acting on a function F on $\mathbb{R}^{2d}$ by

$$\mathcal{U}F(x,\omega) = F(\omega, -x), \quad \forall\, (x,\omega) \in \mathbb{R}^{2d}, \tag{6}$$

then we shall show the identity (Proposition 2.1):

$$A_a^{\varphi_1,\varphi_2} = T_\sigma, \tag{7}$$

with the Kohn-Nirenberg symbol σ given by

$$\sigma = a * \mathcal{UF}(V_{\varphi_1}\varphi_2), \tag{8}$$

and where $\mathcal{F}$ stands for the Fourier transform. We notice that relation (7) expresses a localization operator in terms of pseudodifferential operator by using the STFT of the windows φ_1, φ_2, instead of their cross-Wigner distribution as in (4). An important property of the modulation spaces is that they are invariant under the operator which transforms a Kohn-Nirenberg symbol to a Weyl symbol [16, Lemma 2.1]. Consequently, the relation given by (8) cannot be used to improve the boundedness results obtained in [7] using (4). However, it turns out to be very useful when one handles Kohn-Nirenberg instead of Weyl operators. Moreover, it is the starting point to multilinear pseudodifferential operators [1, 2, 9, 17, 21].

In this paper, we introduce a notion of *multilinear localization operator*, that not only generalizes the linear case, but also yields a subclass of multilinear pseudodifferential operators. Intuitively, one can think of localizing m-fold products of functions. For sake of clarity, we first restrict our attention to the bilinear case, and show how the construction arises naturally from the framework of reproducing formulae and linear localization operators. The general case will be treated in the next section.

Bilinear localization operators. Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$, then the tensor product $(f_1 \otimes f_2)(x_1, x_2) = f_1(x_1)f_2(x_2)$ is a function in $\mathcal{S}(\mathbb{R}^{2d})$. Given four window functions $\varphi_i \in \mathcal{S}(\mathbb{R}^d)$, $i = 1, \dots, 4$, with $\langle\varphi_1, \varphi_3\rangle = \langle\varphi_2, \varphi_4\rangle = 1$, the usual reproducing formula for the functions f_1, f_2 (see, e.g., [15, Corollary 3.2.3]) reads as follows:

$$f_1 = \int_{\mathbb{R}^{2d}} V_{\varphi_1} f_1(z_1, \zeta_1)\, M_{\zeta_1} T_{z_1} \varphi_3 \, d\zeta_1 \, dz_1. \tag{9}$$

$$f_2 = \int_{\mathbb{R}^{2d}} V_{\varphi_2} f_2(z_2, \zeta_2)\, M_{\zeta_2} T_{z_2} \varphi_4 \, d\zeta_2 \, dz_2. \tag{10}$$

The product of both sides of equalities (9) and (10) yields

$$\begin{aligned}(f_1 \otimes f_2)(x_1, x_2) &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} V_{\varphi_1} f_1(z_1, \zeta_1) V_{\varphi_2} f_2(z_2, \zeta_2) M_{\zeta_1} T_{z_1} \varphi_3(x_1) M_{\zeta_2} T_{z_2} \varphi_4(x_2) d\zeta dz \\ &= \int_{\mathbb{R}^{4d}} V_{\varphi_1\otimes\varphi_2}(f_1 \otimes f_2)(z, \zeta)\, M_\zeta T_z(\varphi_3 \otimes \varphi_4)(x_1, x_2)\, d\zeta\, dz\end{aligned}$$

with $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

The previous reproducing formula for the function $f_1 \otimes f_2$ can be localized in the time-frequency plane yielding a localization operator $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ with symbol a (defined on $\mathbb{R}^{4d}$) and windows $\varphi_1 \otimes \varphi_2$, $\varphi_3 \otimes \varphi_4$. In particular, the action of the operator on the function $f_1 \otimes f_2$ is given by

$$A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}(f_1 \otimes f_2)(x_1, x_2)$$
$$= \int_{\mathbb{R}^{4d}} a(z,\zeta)V_{\varphi_1}f_1(z_1,\zeta_1)V_{\varphi_2}f_2(z_2,\zeta_2)M_{\zeta_1}T_{z_1}\varphi_3(x_1)M_{\zeta_2}T_{z_2}\varphi_4(x_2)dzd\zeta.$$

For any symbol $a \in \mathcal{S}'(\mathbb{R}^{4d})$, and window functions φ_j on $\mathcal{S}(\mathbb{R}^d)$, the operator $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ can be seen as a bilinear mapping from the 2-fold product of Schwartz spaces $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into the space $\mathcal{S}'(\mathbb{R}^{2d})$ of tempered distributions. Moreover, if we restrict now our attention to a smoother symbol $a \in \mathcal{S}(\mathbb{R}^{4d})$, we obtain a multilinear mapping from $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$.

In this paper, we are interested in the boundedness properties of the trace of $A_a^{\varphi_1\otimes\varphi_2,\varphi_3\otimes\varphi_4}$ on the diagonal $x_1 = x_2$. This restriction leads to a new kind of localization operator.

Definition 1.1. *Let $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\overrightarrow{f} = (f_1, f_2)$. Given a symbol $a \in \mathcal{S}'(\mathbb{R}^{4d})$ and window functions $\varphi_i \in \mathcal{S}(\mathbb{R}^d)$, with $i = 1, \dots, 4$, the* bilinear localization operator *A_a is given by*

(11)
$$A_a(\overrightarrow{f})(x) = \int_{\mathbb{R}^{4d}} a(z,\zeta)\, V_{\varphi_1}f_1(z_1,\zeta_1)\, V_{\varphi_2}f_2(z_2,\zeta_2)\, M_{\zeta_1}T_{z_1}\varphi_3(x)\, M_{\zeta_2}T_{z_2}\varphi_4(x)\, dz\, d\zeta,$$

where $x \in \mathbb{R}^d$.

Notice that if the symbol $a \in \mathcal{S}'(\mathbb{R}^{4d})$ then the corresponding operator A_a maps $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$.

In order to give a weak definition of the bilinear localization operator A_a, we first introduce the following time-frequency representation. For $\varphi_3, \varphi_4 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, we define $\mathcal{V}_{\varphi_3,\varphi_4}$ by

$$\mathcal{V}_{\varphi_3,\varphi_4}g(z,\zeta) = \int_{\mathbb{R}^d} g(t)\, \overline{M_{\zeta_1}T_{z_1}\varphi_3(t)M_{\zeta_2}T_{z_2}\varphi_4(t)}\, dt, \quad g \in \mathcal{S}(\mathbb{R}^d). \tag{12}$$

Thus, for $\overrightarrow{f} = (f_1, f_2) \in \mathcal{S} \times \mathcal{S}$ and $g \in \mathcal{S}$, the weak definition of (11) is given by

$$\langle A_a(\overrightarrow{f}), g\rangle = \langle a, \overline{V_{\varphi_1\otimes\varphi_2}(f_1 \otimes f_2)}\mathcal{V}_{\varphi_3,\varphi_4}g\rangle. \tag{13}$$

One of our goals in this paper is to understand the boundedness properties of the bilinear localization operators (and their multilinear analogue) on products of modulations spaces. To this end, we will need the representation of these operators as bilinear (or, in general, as multilinear) pseudodifferential operators. This will allow us to use known results on boundedness of multilinear pseudodifferential operators on products of modulation spaces ([1, 2]) to obtain boundedness results of these multilinear localization operators. As a byproduct of these boundedness properties, we construct a class of symbols for (bounded) multilinear pseudodifferential operators starting from a multilinear localization operator. In particular, our results provide a simple way to construct such symbols starting from "almost" any tempered distribution (the symbol of the associated localization operator), by a simple convolution with a function in the modulation space $M^1(\mathbb{R}^{2md})$. Moreover, convolution relation of modulation spaces allows us to play with the smoothness of the windows and the roughness of the symbol (Corollary 5.4). Finally, we show that not only the condition on the localization symbol a is sufficient for the boundedness of the corresponding operator on products of modulation spaces, but this condition is also necessary.

Our paper is organized as follows. In section 2 the time-frequency tools used in our framework are reviewed. The connection between localization and Kohn-Nirenberg operators (7) is showed for both the linear and the bilinear case. In section 3 the definition of the multilinear localization operators is given together with its connection with multilinear pseudodifferential operators. In section 4 we define the modulation spaces and present some of their basic properties. Moreover, we give in this section a seemingly unrelated but important characterization of the space M^1 together with similar results for other modulation spaces. In section 5 we state and prove our first main result which gives a sufficient condition on the symbol a of the multilinear localization operator to be bounded on products of modulation spaces. In section 6, we show that the sufficient condition of section 5 is also necessary. Finally, section 7 is devoted to present specific examples of symbols that yield bounded multilinear pseudodifferential operators on products of modulation spaces.

Notation. We define $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$. We use the brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\, \overline{g(t)}\, dt$ on $L^2(\mathbb{R}^d)$. The space of smooth functions with compact support on $\mathbb{R}^d$ is denoted by $\mathcal{D}(\mathbb{R}^d)$. The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, and its dual, the space of tempered distributions, by $\mathcal{S}'(\mathbb{R}^d)$. We denote by $L^0(\mathbb{R}^{2d})$ the space of all $f \in L^\infty(\mathbb{R}^{2d})$ that vanish at infinity, i.e., for all $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that

$$|f(z)| < \varepsilon, \qquad \text{for a. a. } |z| > R\,.$$

The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)\, e^{-2\pi i t\omega}\, dt$.

Throughout the paper, we shall use the notation $A \lesssim B$ to indicate $A \le cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means that $c^{-1}A \le B \le cA$ for some $c \ge 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the linear space B_1 into B_2.

2. Kohn-Nirenberg representation of bilinear localization operators

We first list some crucial properties of the STFT whose proofs are given in [15, Chapter 3].

Lemma 2.1. *Let $f, g \in L^2(\mathbb{R}^d)$, then we have*
(i) (STFT of the Fourier transform)

$$V_g f(x, \omega) = e^{-2\pi i x\omega}\, V_{\hat{g}} \hat{f}(\omega, -x). \tag{14}$$

(ii)(Fourier transform of the STFT),

$$\mathcal{F}(V_g f)(\omega, x) = e^{2\pi i x\omega}\, f(-x)\, \overline{\hat{g}(\omega)}. \tag{15}$$

To investigate the boundedness properties of the multilinear localization operators, we will need to compute the STFT of a STFT. This is done in the following result [7, Lemma 2.2].

Lemma 2.2. *Fix a nonzero $\gamma \in \mathcal{S}(\mathbb{R}^d)$ and let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Set $\Psi = V_\gamma \gamma \in \mathcal{S}(\mathbb{R}^{2d})$. Then the STFT of $V_g f$ with respect to the window Ψ is given by*

$$V_\Psi(V_g f)(z,\zeta) = e^{-2\pi i z_2 \zeta_2}\, \overline{V_\gamma g(-z_1 - \zeta_2, \zeta_1)}\, V_\gamma f(-\zeta_2, z_2 + \zeta_1), \tag{16}$$

where $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

A linear localization operator can be rewritten as a Kohn-Nirenberg pseudodifferential operator. More precisely, we have:

Proposition 2.1 (**Connection between Kohn-Nirenberg and localization operators**). *If $a \in \mathcal{S}'(\mathbb{R}^{2d})$, $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is the Kohn-Nirenberg operator T_σ, given by* (5), *where the Kohn-Nirenberg symbol σ is given by* (8), *i.e.,*

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega) \hat{f}(\omega) e^{2\pi i x\omega}\, d\omega,$$

with

$$\sigma = a * \mathcal{U}\mathcal{F}(V_{\varphi_1}\varphi_2).$$

Proof. Let $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $a \in \mathcal{S}(\mathbb{R}^{2d})$, then the integral defining $A_a^{\varphi_1,\varphi_2}$ in (3) is absolutely convergent.

We first use (14) in (3) and we get

$$\begin{aligned}
A_a^{\varphi_1,\varphi_2} f(t) &= \int_{\mathbb{R}^{2d}} a(x,\omega)\, V_{\varphi_1} f(x,\omega)\, M_\omega T_x \varphi_2(t)\, dx\, d\omega \\
&= \int_{\mathbb{R}^{2d}} a(x,\omega) e^{-2\pi i x\omega} \left(\int_{\mathbb{R}^d} \hat{f}(y)\, e^{2\pi i x y}\, \overline{\hat{\varphi}_1(y-\omega)}\, dy \right) e^{2\pi i \omega t} \varphi_2(t-x) dx\, d\omega \\
&= \int_{\mathbb{R}^d} \hat{f}(y) \left(\int_{\mathbb{R}^{2d}} a(x,\omega) e^{2\pi i [(t-x)\omega + xy]} \right) \overline{\hat{\varphi}_1(y-\omega)}\, \varphi_2(t-x)\, dx d\omega\, dy.
\end{aligned}$$

Changing variables in the last equation and using (15) yield

$$\begin{aligned}
A_a^{\varphi_1,\varphi_2} f(t) &= \int_{\mathbb{R}^d} \hat{f}(y)\, e^{2\pi i t y} \left(\int_{\mathbb{R}^{2d}} a(t-u, y-v)\, e^{-2\pi i u v}\, \overline{\hat{\varphi}_1(v)}\, \varphi_2(u)\, du\, dv \right) dy \\
&= \int_{\mathbb{R}^d} \hat{f}(y)\, e^{2\pi i t y} \left(\int_{\mathbb{R}^{2d}} a(t-u, y-v)\, \mathcal{F}(V_{\varphi_1}\varphi_2)(v,-u)\, du\, dv \right) dy \\
&= \int_{\mathbb{R}^d} \hat{f}(y)\, e^{2\pi i t y}\, (a * \mathcal{U}\mathcal{F}(V_{\varphi_1}\varphi_2))(t,y)\, dy
\end{aligned}$$

where $\mathcal{U}$ is the rotation defined in (6). Using a standard approximation argument, we then obtain the equivalence of the operators for all $a \in \mathcal{S}'(\mathbb{R}^{2d})$. ∎

Corollary 2.2. *The bilinear localization operator A_a defined in* (11) *is a bilinear pseudodifferential operator with symbol*

$$\sigma(x,u,v) = a * \Phi(x,x,u,v), \quad x,u,v \in \mathbb{R}^d, \tag{17}$$

where

$$\Phi(x,y,u,v) = \mathcal{UF}(V_{\varphi_1}\varphi_3)(x,u)\,\mathcal{UF}(V_{\varphi_2}\varphi_4)(y,v). \tag{18}$$

Proof. The same arguments as in the proof of Proposition 2.1 applied to expression (11) yield

$$A_a(\overrightarrow{f})(x) = T_\sigma(\overrightarrow{f})(x) = \int_{\mathbb{R}^{2d}} \sigma(x,u,v)\,\hat{f}_1(u)\,\hat{f}_2(v)\,e^{2\pi i x(u+v)}\,du\,dv, \tag{19}$$

with σ given by (17) and Φ by (18). ∎

Consequently, the bilinear localization operator A_a has the same boundedness properties as the bilinear pseudodifferential operator T_σ where the symbols a and σ are related by (17). Therefore, we can now appeal to the results of [1, 2] to study boundedness properties of A_a on products of modulation spaces.

3. Multilinear localization operators

Without any further work — just some extra notations — it is straightforward to generalize the above approach to define a multilinear localization operator, and relate it to a multilinear pseudodifferential operator. Thus we are led to make the following definition. Fix $m \in \mathbb{N}$. For every symbol $a \in \mathcal{S}'(\mathbb{R}^{2md})$ and windows φ_i, $i = 1,\ldots,2m$ in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, we introduce the analysis, synthesis window functions $\phi_1, \phi_2 : \mathbb{R}^{md} \to \mathbb{C}$, defined respectively as tensor products of the m analysis, and m synthesis windows, i.e.,

$$\phi_1(t_1,\ldots,t_m) := \varphi_1(t_1)\cdots\varphi_m(t_m), \tag{20}$$

and

$$\phi_2(t_1,\ldots,t_m) := \varphi_{m+1}(t_1)\cdots\varphi_{2m}(t_m). \tag{21}$$

Let $\mathcal{R}$ be the trace mapping that assigns to each function defined on $\mathbb{R}^{md}$ a function defined on $\mathbb{R}^d$ by the following formula:

$$\mathcal{R}F(t) := F_{|\{t_1=t_2=\cdots=t_m=t\}}(t_1,\ldots,t_m) = F(t,\ldots,t), \tag{22}$$

for any $t \in \mathbb{R}^d$.

Definition 3.1. *The multilinear localization operator A_a with symbol $a \in \mathcal{S}'(\mathbb{R}^{2md})$ and windows $\varphi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1,\ldots,2m$ is the multilinear mapping defined on the m-fold product of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ by*

$$\begin{aligned} A_a(\overrightarrow{f})(x) :&= \int_{\mathbb{R}^{2md}} a(z,\zeta)\,V_{\bigotimes_{j=1}^m \varphi_j}\left(\otimes_{j=1}^m f_j\right)(z,\zeta)\,\prod_{j=1}^m M_{\zeta_j}T_{z_j}\varphi_{m+j}(x)\,d\zeta\,dz \\ &= \int_{\mathbb{R}^{2md}} a(z,\zeta)\,V_{\phi_1}\left(\otimes_{j=1}^m f_j\right)(z,\zeta)\,\mathcal{R}M_\zeta T_z\phi_2(x)\,d\zeta\,dz, \end{aligned} \tag{23}$$

where $(z,\zeta)\in\mathbb{R}^{md}\times\mathbb{R}^{md}$, $x\in\mathbb{R}^d$, *and* $\overrightarrow{f}=(f_1,\dots,f_m)\in\mathcal{S}(\mathbb{R}^d)\times\cdots\times\mathcal{S}(\mathbb{R}^d)$.

If $m=1$ we are back to the linear localization operator $A_a^{\varphi_1,\varphi_2}$, whereas the case $m=2$ gives the bilinear localization operator introduced in (11).

Analogously, the weak definition (13), for an arbitrary $m\in\mathbb{N}$, is now given by

$$\langle A_a(\overrightarrow{f}),g\rangle=\langle a,\overline{\prod_{j=1}^m}\,(V_{\varphi_j}f_j)\mathcal{V}_{(\varphi_l)_{l=m+1}^{2m}}g\rangle=\langle a,\overline{V_{\phi_1}(\otimes_{j=1}^m f_j)}\mathcal{V}_{\phi_2}g\rangle \tag{24}$$

where

$$\mathcal{V}_{\phi_2}g(z,\zeta)=\int_{\mathbb{R}^d}g(t)\,\overline{\mathcal{R}M_\zeta T_z\phi_2(t)}\,dt,\qquad (z,\zeta)\in\mathbb{R}^{4d}. \tag{25}$$

We aim at studying the boundedness of A_a on products of modulation spaces by rewriting it as a multilinear pseudodifferential operator T_σ with symbol σ as pointed out earlier for the bilinear case. In the general case we have:

Proposition 3.2. *Let* $a\in\mathcal{S}'(\mathbb{R}^{2md})$ *and* $\varphi_j\in\mathcal{S}(\mathbb{R}^d)$, $j=1,\dots,2m$. *Then the multilinear localization operator* A_a *is the multilinear pseudodifferential operator* T_σ *defined on* $\overrightarrow{f}=(f_j)_{j=1}^m\in\mathcal{S}(\mathbb{R}^d)\times\ldots\times\mathcal{S}(\mathbb{R}^d)$ *by*

$$A_a(\overrightarrow{f})(x)=T_\sigma(\overrightarrow{f})(x):=\int_{\mathbb{R}^{md}}\sigma(x,\xi)\prod_{j=1}^m\hat{f}_j(\xi_j)\,e^{2\pi i x\sum_{j=1}^m\xi_j}\,d\xi \tag{26}$$

where

$$\sigma(x,\xi)=a*\Phi(X,\xi) \tag{27}$$

with $x\in\mathbb{R}^d$, *and* $X=(x,\dots,x)$, $\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^{md}$. *Moreover, the function* Φ *is given by*

$$\Phi(z,\xi)=\prod_{j=1}^m\mathcal{U}\mathcal{F}(V_{\varphi_j}\varphi_{j+m})(z_j,\xi_j), \tag{28}$$

for any $z=(z_1,\dots,z_m)$, $\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^{md}$.

Proof. It is carried out by the same arguments as in Proposition 2.1. ■

4. Modulation spaces

Given $1\le p,q\le\infty$, and given a fixed, non-zero window function $g\in\mathcal{S}(\mathbb{R}^d)$, the modulation space $M^{p,q}=M^{p,q}(\mathbb{R}^d)$ is the space of all distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ for which the following norm is finite:

$$\|f\|_{M^{p,q}}=\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,y)|^p\,dx\right)^{q/p}dy\right)^{1/q}=\|V_gf\|_{L^{p,q}},$$

with the usual modifications if p and/or q are infinite. When $p=q$, we write M^p for $M^{p,p}$. Note that for $p=q=2$, $M^2=L^2$.

Remark 4.1. *The definition of modulation spaces is independent of the choice of the window g in the sense of equivalent norms. It is important to note that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}$ for $1 \le p,q < \infty$. One can also show that the dual of $M^{p,q}$ is $M^{p',q'}$, where $1 \le p,q < \infty$ and $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$.*

We refer to [15] *and the references therein for more details on the modulation spaces.*

Remark 4.2. *Most of our results will be stated even for the cases $p=\infty$ and/or $q=\infty$. To properly deal with duality —which will be used constantly in our proofs— in these cases, we define the related modulation spaces $M^{0,q}, M^{p,0}$, and $M^{0,0}=M^0$. Namely, we have*

$$\begin{aligned} M^{0,q} &= \{f \in M^{\infty,q} : V_g f \in L^0\}, \quad q<\infty, \\ M^{p,0} &= \{f \in M^{p,\infty} : V_g f \in L^0\}, \quad p<\infty, \\ M^{0,0} &= \{f \in M^{\infty} : V_g f \in L^0\}, \end{aligned}$$

equipped with the norms of $M^{\infty,q}, M^{p,\infty}$, and $M^{\infty,\infty}=M^\infty$ respectively. We refer to [1, Lemma 2.2] *for more details about this spaces.*

We recall the convolution property for modulation spaces [7, Proposition 2.4] (see also [26]), that will be used in the proof of one of our main results.

Proposition 4.3. *Let $1 \le p,q,r,s,t \le \infty$. If*

$$\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}, \qquad \textit{and} \qquad \frac{1}{t}+\frac{1}{t'}=1\,,$$

then

$$M^{p,st}(\mathbb{R}^d) * M^{q,st'}(\mathbb{R}^d) \hookrightarrow M^{r,s}(\mathbb{R}^d), \tag{29}$$

with norm inequality $\|f*h\|_{M^{r,s}} \lesssim \|f\|_{M^{p,st}}\|h\|_{M^{q,st'}}$.

The following characterization of the space M^1, which represents a refinement of [15, Proposition 12.1.2], will be crucial in proving our first main result. We refer to [7, Proposition 2.5] for a similar characterziation involving the cross-Wigner distribution.

Proposition 4.4. *The following conditions are equivalent:*
(i) $f \in M^1$.
(ii) $f \in L^2$ and for one/all $g \in M^1$ we have $V_g f \in L^1$.
(iii) $f \in L^2$ and for one/all $g \in M^1$ we have $V_g f \in M^1$, with

$$\|V_g f\|_{M^1(\mathbb{R}^{2d})} \asymp \|g\|_{M^1(\mathbb{R}^d)}\,\|f\|_{M^1(\mathbb{R}^d)}. \tag{30}$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is proved in [15, Proposition 12.1.2].
$(i) \Rightarrow (iii)$.
The STFT of $V_g f$ with respect to the window $\Psi = V_\gamma \gamma$, $\gamma \in \mathcal{S}$ is given by (16). We compute its L^1-norm, i.e., for $z = (z_1, z_2) \in \mathbb{R}^{2d}$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$,

$$\begin{aligned}
\int_{\mathbb{R}^{4d}} |V_\Psi(V_g f)(z,\zeta)| \, dz \, d\zeta &= \int_{\mathbb{R}^{4d}} |V_\gamma g(-z_1 - \zeta_2, \zeta_1)| \, |V_\gamma f(-\zeta_2, z_2 + \zeta_1)| \, dz_1 \, dz_2 \, d\zeta_1 \, d\zeta_2 \\
&= \int_{\mathbb{R}^{4d}} |V_\gamma g(w_1, u_1)| \, |V_\gamma f(u_2, w_2)| \, du_1 \, du_2 \, dw_1 \, dw_2 \\
&= \int_{\mathbb{R}^{d}} |V_\gamma g(w_1, u_1)| \, du_1 \, dw_1 \int_{\mathbb{R}^{d}} |V_\gamma f(u_2, w_2)| \, du_2 \, dw_2 \\
&\asymp \|g\|_{M^1} \|f\|_{M^1} < \infty
\end{aligned}$$

where in the previous computations we have performed some obvious changes of variables. Thus condition (i) implies (iii).
$(iii) \Rightarrow (i)$.
Since the space M^1 is embedded in L^1 [15, Proposition 12.1.4], the converse is achieved by using the implication $(ii) \Rightarrow (i)$. ■

More generally, if $1 \leq p \leq \infty$, we have the following characterization of $M^p(\mathbb{R}^d)$ by the STFT; see [26, Theorem 4.1] for a similar characterization using the cross-Wigner distribution.

Corollary 4.5. *Let $g \in M^1$ and $1 \leq p \leq \infty$ be given. Then $f \in M^p(\mathbb{R}^d)$ if and only if $V_g f \in M^p(\mathbb{R}^{2d})$ with*

$$\|V_g f\|_{M^p} \asymp \|g\|_{M^p} \|f\|_{M^p}. \tag{31}$$

Proof. Notice that $M^1 \subset M^p$ with $\|g\|_{M^p} \leq \|g\|_{M^1} < \infty$, for every $1 \leq p \leq \infty$ [15, Corollary 12.1.10]. The rest of the proof is similar to the one of the previous proposition, and thus is omitted. ■

Observe that the equivalence (31) fails for the space $M^{p,q}$, when $p \neq q$. However, a similar statement holds involving a Wiener amalgam space. More precisely, let $\mathcal{F}L^p$ denote the image of L^p under the Fourier transform, i.e.,

$$\mathcal{F}L^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d), \text{ such that } \exists\, h \in L^p \text{ with } \hat{h} = f\}$$

equipped with the norm $\|f\|_{\mathcal{F}L^p} = \|h\|_p$, with $\hat{h} = f$.

Let $g \in \mathcal{D}(\mathbb{R}^d)$. The Wiener amalgam space $W(\mathcal{F}L^p, L^q)$ is the space of all functions or tempered distributions for which the norm

$$\|f\|_{W(\mathcal{F}L^p, L^q)} = \left(\int_{\mathbb{R}^{2d}} \|f \cdot T_z\, g\|_{\mathcal{F}L^p}^q \, dz \right)^{1/q} \tag{32}$$

is finite. It can be shown that different choices of $g \in \mathcal{D}$ generate the same space and yield equivalent norms. Note that for $p = q$, $W(\mathcal{F}L^p, L^q) = M^p$.

In this setting Corollary 4.5 becomes:

Corollary 4.6. *Let $g \in \mathcal{D}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$ be given. Then $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ if and only if $V_g f \in M^p(\mathbb{R}^{2d})$ with*

$$\|V_g f\|_{M^{p,q}} \asymp \|g\|_{M^1} \|f\|_{W(\mathcal{F}L^p, L^q)}. \tag{33}$$

Proof. Let $f \in W(\mathcal{F}L^p, L^q)$, and $g \in \mathcal{D}(\mathbb{R}^d)$. We estimate the $M^{p,q}$-norm of $V_g f$ as follows:

$$\begin{aligned}
\|V_g f\|_{M^{p,q}} &\asymp \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\Psi (V_g f)(z, \zeta)|^p \, dz \right)^{q/p} d\zeta \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\gamma g(-z_1 - \zeta_2, \zeta_1)|^p | \, V_\gamma f(-\zeta_2, z_2 + \zeta_1)|^p \right.\right. \\
&\qquad\qquad \left. dz_1 \, dz_2)^{q/p} \, d\zeta_1 \, d\zeta_2 \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} |V_\gamma g(w_1, u_1)|^p \, dw_1 \right)^{q/p} du_1 \right)^{1/q} \\
&\qquad \cdot \left(\int_{\mathbb{R}^{2d}} \left(\int_{\mathbb{R}^{2d}} V_\gamma f(u_2, w_2)|^p \, dw_2 \right)^{q/p} du_2 \right)^{1/q} \\
&\asymp \|g\|_{M^{p,q}} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\gamma f(u_2, w_2)|^p \, dw_2 \right)^{q/p} du_2 \right)^{1/q} \\
&\asymp \|g\|_{M^{p,q}} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |(f \cdot T_{u_2} \overline{\gamma})\widehat{\ }(w_2)|^p \, dw_2 \right)^{q/p} du_2 \right)^{1/q} \\
&\asymp \|g\|_{M^{p,q}} \left(\int_{\mathbb{R}^d} \|f \cdot T_{u_2} \overline{\gamma}\|_{\mathcal{F}L^p}^q \, du_2 \right)^{1/q} \\
&\asymp \|g\|_{M^{p,q}} \|f\|_{W(\mathcal{F}L^p, L^q)}.
\end{aligned}$$

Thus the proof of the lemma is completed. ■

5. Sufficient conditions for boundedness of multilinear localization operators

The tools from the previous sections enable us to obtain boundedness properties for multilinear localization operators. As pointed out earlier, this result will be achieved by passing to the associated multilinear pseudodifferential operators, and using the following result [1, Theorem 3.1].

Theorem 5.1. *If $\sigma \in M^{\infty,1}(\mathbb{R}^{(m+1)d})$, then the m-linear pseudodifferential operator T_σ defined by (26) extends to a bounded operator from $M^{p_1,q_1}(\mathbb{R}^d) \times \cdots \times M^{p_m,q_m}(\mathbb{R}^d)$*

into $M^{p_0,q_0}(\mathbb{R}^d)$*, when*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p_0},\ \frac{1}{q_1}+\cdots+\frac{1}{q_m}=m-1+\frac{1}{q_0},$$

and $1\leq p_j,q_j\leq\infty$*, for* $j=0,\dots,m$*. Moreover, we have the following norm estimate*

$$\|T_\sigma\|\leq C\,\|\sigma\|_{M^{\infty,1}}, \tag{34}$$

where the positive constant C *is independent of* σ*.*

Next, we prove a proposition that gives a modulation norm estimate on the symbol σ of the multilinear pseudodifferential operator T_σ in terms of a modulation norm estimate of the symbol a of the corresponding multilinear localization operator A_a.

Proposition 5.2. *Let* $a\in M^\infty(\mathbb{R}^{2md})$*,* $\varphi_j\in M^1(\mathbb{R}^d)$*,* $j=1,\dots,2m$*. Let* $x\in\mathbb{R}^d$ *and* $\xi=(\xi_1,\dots,\xi_m)\in\mathbb{R}^{md}$*, and denote* $X=(x,x,\dots,x)\in\mathbb{R}^{md}$*. Then we have*
(i) $\Phi\in M^1(\mathbb{R}^{2md})$*, where* Φ *is the function defined in* (28)*.*
(ii) The symbol $\sigma(x,\xi)=a*\Phi(X,\xi)$ *(see* (27)*), is in* $M^{\infty,1}(\mathbb{R}^{(m+1)d})$*, with the following norm estimate:*

$$\|\sigma\|_{M^{\infty,1}(\mathbb{R}^{(m+1)d})}\leq C\,\|a\|_{M^\infty(\mathbb{R}^{2md})}\prod_{j=1}^{2m}\|\varphi_j\|_{M^1(\mathbb{R}^d)};$$

where the positive constant C *is independent of* a *and of* $\varphi_j, j=1,\dots,2m$*.*

Proof. (i) We claim that the function

$$\Phi(z,\xi)=\prod_{j=1}^{m}\mathcal{U}\mathcal{F}(V_{\varphi_j}\varphi_{j+m})(z_j,\xi_j)$$

belongs to $M^1(\mathbb{R}^{2md})$. This is achieved by the following arguments. First, simple calculations show that the modulation space M^1 is invariant under the rotation operator $\mathcal{U}$. Secondly, since $M^1(\mathbb{R}^{2md})$ is invariant under the Fourier transform [15, Theorem 11.3.5], it will be enough to show that, for each j, $V_{\varphi_j}\varphi_{j+m}\in M^1(\mathbb{R}^{2d})$. But this has already been proven in (30).
(ii) Using the convolution relation for modulation spaces (29), with $p=r=t=\infty$, and $s=q=t'=1$, we conclude that $a*\Phi\in M^{\infty,1}(\mathbb{R}^{2md})$ with

$$\|a*\Phi\|_{M^{\infty,1}(\mathbb{R}^{2md})}\leq C\,\|a\|_{M^\infty(\mathbb{R}^{2md})}\,\|\Phi\|_{M^1(\mathbb{R}^{2md})}\leq C\prod_{j=1}^{2m}\|\varphi_j\|_{M^1(\mathbb{R}^d)}\,\|a\|_{M^\infty(\mathbb{R}^{2md})}.$$

Now, let R_1 be any linear invertible transformation on $\mathbb{R}^{md}$ (different from the identity) such that $R_1e_1=(1,1,\dots,1)$ where $e_1=(1,0,\dots,0)$ is the first unit vector in $\mathbb{R}^{md}$. Consider the linear (invertible) transformation R on $\mathbb{R}^{2md}$ defined by $R(z,\xi)=(R_1z,\xi)$ where $z,\xi\in\mathbb{R}^{md}$. R induces a mapping $\tilde{R}$ on functions defined on $\mathbb{R}^{2md}$ through $\tilde{R}F(z,\xi)=F(R(z,\xi))=F(R_1z,\xi)$. Denote by P the projection of $\mathbb{R}^{md}$ into the hyperplane $\{z_2=z_3=\ldots=z_m=0\}$, i.e., $P(z)=(z_1,0,\dots,0)$ for

$z=(z_1,\dots,z_m)\in\mathbb{R}^{md}$. Similarly, P induces a mapping $\tilde{P}$ on functions defined on $\mathbb{R}^{2md}$ by $\tilde{P}F(z,\xi)=F(P(z),\xi)$. It is easy to see that $\tilde{R}$ leaves invariant all the modulation spaces, since its action is merely a change of coordinates in the time-frequency plane. In particular, $M^{\infty,1}(\mathbb{R}^{2md})$ is invariant under $\tilde{R}$. Moreover, the action of $\tilde{P}$ on functions defined on $\mathbb{R}^{2md}$ is equivalent to restricting the functions to the hyperplane $\{z_2=z_3=\ldots=z_m=0\}$. It is shown in [27, Theorem 3.1.] that this restriction operator is bounded from $M^{p,1}(\mathbb{R}^{2md})$ into $M^{p,1}(\mathbb{R}^{(m+1)d})$ for all $1\le p\le\infty$. In particular, $\tilde{R}$ maps boundedely $M^{\infty,1}(\mathbb{R}^{2md})$ into $M^{\infty,1}(\mathbb{R}^{(m+1)d})$. But

$$\sigma(x,\xi)=a*\Phi(X,\xi)=\tilde{P}\circ\tilde{R}(a*\Phi)(X,\xi).$$

Thus $\sigma\in M^{\infty,1}(\mathbb{R}^{(m+1)d})$ with the norm estimate given in the lemma. ■

Remark 5.3. *If we want to use rougher windows φ_i, we need to consider a smoother symbol a to get a pseudodifferential symbol $\sigma\in M^{\infty,1}(\mathbb{R}^{(m+1)d})$. In particular, the following special case of Proposition 4.3 : $M^p*M^{p'}\hookrightarrow M^{\infty,1}$, with $1/p+1/p'=1$, and arguments similar to the one presented in the proof of the preceding proposition give the following result.*

Corollary 5.4. *Let $a\in M^p(\mathbb{R}^{2md})$, $\varphi_j\in M^{p'}(\mathbb{R}^d)$, $j=1,\dots,2m$, and $1\le p\le\infty$. Then:*
(i) $\Phi\in M^{p'}(\mathbb{R}^{2md})$.
(ii) $\sigma(x,\xi)\in M^{\infty,1}(\mathbb{R}^{(m+1)d})$, with

$$\|\sigma\|_{M^{\infty,1}(\mathbb{R}^{(m+1)d})}\le C\,\|a\|_{M^p(\mathbb{R}^{2md})}\prod_{j=1}^{2m}\|\varphi_j\|_{M^{p'}(\mathbb{R}^d)}.$$

Proposition 5.2 is the key ingredient to prove our first main result:

Theorem 5.5. *Let $m\in\mathbb{N}$, a symbol $a\in M^\infty(\mathbb{R}^{2md})$, and window functions $\varphi_j\in M^1(\mathbb{R}^d)$, $j=1,\dots,2m$, be given. Then the m-linear localization operator A_a defined by (23) extends to a bounded operator from $M^{p_1,q_1}(\mathbb{R}^d)\times\cdots\times M^{p_m,q_m}(\mathbb{R}^d)$ into $M^{p_0,q_0}(\mathbb{R}^d)$, when*

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p_0},\ \frac{1}{q_1}+\cdots+\frac{1}{q_m}=m-1+\frac{1}{q_0},$$

and $1\le p_j,q_j\le\infty$, for $j=0,\dots,m$. Moreover, we have the following norm estimate

$$\|A_a\|\le C\|a\|_{M^\infty(\mathbb{R}^{2md})}\prod_{i=1}^{2m}\|\varphi_i\|_{M^1(\mathbb{R}^d)}, \tag{35}$$

where the positive constant C is independent of a and of φ_j, $j=1,\dots,2m$.

Proof. By (26) it suffices to prove that the multilinear pseudodifferential operator $T_\sigma = A_a$ is bounded under the hypothesis of the theorem, where the symbol σ is related to a by (27). By Proposition 5.2, $a \in M^\infty(\mathbb{R}^{2md})$ implies that $\sigma \in M^{\infty,1}(\mathbb{R}^{(m+1)d})$. The result now follows from Theorem 5.1. ■

Making use of Corollary 5.4, a similar result can be achieved. Namely,

Corollary 5.6. *The same result stated in Theorem 5.5 is obtained under the following assumptions: $a \in M^p(\mathbb{R}^{2md})$, $\varphi_j \in M^{p'}(\mathbb{R}^d)$, $j = 1, \dots, 2m$, and $1 \le p \le \infty$.*

It is worth mentioning that Theorem 5.5 is a special case of the following more general result whose proof is almost identical to the previous one, and is based on [2, Theorem 1]. We leave it to the interested reader to fill in the details of the proof.

Theorem 5.7. *Let $m \in \mathbb{N}$ and $1 \le p \le \infty$ be given. Consider a symbol $a \in M^{p,\infty}(\mathbb{R}^{2md})$, and window functions $\varphi_j \in M^1(\mathbb{R}^d)$, $j = 1, \dots, 2m$. Then the m-linear localization operator A_a defined by (23) extends to a bounded operator from $M^{p_1,q_1}(\mathbb{R}^d) \times \dots \times M^{p_m,q_m}(\mathbb{R}^d)$ into $M^{p_0,q_0}(\mathbb{R}^d)$, when*

$$\frac{1}{p} + \frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p_0}, \quad \frac{m}{p} + \frac{1}{q_1} + \dots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0},$$

and $1 \le p_j, q_j \le \infty$, for $j = 0, \dots, m$. Moreover, we have the following norm estimate

$$\|A_a\| \le C \|a\|_{M^p(\mathbb{R}^{2md})} \prod_{i=1}^{2m} \|\varphi_i\|_{M^1(\mathbb{R}^d)}, \tag{36}$$

where the positive constant C is independent of a and of φ_j, $j = 1, \dots, 2m$.

Remark 5.8. *We point out that in the previous theorems one or more of the indices p_j or q_j can be infinite; in which case the results still hold with the corresponding modulation spaces replaced by the ones given in Remark 4.2.*

6. Necessary conditions for boundedness of multilinear localization operators

Here we give a (almost) converse of Theorem 5.5. This guarantees the optimality of the choice of modulation spaces in the framework of multilinear localization operators as is the case in the linear setting [8]. First we need the following preliminary result. Recall that for $\varphi_j \in \mathcal{S}(\mathbb{R}^d)$, $j = 1, \dots, 2m$, ϕ_1 and ϕ_2 are given respectively by (20), (21).

Lemma 6.1. *Let $f_l, g \in \mathcal{S}(\mathbb{R}^d)$, for $l = 1, \dots, m$, and consider the analysis, synthesis window functions ϕ_1, ϕ_2. Then the function Ψ on $\mathbb{R}^{md} \times \mathbb{R}^{md}$, given by*

$$\Psi(z, \zeta) := (\overline{V_{\phi_1}(\otimes_{l=1}^m f_l)} \mathcal{V}_{\phi_2} g)(z, \zeta), \tag{37}$$

belongs to $\mathcal{S}(\mathbb{R}^{2md})$.

Proof. Since the m-fold tensor product is a continuous mapping from $\mathcal{S}(\mathbb{R}^d)\times\cdots\times\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{md})$ and $V_g f\in\mathcal{S}(\mathbb{R}^{2d})$ when $f,g\in\mathcal{S}(\mathbb{R}^d)$ [15, Theorem 11.2.5], we have

$$\overline{V_{\phi_1}(\otimes_{l=1}^m f_l)}(z,\zeta)=\prod_{j=1}^m V_{\varphi_j}f_j(z_j,\zeta_j)\in\mathcal{S}(\mathbb{R}^{2md}).$$

It remains to check that $\mathcal{V}_{\phi_2}g(z,\zeta)\in\mathcal{S}(\mathbb{R}^{2md})$. Observe that, the tensor product $\phi_2\in\mathcal{S}(\mathbb{R}^{md})$, and the time-frequency shifts $M_\zeta T_z$ are continuous mappings on $\mathcal{S}(\mathbb{R}^{md})$ to itself [15, Theorem 11.3.5]. To complete the proof of the lemma, it only remains to show that $\mathcal{R}F\in\mathcal{S}(\mathbb{R}^d)$, whenever $F\in\mathcal{S}(\mathbb{R}^{md})$, and where $\mathcal{R}$ is given by (22). For sake of simplicity, we study the case $d=1$, and leave the general case $d>1$ to the interested reader. Since

$$\frac{d^k}{dx^k}(\mathcal{R}F)(x)=\sum_{j_1+\cdots+j_m=k}\frac{k!}{j_1!\ldots j_m!}(\partial^{j_1}\ldots\partial^{j_m}F)(x,\ldots,x),$$

we can write

$$\|x^l\frac{d^k}{dx^k}(\mathcal{R}F)(x)\|_{L^\infty(\mathbb{R})}\le\sum_{j_1+\cdots+j_m=k}\frac{k!}{j_1!\ldots j_m!}\||\cdot|^l(\partial^{j_1}\ldots\partial^{j_m}F)\|_{L^\infty(\mathbb{R}^m)}<\infty.$$

Hence $\Psi=(\overline{V_{\phi_1}(\otimes_{j=1}^m f_j)}\mathcal{V}_{\phi_2}g)\in\mathcal{S}(\mathbb{R}^{2md})$ as desired. ∎

We can now prove our second main result, which is essentially a converse to Theorem 5.5.

Theorem 6.1. *Let $m\in\mathbb{N}$, and $a\in\mathcal{S}'(\mathbb{R}^{2md})$ be given. Assume that:*
(i) the m-linear localization operator A_a is a bounded operator from $M^{p_1,q_1}(\mathbb{R}^d)\times\cdots\times M^{p_m,q_m}(\mathbb{R}^d)$ into $M^{p_0,q_0}(\mathbb{R}^d)$, where

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p_0},\ \frac{1}{q_1}+\cdots+\frac{1}{q_m}=m-1+\frac{1}{q_0},$$

and $1\le p_j,q_j\le\infty$, for $j=0,\ldots,m$, and moreover,
(ii) A_a satisfies the following norm estimate

$$\|A_a\|\le C(a)\prod_{i=1}^{2m}\|\varphi_i\|_{M^1(\mathbb{R}^d)},\quad\forall\varphi_i\in\mathcal{S}(\mathbb{R}^d),\ i=1,\ldots,2m,\tag{38}$$

where the positive constant $C(a)$ depends only on the symbol a. Then the symbol a belongs necessarily to $M^\infty(\mathbb{R}^{2md})$.

Proof. **Step 1.** We first compute the time-frequency shifts of the function Ψ defined in (37). If $u=(u_1,u_2)\in\mathbb{R}^{md}\times\mathbb{R}^{md}$, $w=(w_1,w_2)\in\mathbb{R}^{md}\times\mathbb{R}^{md}$, the time shift $T_w\Psi$ gives

$$T_{(w_1,w_2)}(\overline{\prod_{j=1}^m\left(V_{\varphi_j}f_j\right)}\mathcal{V}_{\phi_2}g)(z,\zeta)=\overline{\prod_{j=1}^m\left(V_{\varphi_j}f_j\right)(z-w_1,\zeta-w_2)}\,(\mathcal{V}_{\phi_2}g)(z-w_1,\zeta-w_2).$$

We fix $j \in \{1, \dots, m\}$ and we compute the time shifts related to the STFT of order j, that is

$$T_{(w_{1_j}, w_{2_j})} V_{\varphi_j} f_j(z_j, \zeta_j) = \langle f_j, M_{\zeta_j - w_{1_j}} T_{z_j - w_{2_j}} \varphi_j \rangle = e^{2\pi i z_j w_{2_j}} \langle f_j, M_{\zeta_j} T_{z_j} (M_{-w_{2_j}} T_{-w_{1_j}} \varphi_j) \rangle.$$

Next we compute the time shifts of $\mathcal{V}_{\phi_2} g$, that is

$$\begin{aligned} \mathcal{V}_{\phi_2} g(z - w_1, \zeta - w_2) &= \int_{\mathbb{R}^d} g(t) \overline{\prod_{j=1}^m M_{\zeta_j - w_{2_j}} T_{z_j - w_{1_j}} \varphi_{j+m}(t)} \, dt \\ &= e^{-2\pi i \sum_{j=1}^m z_j w_{2_j}} \int_{\mathbb{R}^d} g(t) \overline{\prod_{j=1}^m (M_{\zeta_j} T_{z_j} M_{-w_{2_j}} T_{-w_{1_j}} \varphi_{j+m})(t)} \, dt. \end{aligned}$$

Hence, the time shifts of Ψ are given by

$$T_w \Psi(z, \zeta) = \overline{\prod_{j=1}^m \left(V_{(M_{-w_{2_j}} T_{-w_{1_j}} \varphi_j)} f_j \right) (z, \zeta)} \, \mathcal{V}_{M_{-w_2} T_{-w_1} \phi_2} g(z, \zeta), \tag{39}$$

where $\mathcal{V}_{M_{-w_2} T_{-w_1} \phi_2} g(z, \zeta) = \int_{\mathbb{R}^d} g(t) \overline{\prod_{j=1}^m (M_{\zeta_j} T_{z_j} M_{-w_{2_j}} T_{-w_{1_j}} \varphi_{j+m})(t)} \, dt$.

The frequency shifts M_p are computed similarly, and yield

$$\begin{aligned} M_{(u_1, u_2)} \big(\overline{V_{\phi_1} (\otimes_{j=1}^m f_j)} \, \mathcal{V}_{\phi_2} g \big)(z, \zeta) &= \overline{e^{-2\pi i \sum_{j=1}^m (z_j u_{1_j} + \zeta_j u_{2_j})} \prod_{j=1}^m \big(V_{\varphi_j} f_j \big)} \mathcal{V}_{\phi_2} g(z, \zeta) \\ &= \overline{\prod_{j=1}^m \Big(V_{M_{-u_{1_j}} T_{u_{2_j}} \varphi_j} M_{-u_{1_j}} T_{u_{2_j}} f_j \Big)} \mathcal{V}_{\phi_2} g(z, \zeta). \end{aligned} \tag{40}$$

Gluing together (39) and (40) we obtain

$$M_u T_v (\Psi) = \overline{\prod_{j=1}^m V_{(M_{-u_{1_j}} T_{u_{2_j}} M_{-v_{2_j}} T_{-v_{1_j}} \varphi_j)} \Big(M_{-u_{1_j}} T_{u_{2_j}} f_j \Big)} \mathcal{V}_{M_{-v_2} T_{-v_1} \phi_2} g. \tag{41}$$

Step 2. To verify that $a \in \mathcal{S}'(\mathbb{R}^{2md})$ is in $M^\infty(\mathbb{R}^{2md})$, we consider the window function Ψ of the previous step and we show that $V_\Psi a$ is in $L^\infty(\mathbb{R}^{4md})$. We use the assumption (38), the weak definition of A_a and (41) to conclude the proof. More precisely, if we denote by A_a the multilinear localization operator with analysis windows $M_{-u_{1_j}} T_{u_{2_j}} M_{-v_{2_j}} T_{-v_{1_j}} \varphi_j$, and synthesis windows $M_{-v_{2_j}} T_{-v_{1_j}} \varphi_{j+m}$, $j =$

$1, \ldots, m$, we get

$$\begin{aligned}
|V_\Psi a(v,u)| &= |\langle a, M_u T_v \Psi\rangle| \\
&= |\langle a, \overline{\prod_{j=1}^m V_{(M_{-u_{1_j}} T_{u_{2_j}} M_{-v_{2_j}} T_{-v_{1_j}} \varphi_j)} \left(M_{-u_{1_j}} T_{u_{2_j}} f_j\right) \mathcal{V}_{M_{-v_2} T_{-v_1} \phi_2} g}\rangle| \\
&= |\langle A_a \overrightarrow{f}, g\rangle| \\
&\leq C(a) \prod_{j=1}^m \|M_{-u_{1_j}} T_{u_{2_j}} M_{-v_{2_j}} T_{-v_{1_j}} \varphi_j\|_{M^1(\mathbb{R}^d)} \|M_{-v_{2_j}} T_{-v_{1_j}} \varphi_{j+m}\|_{M^1(\mathbb{R}^d)} \times \\
&\qquad \|M_{-u_{1_j}} T_{u_{2_j}} f_j\|_{M^{p_j,q_j}} \|g\|_{M^{p_0', q_0'}} \\
&\lesssim \prod_{j=1}^{2m} \|\varphi_j\|_{M^1} \prod_{j=1}^m \|f_j\|_{M^{p_j,q_j}} \|g\|_{M^{p_0', q_0'}} < \infty.
\end{aligned}$$

where $\overrightarrow{f} = (M_{-u_{1_j}} T_{u_{2_j}} f_j)_{j=1}^m$. Since, $u, v \in \mathbb{R}^{md}$ are arbitrary, the last inequality concludes the proof. ∎

7. An application: symbols for multilinear bounded Kohn-Nirenberg operators

For the sake of simplicity, we provide examples for the bilinear case, and when the dimension $d = 1$. The multilinear, higher dimensional case is dealt with similarly. Two steps are needed to construct symbols in $M^{\infty,1}(\mathbb{R}^3)$, as indicated in section 5. First, we need to choose suitable windows φ_i, $i = 1, \ldots, d$, and to compute the function Φ defined in (17). Secondly, we look for symbols a on $\mathbb{R}^4$ that, when convolved with Φ, yield the Kohn-Nirenberg symbols σ as explained in (18).

Construction of Φ. It is certainly clear that, to give interesting examples of non-smooth Kohn-Nirenberg symbols σ, the starting point can not be the Gaussian $\varphi_0 = e^{-\pi t^2}$, or as a matter of fact any function in $\mathcal{S}$. Thus, we look for windows $\varphi_i \in M^1(\mathbb{R})$ (with minimal smoothness) such that by Lemma 4.4, we obtain $\Phi \in M^1(\mathbb{R}^4)$.

(i) Let $A > 0$ and for every $i = 1, \ldots, 4$, choose $\varphi_i(t) = \varphi(t) = e^{-A|t|}$. Its Fourier transform is given by $\hat{\varphi}(\omega) = \dfrac{2A}{A^2 + 4\pi\omega^2}$. Using the Fourier transform of the STFT in (15), we construct a first example of window $\Phi = \Phi_1$:

$$\begin{aligned}
(42)\quad \Phi_1(x,y,u,v) &= \mathcal{U}\mathcal{F}(V_\varphi \varphi)(x,u) \mathcal{U}\mathcal{F}(V_\varphi \varphi)(y,v) \\
&= \mathcal{U}\left(e^{2\pi i x u} e^{-A|x|} \frac{2A}{A^2 + 4\pi u^2}\right) \mathcal{U}\left(e^{2\pi i y v} e^{-A|y|} \frac{2A}{A^2 + 4\pi v^2}\right) \\
&= e^{-2\pi i (xu + yv)} e^{-A(|u|+|v|)} \frac{4A^2}{(A^2 + 4\pi x^2)(A^2 + 4\pi y^2)}.
\end{aligned}$$

(ii) Let $A > 0$ and for every $i = 1, \ldots, 4$, we set $\varphi_i(t) = \varphi(t) = Ate^{-A|t|}h(t)$, where h is the step function:

$$h(t) = \begin{cases} 0, & \text{if } t < 0; \\ 1, & \text{if } t \geq 0. \end{cases} \tag{43}$$

An easy computation shows: $\hat{\varphi}(\omega) = \dfrac{A}{(A + 2\pi i\omega)^2}$. A second example is the function $\Phi = \Phi_2$ given by:

$$\Phi_2(x, y, u, v) = A^4 uve^{-2\pi i(xu+yv)} \frac{e^{A(u+v)}}{(A^2 + 4\pi x)^2(A^2 + 4\pi y)^2}(h \otimes h)(-u, -v). \tag{44}$$

(iii) Note that rougher windows can be considered as well. For instance, take $A > 0$ and for every $i = 1, \ldots, 4$, set $\varphi_i(t) = \varphi(t) = e^{-A|t|}h(t)$, where h is defined in (43). Then $\varphi \notin M^1$ (φ is not even continuous). Nevertheless, $\varphi \in L^2 = M^2$ and its Fourier transform is given by $\hat{\varphi}(\omega) = \dfrac{1}{A + 2\pi i\omega}$. In this case, by Corollary 5.4, we obtain $\Phi_3 \in M^2(\mathbb{R}^4) = L^2(\mathbb{R}^4)$, where

$$\Phi_3(x, y, u, v) = e^{-2\pi i(xu+yv)} \frac{e^{A(u+v)}}{(A - 2\pi ix)(A - 2\pi iy)}(h \otimes h)(-u, -v). \tag{45}$$

Of course, a host of other examples of functions Φ can be obtained by playing with different windows φ_i, instead of assuming that all the windows are equal.

Construction of the Kohn-Nirenberg symbols σ. For any $(c_k) \in \ell^\infty$, measures of the form $a = a_1 = \sum_{k \in \mathbb{Z}^4} c_k \delta_k$ belong to $M^\infty(\mathbb{R}^4)$.

By Lemma 5.2, the Kohn-Nirenberg symbol $\sigma \in M^{\infty,1}(\mathbb{R}^3)$, provided that $\Phi \in M^1(\mathbb{R}^4)$ and $a_1 \in M^\infty(\mathbb{R}^4)$.

i) The first example of $\sigma = \sigma_1(x, u, v)$ is given by the convolution of Φ_1 with a_1:

$$\begin{aligned} \sigma_1(x, u, v) &= \left(\sum_{k \in \mathbb{Z}^4} c_k \delta_k * \Phi_1 \right)(x, x, u, v) \\ &= 4A^2 \sum_{(k_1,k_2,k_3,k_4) \in \mathbb{Z}^4} c_{(k_1,k_2,k_3,k_4)} \left[T_{(k_1,k_2,k_3,k_4)} e^{-2\pi i(xu+yv)} \right]_{|y=x} \times \\ &\qquad \frac{T_{(k_3,k_4)} e^{-A(|u|+|v|)}}{\left[T_{(k_1,k_2)}((A^2 + 4\pi x^2)(A^2 + 4\pi y^2)) \right]_{|y=x}}. \end{aligned}$$

ii) Similarly, a second example $\sigma_2(x, u, v)$ is obtained by convolving the measure a_1 with Φ_2:

$$
\begin{aligned}
\sigma_2(x,u,v) &= \left(\sum_{k\in\mathbb{Z}^4} c_k\delta_k * \Phi_2\right)(x,x,u,v)\\
&= A^4 \sum_{(k_1,k_2,k_3,k_4)\in\mathbb{Z}^4} c_{(k_1,k_2,k_3,k_4)}(u-k_3)(v-k_4)(h\otimes h)(-u+k_3,-v+k_4)\times\\
&\qquad \frac{\left[T_{(k_1,k_2,k_3,k_4)}e^{-2\pi i(xu+yv)}\right]_{|y=x} T_{(k_3,k_4)}e^{A(u+v)}}{\left[T_{(k_1,k_2)}((A^2+4\pi x)^2(A^2+4\pi y)^2)\right]_{|y=x}}.
\end{aligned}
$$

iii) One can apply the same arguments as before, starting from any other symbol a in M^∞. For instance, take $a=\chi_E$, the indicator function of any bounded subset of $\mathbb{R}^4$.

iv) Smoother examples of functions a can be used e.g.,

$$
a_2(x_1,x_2,x_3,x_4)=\prod_{k=1}^4\frac{1}{1+x_k^2},\qquad a_3(x_1,x_2,x_3,x_4)=\prod_{k=1}^4\frac{1}{A_k+2\pi i x_k}.
$$

Both examples define functions $a_j\in L^2(\mathbb{R}^4)$, which can be convolved with any function $\Phi\in L^2(\mathbb{R}^4)$, e.g., the function Φ_3 computed in (45). Then Corollary 5.6 gives symbols $\sigma\in M^{\infty,1}(\mathbb{R}^3)$.

8. Acknowledgment

The authors thank H. Feichtinger and K. Gröchenig for very helpful and interesting discussions while completing this work. This work was developed while both authors were visiting the Erwin Schrödinger Institute in Vienna whose support, hospitality and great working conditions are gratefully acknowledged.

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Department of Mathematics, University of Torino, Italy
E-mail address: cordero@dm.unito.it

Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853, USA
E-mail address: kasso@math.cornell.edu