A CLASS OF FOURIER MULTIPLIERS FOR MODULATION SPACES

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Abstract. We prove the boundedness of a general class of Fourier multipliers, in particular of the Hilbert transform, on modulation spaces. In general, however, the Fourier multipliers in this class fail to be bounded on L^p spaces. The main tools are Gabor frames and methods from time-frequency analysis.

1. Introduction

In this note we explore the boundedness properties of certain translation invariant operators (initially defined on the class of Schwartz rapidly decreasing smooth functions). Namely, if $b > 0$ and $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$ is a bounded sequence of complex numbers, we are interested in the operator $H_{b,\mathbf{c}}$ (formally) defined by

$$H_{b,\mathbf{c}} = \frac{c_0}{2}(H - M_b H M_{-b}) + \sum_{n\neq 0} c_n (M_{bn} H M_{-bn} - M_{b(n+1)} H M_{-b(n+1)}). \tag{1}$$

Here, M_b denotes the modulation by b and H is the Hilbert transform, that is,

$$M_b f(t) = e^{2\pi i t\cdot b} f(t)$$

and

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \int_{|t|>\epsilon} \frac{f(x-t)}{t}\, dt.$$

The operators $H_{b,\mathbf{c}}$ are better understood if viewed on the Fourier side as multiplier operators

$$\widehat{T_{b,\mathbf{c}} f} = m_{b,\mathbf{c}} \widehat{f}, \tag{2}$$

with Fourier multipliers

$$m_{b,\mathbf{c}} = c_0 \chi_{(0,b)} + \sum_{n\neq 0} c_n \chi_{(bn, b(n+1))}; \tag{3}$$

$\chi_{(a,b)}$ denotes the characteristic function of the real interval (a, b). It is easy to see that the Hilbert transform is a particular case of operator $H_{b,\mathbf{c}}$. Indeed, if we recall that $m(\xi) = -i\operatorname{sgn}\xi$ is the multiplier of H, then $H = -iH_{b,\mathbf{c}}$ for any $b > 0$ and $\mathbf{c} = (c_n)_{n\in\mathbb{Z}}$, with $c_0 = 1$ and $c_n = \operatorname{sgn} n$ for $n \neq 0$.

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There are many proofs for the boundedness of H on L^p spaces, $1 < p < \infty$. For example, in the context of Calderón-Zygmund theory, the boundedness of H follows from its boundedness on L^2 (a consequence of Plancherel's theorem), a weak type $(1,1)$ estimate, and interpolation. A natural question then is whether the more general operators $H_{b,\mathbf{c}}$ are also bounded on the Lebesgue spaces $L^p, 1 < p < \infty$. We note right away that the boundedness of the sequence $\mathbf{c}$ and Plancherel's theorem guarantee that $H_{b,\mathbf{c}}$ maps L^2 into L^2. But unlike the kernel of the Hilbert transform, the convolution kernels of the operators $H_{b,\mathbf{c}}$ do not possess the required amount of smoothness to fall under the scope of Calderón-Zygmund theory. Moreover, the multipliers $m_{b,\mathbf{c}}$ are not of bounded variation in general, nor do they satisfy the conditions of some Fourier multiplier theorem. In fact, the operators $H_{b,\mathbf{c}}$ fail to be bounded on $L^p, p \neq 2$.

Consider for example the operators $H_{1,\mathbf{c}}$, $\mathbf{c} = (c_n)_{n\in\mathbb{Z}} \in \ell^\infty$, and assume that an estimate of the form

$$\|H_{1,\mathbf{c}} f\|_p \le C\|\mathbf{c}\|_{\ell^\infty} \|f\|_p \tag{4}$$

were true for some $1 < p < 2$ and a constant C independent of $\mathbf{c}$ and f. This ineqality remains true, if we replace c_n with $c_n r_n(t)$, where $r_n(t)$ is the nth Rademacher function. (For the exact definition of the Rademacher system we refer the reader to the book by Duandikoetxea [2, p. 177] for further details.) Define the operator S_n by $\widehat{S_n f} = c_n \chi_{[n,n+1]}\hat{f}$. If we now integrate over $t \in [0,1]$, we can deduce from (4), e.g., by the estimates in [9, Appendix C], that for every $f \in L^p$

$$\|(\sum_{n\in\mathbb{Z}} |S_n f|^2)^{1/2}\|_{L^p} \le C\|f\|_{L^p}\,. \tag{5}$$

For the choice $c_n = 1$ this averaging procedure over the plus and minus signs yields a square function, and estimate (5) is known to hold true only if $p \ge 2$; see e.g. [9, Sec. 10.2]. This contradicts our initial assumption $1 < p < 2$. By duality we can also exclude the case $p > 2$, because the space of Fourier multipliers on L^p coincides with those on $L^{p'}$, where $p' = p/(p-1)$ is the dual exponent of p. We conclude that an estimate of the form (4) holds true if and only if $p = 2$.

Nevertheless, we will prove below that the operators $H_{b,\mathbf{c}}$ are bounded on a different class of function spaces, the so-called modulation spaces $\mathcal{M}^{p,q}, 1 < p < \infty, 1 \le q \le \infty$. These spaces include $L^2 = \mathcal{M}^{2,2}$ and are defined by their phase-space distribution (instead of their Littlewood-Paley decomposition). The modulation spaces occur naturally in time-frequency analysis (or phase-space analysis) and have found numerous applications to pseudodifferential operators, signal analysis, non-linear approximation, and the formulation of uncertainty principles; see, e.g., [8], [11], [12], [14], [15], [17], [18], [19], [23].

We will prove the boundedness of the Fourier multipliers by studying their matrix with respect to a so-called Gabor frame (often called Weyl-Heisenberg frame) and by using the Gabor expansion of functions. In this manner we can convert the question of boundedness of Fourier multipliers on $\mathcal{M}^{p,q}$ into a problem about the boundedness of an infinite matrix acting on certain sequence spaces. It is perhaps not too surprising that this discretization leads to the discrete Hilbert transform that was first studied by Hilbert in ℓ^2 setting and by M. Riesz [21] and Titchmarsh [22] on general ℓ^p spaces.

In particular, we obtain the boundedness of the Hilbert transform on modulation spaces. This fact was first observed by Okoudjou in [20]. In this paper we treat a natural extension and also close a gap in the original proof. We also point out that a similar class of (pointwise) multipliers was considered on the so-called amalgam spaces in [24]. Since the modulation spaces are the Fourier transforms of certain amalgam spaces, some of the results in [24, Theorem 3.6] overlap with our main result. However, our techniques and the ones used in [24] are completely different. In addition, the use of time-frequency techniques in "hard analysis" seems of independent interest. As for the endpoint modulation spaces, the situation is similar to the case of Lebesgue spaces. For example, the Hilbert transform fails to be bounded on the Feichtinger algebra $\mathcal{M}^1$.

The remainder of the paper is organized as follows. The next section is devoted to some basic facts about modulation spaces and Gabor frames. The main result is stated and proved in Section 3, as well as some counter-example.

2. Preliminaries

2.1. **General notation.** We will be working on the real line $\mathbb{R}$. The operators of translation and modulation of a function f with domain $\mathbb{R}$ are defined by

$$T_x f(t) = f(t-x) \qquad \text{and} \qquad M_y f(t) = e^{2\pi i y t} f(t).$$

The Fourier transform of $f \in L^1(\mathbb{R})$ is $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i t\cdot\omega}\, dt$, $\omega \in \mathbb{R}$. The Fourier transform is an isomorphism of the Schwartz space $\mathbb{S} = \mathbb{S}(\mathbb{R})$ onto itself, and extends to the space $\mathbb{S}' = \mathbb{S}'(\mathbb{R})$ of tempered distributions by duality.

The inner product of two functions $f, g \in L^2$ is $\langle f, g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt$, and its extension to $\mathbb{S}' \times \mathbb{S}$ will be also denoted by $\langle \cdot, \cdot \rangle$.

The Short-Time Fourier Transform (STFT) of a function f with respect to a window g is

$$V_g f(x,y) = \langle f, M_y T_x g\rangle = \int_{\mathbb{R}} e^{-2\pi i y\cdot t}\, \overline{g(t-x)}\, f(t)\, dt,$$

whenever the integral makes sense. Analogously to the Fourier transform, the STFT extends in a distributional sense to f, g in the space of tempered distributions $\mathbb{S}'$; see e.g. Folland's book [7, Prop. 1.42].

We let $L^{p,q} = L^{p,q}(\mathbb{R}\times\mathbb{R})$ be the spaces of measurable functions $f(x,y)$ for which the mixed norm

$$\|f\|_{L^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)|^p\, dx\right)^{q/p} dy\right)^{1/q}$$

is finite. If $p = q$, we have $L^{p,p} = L^p$, the usual Lebesgue spaces. By $\ell^{p,q}(\mathbb{Z}\times\mathbb{Z})$ we denote the spaces of sequences $\mathbf{a} = (a_{k,l})_{k,l\in\mathbb{Z}}$ for which the mixed norm

$$\|\mathbf{a}\|_{\ell^{p,q}} = \left(\sum_{l\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |a_{k,l}|^p\right)^{q/p}\right)^{1/q}$$

is finite. If $p = q$, we recover the sequence spaces $\ell^p(\mathbb{Z} \times \mathbb{Z})$.

2.2. Modulation spaces.

Definition 1. Given $1 \le p, q \le \infty$, and given a window function $g \in \mathbb{S}$, the modulation space $\mathcal{M}^{p,q} = \mathcal{M}^{p,q}(\mathbb{R})$ is the space of all distributions $f \in \mathbb{S}'$ for which the following norm is finite:

$$\|f\|_{\mathcal{M}^{p,q}} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |V_g f(x,y)|^p \, dx \right)^{q/p} dy \right)^{1/q} = \|V_g f\|_{L^{p,q}}, \tag{6}$$

with the usual modifications if p and/or q are infinite. When $p = q$, we will write $\mathcal{M}^p$ for the modulation space $\mathcal{M}^{p,p}$.

Remark 1. The definition is independent of the choice of the window g in the sense of equivalent norms. If $1 \le p, q < \infty$, then $\mathcal{M}^1$ is densely embedded into $\mathcal{M}^{p,q}$. In fact, the Schwartz class $\mathbb{S}$ is dense in $\mathcal{M}^{p,q}$ for $1 \le p, q < \infty$. One can also show that the dual of $\mathcal{M}^{p,q}$ is $\mathcal{M}^{p',q'}$, where $1 \le p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. We refer to [10] and the references therein for more details about modulation spaces.

Remark 2. The modulation space $\mathcal{M}^1$, also called the Feichtinger algebra, is a Banach algebra under both pointwise multiplication and convolution and is invariant under Fourier transform. It plays also an important role in the theory of Gabor frames where it serves as a convenient class of windows that generate Gabor frames for the whole class of modulation spaces.

2.3. Gabor Frames.

Definition 2. Given a window function $\phi \in L^2(\mathbb{R})$ and constants $\alpha, \beta > 0$, we say that $\{M_{\beta n} T_{\alpha k} \phi\}_{k,n \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ (called frame bounds) such that

$$A \|f\|_{L^2(\mathbb{R})} \le \sum_{k,n \in \mathbb{Z}} |\langle f, M_{\beta n} T_{\alpha k} \phi \rangle|^2 \le B \|f\|_{L^2(\mathbb{R})}, \quad \forall f \in L^2(\mathbb{R}).$$

We refer to the works by Daubechies [1], Gröchenig [10], and Heil and Walnut [16] for extensive treatments of frames and Gabor frames.

The theory of Gabor frames can be generalized from the pure L^2-theory to the whole class of modulation spaces as explained in [5], [6], [10]. The next theorem provides a characterization of modulation spaces by means of Gabor frames and will be used heavily in the sequel.

Theorem A. *Let $\phi \in \mathcal{M}^1$ be such that $\{M_{\beta n} T_{\alpha k} \phi\}_{k,n \in \mathbb{Z}}$ is a Gabor frame for L^2, and let $1 \le p, q \le \infty$. Then there exists a (canonical) dual $\gamma \in \mathcal{M}^1$ such that every tempered distribution in $\mathcal{M}^{p,q}$ has a Gabor expansion that converges unconditionally (or weak* unconditionally if $p = \infty$ or $q = \infty$), namely*

$$f = \sum_{k,n \in \mathbb{Z}} \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle \, M_{\beta n} T_{\alpha k} \phi, \qquad \forall f \in \mathcal{M}^{p,q}(\mathbb{R});$$

moreover, we have the following norm equivalences

$$\|f\|_{\mathcal{M}^{p,q}} \asymp \|\langle f, M_{\beta n} T_{\alpha k} \phi \rangle\|_{\ell^{p,q}} \asymp \|\langle f, M_{\beta n} T_{\alpha k} \gamma \rangle\|_{\ell^{p,q}}.$$

To summarize, a tempered distribution f belongs to the modulation space $\mathcal{M}^{p,q}(\mathbb{R})$ if and only if the sequence of its Gabor coefficients defined as

$$C_\phi f = (\langle f, M_{\beta n} T_{\alpha k} \phi \rangle)_{k,n \in \mathbb{Z}}$$

belongs to the sequence space $\ell^{p,q}(\mathbb{Z} \times \mathbb{Z})$. Moreover, the norm of f is equivalent to the norm of its Gabor coefficients.

3. Boundedness of $H_{b,\mathbf{c}}$ on modulation spaces

Our main result can be stated as follows:

Theorem 1. *For any $b > 0$ and $\mathbf{c} \in \ell^\infty(\mathbb{C})$, the operators $H_{b,\mathbf{c}}$ are bounded from $\mathcal{M}^{p,q}$ into $\mathcal{M}^{p,q}$ for $1 < p < \infty$, $1 \le q \le \infty$ with a norm estimate*

$$\|H_{b,\mathbf{c}} f\|_{\mathcal{M}^{p,q}} \le C \, \|\mathbf{c}\|_\infty \, \|f\|_{\mathcal{M}^{p,q}}$$

for some constant depending only on b, p, and q. In particular, the Hilbert transform H is bounded on $\mathcal{M}^{p,q}$ for $1 < p < \infty$ and $1 \le q \le \infty$

Proof. Since the modulation spaces are invariant under dilations, we may assume without loss of generality that $b = 1$ by conjugating $H_{b,\mathbf{c}}$ with a suitable dilation. Thus from now on we will only consider the multiplier (1) with $b = 1$.

Next we choose a Gabor frame that is taylored to the analysis of our particular class of Fourier multipliers. Let

$$\phi(x) = \Big(\frac{\sin \pi x}{\pi x}\Big)^2, \text{ or equivalently, } \hat{\phi}(\omega) = \chi_{[-1/2,1/2]} * \chi_{[-1/2,1/2]}(\omega) = \max(0, 1 - |\omega|).$$

It follows that $\phi \in \mathcal{M}^1$ [3] or [10, Prop. 12.1.6], and that $\{M_n T_{\frac{k}{2}} \phi\}_{k,n \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ [10, Thm. 6.4.1].

We expand f with respect to a Gabor frame, i.e., $f = \sum_{k,n \in \mathbb{Z}} \langle f, M_{\beta n} T_{\alpha k} \gamma \rangle \, M_{\beta n} T_{\alpha k} \phi$, and then take the coefficients of $H_{1,\mathbf{c}}$. We find that

$$\begin{aligned} (C_\phi H_{1,\mathbf{c}} f)(k,n) &= \langle H_{1,\mathbf{c}} f, M_n T_{\frac{k}{2}} \phi \rangle \\ &= \sum_{k',n' \in \mathbb{Z}} \langle f, M_{n'} T_{\frac{k'}{2}} \gamma \rangle \, \langle H_{1,\mathbf{c}} M_{n'} T_{\frac{k'}{2}} \phi, M_n T_{\frac{k}{2}} \phi \rangle \,. \end{aligned} \tag{7}$$

Recall that by Theorem A $f \in \mathcal{M}^{p,q}$ if and only if $C_\gamma f = \big(\langle f, M_n T_{\frac{k}{2}} \gamma \rangle\big)_{k,n \in \mathbb{Z}} \in \ell^{p,q}(\mathbb{Z}^2)$ and $H_{1,\mathbf{c}} f \in \mathcal{M}^{p,q}$ if and only if $C_\phi H_{1,\mathbf{c}} f = \big(\langle H_{1,\mathbf{c}} f, M_n T_{\frac{k}{2}} \phi \rangle\big)_{k,n \in \mathbb{Z}} \in \ell^{p,q}(\mathbb{Z}^2)$. Consequently to verify the boundedness of $H_{1,\mathbf{c}}$, it suffices to show that the matrix Υ whose entries are defined by

$$\Upsilon_{(k,n),(k',n')} = \langle H_{1,\mathbf{c}} M_{n'} T_{\frac{k'}{2}} \phi, M_n T_{\frac{k}{2}} \phi \rangle$$

is bounded on $\ell^{p,q}(\mathbb{Z}^2)$. The commutative diagram 8 illustrates the situation.

$$\begin{array}{ccc} \mathcal{M}^{p,q} & \xrightarrow{H_{1,\mathbf{c}}} & \mathcal{M}^{p,q} \\ \downarrow C_\gamma & & \downarrow C_\phi \\ \ell^{p,q} & \xrightarrow{\Upsilon} & \ell^{p,q} \end{array} \tag{8}$$

We now compute the entries of the matrix Υ. Here we exploit the special properties of the basic function ϕ and the algebra of time-frequency shifts. We have:

$$\begin{aligned} \Upsilon_{(k,n),(k',n')} &= \langle H_{1,\mathbf{c}} M_{n'} T_{\frac{k'}{2}} \phi, M_n T_{\frac{k}{2}} \phi \rangle \\ &= \langle H_{1,\mathbf{c}} \widehat{M_{n'} T_{\frac{k'}{2}}} \phi, \widehat{M_n T_{\frac{k}{2}}} \phi \rangle \\ &= \langle \sum_{m\in\mathbb{Z}} c_m \, \chi_{[m,m+1]} \cdot T_{n'} M_{-\frac{k'}{2}} \hat{\phi}, T_n M_{-\frac{k}{2}} \hat{\phi} \rangle \\ &= \sum_{m\in\mathbb{Z}} c_m \, e^{\pi i k'(n'-n)} \, \langle (T_{-n}\chi_{[m,m+1]}) \, M_{\frac{k-k'}{2}} T_{n'-n} \hat{\phi}, \hat{\phi} \rangle \\ &= \sum_{m\in\mathbb{Z}} c_m \, (-1)^{k'(n'-n)} \, \langle M_{\frac{k-k'}{2}} \chi_{[m-n,m-n+1]} \, T_{n'-n} \hat{\phi}, \hat{\phi} \rangle . \end{aligned} \tag{9}$$

Because the function $\hat{\phi}$ is compactly supported on $[-1,1]$, the sum in (9) is finite, and only the terms $m = n-1$ and $m = n$ occur:

$$\begin{aligned} \Upsilon_{(k,n),(k',n')} = c_{n-1} \, (-1)^{k'(n'-n)} \, \langle M_{\frac{k-k'}{2}} \chi_{[-1,0]} T_{n'-n} \hat{\phi}, \hat{\phi} \rangle \\ + c_n \, (-1)^{k'(n'-n)} \, \langle M_{\frac{k-k'}{2}} \chi_{[0,1]} T_{n'-n} \hat{\phi}, \hat{\phi} \rangle . \end{aligned} \tag{10}$$

Furthermore, the support condition on $\hat{\phi}$ implies that the indices n, n' are related by $n-1 \le n' \le n+1$ and we only need to calculate the following integrals explicitly:

$$\begin{aligned} \langle M_l \chi_{[0,1]} \hat{\phi}, \hat{\phi} \rangle &= \langle M_{-l} \chi_{[-1,0]} \hat{\phi}, \hat{\phi} \rangle = \int_0^1 (1-\omega)^2 e^{-2\pi i l \omega} \, d\omega \, , \\ \langle M_l \chi_{[0,1]} T_1 \hat{\phi}, \hat{\phi} \rangle &= \langle M_{-l} \chi_{[-1,0]} T_{-1} \hat{\phi}, \hat{\phi} \rangle = \int_0^1 \omega \, (1-\omega) \, e^{-2\pi i l \omega} \, d\omega \, , \end{aligned}$$

whereas

$$\langle M_l \chi_{[0,1]} T_{-1} \hat{\phi}, \hat{\phi} \rangle = \langle M_{-l} \chi_{[-1,0]} T_1 \hat{\phi}, \hat{\phi} \rangle = \int_0^1 (1-\omega)^2 e^{-2\pi i l \omega} \, d\omega = 0 \, .$$

The evaluation of these elementary integrals suggests the following auxiliary sequences:

$$\delta_k = \begin{cases} 0 & \text{if} \quad k \neq 0 \\ 1 & \text{if} \quad k = 0, \end{cases}$$

$$\rho_k = \begin{cases} \frac{1}{\pi i k} + \frac{2}{\pi^2 k^2} + \frac{2((-1)^k - 1)}{i \pi^3 k^3} & \text{if} \quad k \neq 0 \\ 0 & \text{if} \quad k = 0 \end{cases}$$

and

$$\epsilon_k = \begin{cases} -\frac{(-1)^k + 1}{\pi^2 k^2} + \frac{2((-1)^k - 1)}{i \pi^3 k^3} & \text{if} \quad k \neq 0 \\ 0 & \text{if} \quad k = 0. \end{cases}$$

Using these sequences we can compute the entries of Λ as follows:

- Case 1: If $n' = n$, then
$$\Upsilon_{(k,n),(k',n)} = c_{n-1}\,(\rho_{k-k'} + \frac{1}{3}\delta_{k-k'}) + c_n\,(\rho_{k'-k} + \frac{1}{3}\delta_{k'-k}).$$
- Case 2: If $n' = n+1$, then
$$\Upsilon_{(k,n),(k',n')} = c_n\,(-1)^{k'}\,(\epsilon_{k-k'} + \frac{1}{6}\,\delta_{k-k'}).$$
- Case 3: If $n' = n-1$, then
$$\Upsilon_{(k,n),(k',n')} = c_{n-1}\,(-1)^{k'}\,(\epsilon_{k'-k} + \frac{1}{6}\,\delta_{k'-k}).$$

Consequently, letting $a_{k',n'} = \langle f, M_{n'}T_{\frac{k'}{2}}\gamma\rangle$, we can rewrite (7) as follows:

$$\begin{aligned}(C_\phi H_{1,\mathbf{c}}(f))(k,n) &= \sum_{k',n'\in\mathbb{Z}} \Upsilon_{(k,n),(k',n')}\,a_{k',n'}\\
&= \sum_{k'\in\mathbb{Z}} a_{k',n}\,c_n(\rho_{k'-k} + \frac{1}{3}\,\delta_{k'-k}) + \sum_{k'\in\mathbb{Z}} a_{k',n}\,c_{n-1}(\rho_{k-k'} + \frac{1}{3}\,\delta_{k-k'})\\
&+ \sum_{k'\in\mathbb{Z}} a_{k',n+1}\,c_n\,(-1)^{k'}\,(\epsilon_{k-k'} + \frac{1}{6}\,\delta_{k-k'})\\
&+ \sum_{k'\in\mathbb{Z}} a_{k',n-1}\,c_{n-1}\,(-1)^{k'}\,(\epsilon_{k'-k} + \frac{1}{6}\,\delta_{k'-k}).\end{aligned}\tag{11}$$

The action of the matrix Υ can now be expressed in terms of convolutions with the sequences ρ, ϵ and δ defined above. More precisely,

$$\begin{aligned}(C_\phi H_{1,\mathbf{c}}(f))(k,n) &= c_n(a_{\cdot,n} * \tilde{\rho}(k) + \frac{1}{3}a_{k,n}) + c_{n-1}(a_{\cdot,n} * \rho(k) + \frac{1}{3}a_{k,n})\\
&+ c_n((-1)^{\cdot}a_{\cdot,n+1} * \epsilon(k) + \frac{1}{6}a_{k,n+1})\\
&+ c_{n-1}((-1)^{\cdot}a_{\cdot,n-1} * \tilde{\epsilon}(k) + \frac{1}{6}a_{k,n-1}),\end{aligned}\tag{12}$$

where $\tilde{x}(k) = x(-k)$ and $(-1)^{\cdot}a_{\cdot,n} * \epsilon$ denotes the sequence $\left(\sum_{k'\in\mathbb{Z}}(-1)^{k'}\,a_{k',n}\epsilon_{k-k'}\right)_{k\in\mathbb{Z}}$ for any fixed n.

The sequence ϵ belongs to $\ell^1(\mathbb{Z})$, thus we can use Young's inequality to take care of the convolution terms involving ϵ. However, ρ is the sum of three sequences, two of which are in $\ell^1(\mathbb{Z})$, but the third one, $\frac{1}{\pi i k}$, is clearly not summable. Fortunately, the convolution with $\frac{1}{\pi i k}$ (the discrete Hilbert transform) is bounded on $\ell^p(\mathbb{Z})$ for $1 < p < \infty$; see the book by Hardy, Littlewood and Pólya [13, Section 8.12], also

[21], [22]. Consequently, convolution with ρ is also bounded on $\ell^p(\mathbb{Z})$ for $1 < p < \infty$, and using Young's inequality we obtain

$$\Big(\sum_{k\in\mathbb{Z}} |C_\phi H_{1,\mathbf{c}}(f)(k,n)|^p\Big)^{1/p} \leq K\, \|\mathbf{c}\|_\infty\, \|a_{\cdot,n}\|_p$$

for some positive constant K and every $n \in \mathbb{Z}$. Hence, by taking the q-norm with respect to the variable n we obtain

$$\|C_\phi H_{1,\mathbf{c}}(f)\|_{\ell^{p,q}(\mathbb{Z}^2)} \leq K\, \|\mathbf{c}\|_\infty\, \|a\|_{\ell^{p,q}(\mathbb{Z}^2)},$$

for all $1 \leq q \leq \infty$. This concludes the proof. □

Remark 3. In general, the operators $H_{b,\mathbf{c}}$ are not bounded on $\mathcal{M}^1$. In fact, this is already the case for the Hilbert transform H. Assume on the contrary that $Hf \in \mathcal{M}^1$ for every $f \in \mathcal{M}^1$. Since $\mathcal{M}^1$ is invariant under the Fourier transform, this would imply that $\widehat{Hf} = -i\mathrm{sgn}\,(\cdot)\widehat{f} \in \mathcal{M}^1$ and $\widehat{Hf}$ would have a discontinuity at the origin whenever $\hat{f}(0) \neq 0$. But this contradicts the fact that every function in $\mathcal{M}^1$ is uniformly continuous. By duality, H cannot be bounded from $\mathcal{M}^\infty$ into $\mathcal{M}^\infty$ either.

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