Research Statement

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Summary

My research interests lie in the general area of harmonic analysis, with a particular emphasis on time-frequency analysis, frame theory, pseudodifferential operators, and analysis on fractals. My work to date has proceeded in several distinct but related directions.

• **Time-frequency analysis of function spaces.** First, I am interested in a comprehensive study of various Banach spaces of functions, or distributions. More specifically, I seek to understand the interplay between regularity and time-frequency concentration of distributions. This is usually achieved via embedding relations between certain function spaces. In particular, I proved embeddings between Banach spaces of functions characterized by their regularity (e.g., the Besov, Sobolev spaces) and a class of Banach spaces of functions, called modulation spaces, defined by means of their time-frequency content. The modulation spaces play also key roles in recent developments of pseudodifferential operators as well as Gabor analysis. However, these spaces have a rather implicit definition, thus my embedding results give useful sufficient conditions for membership in the modulation spaces.

• **Discrete characterization of function spaces.** A second aspect of my work which is related to the previous one is concerned with a thorough understanding of the fine properties of distributions, e.g., time-frequency concentration, or regularity using frame theory. Frames are basis-like objects used in particular to characterize various functions spaces. Such characterizations are very important in approximation theory, and are also widely used in signal processing. In particular, I used Gabor frames to characterize a class of function spaces known as the Wiener amalgam spaces which play important roles in sampling theory as well as in time frequency analysis. I am also interested in the theory of finite frames which has many applications in engineering, where different applications require construction of frames with specific structures. For example, I gave a complete characterization of finite tight frames with a convolutional structure. Such frames are related to filter banks and thus can be implemented using fast algorithms.

• **Time-frequency of operators.** Another aspect of my research deals with a time-frequency analysis of multilinear pseudodifferential operators and Fourier multipliers in the realm of the modulation spaces. These spaces arise again naturally in many applications such as mobile communications, seismic data image processing, and appear as substitutes to settle continuity properties of certain operators that are known to be unbounded on other classical spaces. In my work I have shown such results for a class of Fourier multipliers which are generally unbounded on the Lebesgue spaces. Additionally, I have been investigating continuity properties of multilinear pseudodifferential operators on modulation spaces, for which I formulated and proved some multilinear Calderón-Vaillancourt-type theorems.

• **Analysis on fractals.** The last aspect of my research is centered around the developing area of analysis on fractals. In particular, I am interested in investigating the analogue of certain results from classical analysis in the fractal setting. For example, I formulated and proved an analogue of Heisenberg's uncertainty principle on a large class of fractals, including the Sierpinski gasket, and the Sierpinski Carpet.

What follows is a more detailed discussion on several specific problems I work on, progress to date, and plans for the future.

General notations

• **Modulation spaces.** For x, ω, and $t \in \mathbb{R}^d$, let $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i t\cdot\omega} f(t)$. The short time Fourier transform (STFT) of $f \in \mathbb{S}'$ with respect to a window $g \in \mathbb{S}$ is the function $V_g f$ which provides local frequency information of f and is defined by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i t\cdot\omega}\,dt \qquad (x,\omega) \in \mathbb{R}^{2d}.$$

For $\alpha, \beta > 0$ and $g \in L^2$ the family $\mathcal{G}(g,\alpha,\beta) = \{M_{\beta n} T_{\alpha m} g\}_{m,n\in\mathbb{Z}^d}$, is a Gabor frame for L^2 ([19], [37]) if there exist $A, B > 0$ such that

$$A\|f\|_{L^2}^2 \le \sum_{m,n\in\mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha m} g\rangle|^2 \le B\|f\|_{L^2}^2 \qquad \forall f \in L^2. \tag{1}$$

It follows from (1) that there exists a dual Gabor frame $\mathcal{G}(\tilde{g},\alpha,\beta)$ such that every $f \in L^2$ has the following L^2-convergent series expansion

$$f = \sum_{n,m\in\mathbb{Z}^d} \langle f, M_{\beta n} T_{\alpha m} \tilde{g}\rangle M_{\beta n} T_{\alpha m} g. \tag{2}$$

Given $1 \le p, q \le \infty$, the modulation space $M^{p,q}$ introduced by Feichtinger [23], is the Banach space of all $f \in \mathbb{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p\,dx\right)^{q/p} d\omega\right)^{1/q} < \infty,$$

with usual adjustments when p, or q is infinite. Moreover, Feichtinger and Gröchenig showed that every $f \in M^{p,q}(\mathbb{R}^d)$ has a Gabor expansion (2) with convergence in the $M^{p,q}$ norm, and, moreover, $A\|f\|_{M^{p,q}} \le \big(\sum_{n\in\mathbb{Z}^d}\big(\sum_{m\in\mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha m} g\rangle|^p\big)^{q/p}\big)^{1/q} \le B\|f\|_{M^{p,q}}$, for some $0 < A \le B < \infty$ [26, 27].

• **Wiener amalgam spaces.** For $1 \le p, q \le \infty$, the Wiener amalgam spaces can be intuitively thought as the spaces of functions that are locally L^p and globally ℓ^q, and were first used by Wiener [61] in his development of generalized harmonic analysis. More specifically, let $Q_\alpha = [0,\alpha)^d$ where $\alpha > 0$ and denote by χ_E the characteristic function of the set E. A measurable function on $\mathbb{R}^d$ belongs to the Wiener amalgam space $\mathcal{W}(L^p, \ell^q)$ if and only if

$$\|f\|_{\mathcal{W}(L^p,\ell^q)} = \big(\sum_{n\in\mathbb{Z}^d} \|f\cdot T_{\alpha n}\chi_{Q_\alpha}\|_{L^p}^q\big)^{1/q} < \infty.$$

• **Background on analysis on fractals.** Analytical tools such as the energy, the Laplacian, and harmonic functions were introduced by Kigami on the class of self-similar post-critically finite (p.c.f.) fractals which includes the Sierpinski gasket [45]; see also [3, 54]. More precisely, Kigami's theory is developed on certain fractals K, where K is p.c.f. self-similar set with boundary V_0, self-similar measure μ, and self-similar energy $\mathcal{E}$. In particular, assuming that K is embedded in some Euclidean space, K can be shown to be the invariant set for a contractive linear iterated function system $\{F_i\}_{i=1}^{n_0}$. Let q_i denote the fixed point of F_i and let $V_0 = \{q_i\}_{i=1}^{n_1}$ for some $n_1 \leq n_0$. The p.c.f. condition is that K is connected and $F_iK \cap F_jK \subset F_iV_0 \cap F_jV_0$ for $i \neq j$.

Time-Frequency Analysis of Function Spaces

Characterization of functions spaces by bases or frames which are simply generated such as Gabor or wavelet systems is essential in many applications as well as in approximation theory [21, 42]. For example, the Besov and Triebel-Lizorkin spaces are naturally characterized by wavelet bases [31], while as pointed out earlier, the modulation spaces are completely described by Gabor frames. However, no comprehensive comparison between time-scale and time-frequency analysis exists in the literature. More specifically, it is still unknown in general, whether the Besov or Triebel-Lizorkin spaces can be characterized by time-frequency methods or if the modulation spaces can be described by time-scale systems. For example, the Lebesgue spaces L^p which are examples of Triebel-Lizorkin spaces, are not modulation spaces if $p \neq 2$ [28]. Therefore, their characterization by Gabor frames was not expected. Nevertheless, in a joint work with K. Gröchenig, C. Heil, we proved that, by reinterpreting (2) as an iterated series, one could characterize the Wiener amalgam spaces of which the Lebesgue spaces are particular cases, by Gabor frames [41]. However, functions in these spaces are no longer characterized solely by the magnitude of their Gabor coefficients, but, rather, the phases of these coefficients are now important. As corollaries, we recover and extend results which were obtained in [2, 35, 40]. Amalgam spaces serve as the appropriate mathematical tool in sampling theory [1, 24] and play important role in the theory of Gabor frames [37].

While frame theory has numerous applications in engineering, it also generates some interesting questions in pure mathematics. For example, Feichtinger conjectured that any bounded (Gabor) frame can be decomposed into finite union of Riesz basic sequences. This conjecture is still unsolved and has been shown to be equivalent to the 1959 outstanding Kadison-Singer problem [14]. However, as mentioned earlier, it is the finite frames that one really implements in applications. Furthermore, the pioneering work of Benedetto and Fickus [6] in which normalized tight frames (i.e., finite frames of unit vectors) were completely characterized, ignited new investigations on finite frames. In general, for different applications in signal processing, or coding theory [12, 13, 15], finite frames with additionally structure are sought. For example, in a joint work with M. Fickus, B. D. Johnson, and K. Kornelson we completely characterized all (finite) tight frames with a convolutional structure [29], by showing that they are exactly the minimizers of the frame potential. Due to their connection with filter banks such frames have potential applications in signal processing, where fast algorithms for their implementation are available.

Regularity versus Time-Frequency Concentration

A distribution belongs to a modulation space if it has certain joint decay in time and in frequency. However, it is rather difficult to assert this from the definition of the modulation spaces. One can then ask how other properties of a distribution such as regularity, relate to its time-frequency content, and thus imply its membership into a particular modulation space. Consequently, it is of fundamental importance to investigate embeddings between Banach spaces of tempered distributions that measure regularity and the modulation spaces. Moreover, embeddings between modulation spaces and Besov and Sobolev spaces are not only also extremely important in studying Schatten properties of linear pseudodifferential operators [57], but they have also been used to formulate and prove new uncertainty relations [37]. We formulated and proved in [49], that certain Besov and Triebel-Lizorkin spaces ([53, 58]) embed into modulation spaces. The proofs of my results rely on a delicate use of the various equivalent norms defining the Besov and Triebel-Lizorkin spaces as well as a careful exploitation of the algebraic structure of the STFT. Two special cases of my results deserve specific mention. First, we gave a new proof of the embedding $\mathcal{C}^s(\mathbb{R}^d) \subset M^{\infty,1}(\mathbb{R}^d)$ for $s > d$ which was first proved in [43]. The interest in the modulation space, $M^{\infty,1}$ —also known as the Sjöstrand's algebra [52]— stems from the important role it has been playing in recent developments involving pseudodifferential operators on modulation spaces [39, 43, 56, 57]. Second, we also proved a conjecture of Feichtinger that the Sobolev space $L^1_2(\mathbb{R})$ is contained in $M^1(\mathbb{R})$. This last embedding is particularly useful due to the critical role that the Feichtinger algebra M^1 plays in the theory of Gabor frames, as it gives a fairly easy sufficient condition for membership in $M^1(\mathbb{R})$.

Time-Frequency Analysis of Fourier Multipliers

The pseudodifferential operator with a symbol $\sigma \in \mathbb{S}'(\mathbb{R}^{2d})$ is defined by the Kohn-Nirenberg correspondence as the operator K_σ mapping $\mathbb{S}$ into $\mathbb{S}'$, such that for $f \in \mathbb{S}$,

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\xi)\, \hat{f}(\xi)\, e^{2\pi i x\cdot\xi}\, d\omega. \tag{3}$$

These operators arise naturally not only in PDEs [30, 34, 53], but also in mobile communications [55], as well as in seismic image processing [33] . A basic but important question is to find conditions on the symbol σ so that the corresponding operator is bounded on certain function spaces. For instance, Caldéron-Vaillancourt [16] proved that if σ belongs to the Hörmander class $S^0_{0,0}$ i.e., if σ satisfies the following estimates

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta}, \qquad \forall\, \alpha, \beta \geq 0, \tag{4}$$

then K_σ is bounded on L^2. However, this condition does not guarantee the boundedness of K_σ on L^p for $p \neq 2$ [5]. Recently, the modulation space $M^{\infty,1}$ has been introduced as class of non-smooth symbols that yield bounded pseudodifferential operators on $M^{p,q}$ for $1 \leq p, q \leq \infty$, and, hence on L^2 [39, 43, 52, 56]. Notice that $S^0_{0,0}(\mathbb{R}^{2d}) \subset \mathcal{C}^s(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^{2d})$ if $s > 2d$, and so the last boundedness result recovers and extends Caldéron-Vaillancourt theorem. Furthermore $M^{\infty,1}$ is a Banach algebra under the so-called twisted convolution, and moreover, if $\sigma \in M^{\infty,1}$ is such that K_σ is invertible on $L^2(\mathbb{R}^d)$ then $K_\sigma^{-1} = K_{\tilde{\sigma}}$ for some $\tilde{\sigma} \in M^{\infty,1}$ [38, 52]. Additionally, for applications in signal processing or in seismic image data processing, it is more natural to impose time-frequency concentration conditions on the symbols of the pseudodifferential operators. These are some of the reasons why the modulation spaces are natural setting to study these operators.

A special case of pseudodifferential operator occurs when the symbol takes the form $\sigma(x,\xi) = m(\xi)$. In this case, $K_\sigma = H_m$ is a Fourier multiplier with symbol $m \in \mathbb{S}'(\mathbb{R}^d)$ and (3) reads $\widehat{H_m f}(\xi) = m(\xi)\, \hat{f}(\xi)$. These operators are typical models of filters or equivalently linear time invariant systems in signal processing. The complete characterization of the Fourier multipliers for the Lebesgue spaces is only known for L^1, L^2, and L^∞. Nonetheless, for $p \neq 1, 2, \infty$, a few *sufficient* conditions for boundedness of Fourier multipliers are known. For example, the Hömander multiplier theorem [34, 44, 53], is a typical result on the boundedness of Fourier multipliers on Lebesgue spaces. In view of the increasing role the modulation spaces play in time-frequency analysis and signal processing, it is not only important to characterize their Fourier multipliers, but also to understand the differences between their Fourier multipliers and those of other function spaces such as the Lebesgue space. In particular, in a joint work with A. Bényi, L. Grafakos, and K. Gröchenig we proved the boundedness on the modulation spaces of a class of Fourier multipliers that are generally unbounded on Lebesgue spaces and whose symbols are not in $M^{\infty,1}$ [11]. More specifically, using the atomic decomposition of modulation spaces by Gabor frames, we showed that the continuity of this class of Fourier multipliers reduces to the boundedness of the *discrete Hilbert transform.* As a byproduct of our results, we show that the Hilbert transform is bounded on all modulation spaces $M^{p,q}(\mathbb{R})$ with $1 < p < q$ and $1 \leq q \leq \infty$. Moreover, in an ongoing work with A. Bényi, K. Gröchenig, and L. Rogers, we are investigating the continuity of Fourier multipliers related to certain evolution equations such as the Schrödinger and the wave equations, as well as the problem of the ball multiplier on the modulation spaces. More precisely, we are interested in the family of Fourier multipliers whose symbols are given by $m_\alpha(\xi) = e^{i|\xi|^\alpha}, \xi \in \mathbb{R}^d$, and $\alpha \geq 0$. The cases $\alpha = 1$ and 2 are the multipliers corresponding to the wave and Schrödinger equations, respectively, and are known to be unbounded on Lebesgue spaces except when $p = 2$ or $d = 1$ [44, 47]. Similarly, Fefferman showed that if $m = \chi_B$ where B is the unit ball in $\mathbb{R}^d$, then H_m is unbounded on L^p if $p \neq 2$ and $d \geq 2$ [22]. Preliminary results indicate that for certain range of α, $m_\alpha \in M^{\infty,1}(\mathbb{R}^d)$, in which case the boundedness result is easily established. However, this is not a necessary condition. New insight about the Sjöstrand algebra is emerging from our investigation.

Time-Frequency Analysis of Multilinear Pseudodifferential Operators

An m-linear pseudodifferential operator with symbol $\sigma \in \mathbb{S}'(\mathbb{R}^{(m+1)d})$ is the mapping K_σ from $\mathbb{S}(\mathbb{R}^d) \times \cdots \times \mathbb{S}(\mathbb{R}^d)$ into $\mathbb{S}'(\mathbb{R}^d)$ given by

$$(5) \qquad \begin{aligned} &K_\sigma(f_1, \ldots, f_m)(x) \\ &= \int_{(\mathbb{R}^d)^m} \sigma(x, \xi_1, \ldots, \xi_m)\, \hat{f}_1(\xi_1) \cdots \hat{f}_m(\xi_m)\, e^{2\pi i x\cdot(\xi_1+\cdots+\xi_m)}\, d\xi_1 \cdots d\xi_m, \end{aligned}$$

for $f_1, \ldots, f_m \in \mathbb{S}(\mathbb{R}^d)$. The trivial case $\sigma \equiv 1$ corresponds to the pointwise product $f_1 \cdots f_m$, and Hölder's inequality regulates its boundedness on Lebesgue spaces.

Multilinear pseudodifferential operators are being extensively investigated not only due to their many applications to linear and non-linear partial differential equations, but also due to the deep result of Lacey and Thiele on the boundedness of the bilinear Hilbert transform [18, 36, 46, 48]. Finding conditions on the symbols of such operators that guarantee their boundedness on products of certain Banach spaces is still being thoroughly investigated. In particular, assume that $m = 2$ and that the symbol σ of the bilinear pseudodifferential operator belongs to the (bilinear) Hörmander class $S^0_{0,0}$, i.e.,

$$(6) \qquad |\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma(x, \xi, \eta)| \leq C_{\alpha,\beta,\gamma}, \qquad \forall\, \alpha, \beta, \gamma \geq 0.$$

It was shown in [8] that (6) does not imply the boundedness of the corresponding operator from $L^2 \times L^2$ into L^1, unless additional size conditions are imposed on σ. However, as mentioned earlier for the linear case ($m = 1$), condition (6) (or equivalently condition (4)) implies that the symbol $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ which is sufficient for the boundedness of K_σ on all modulation spaces. In a joint work with A. Bényi, we introduced the modulation spaces as class of symbols for bilinear pseudodifferential operators. In particular, using the atomic decomposition of modulation spaces via Gabor frames, we proved in [7] that the Feichtinger algebra M^1 is a class of non-smooth symbols that yield bounded bilinear pseudodifferential operators. Moreover, A. Bényi, K. Gröchenig, C. Heil, and I proved Calderón-Vaillancourt-type theorems for multilinear pseudodifferential operators with non-smooth symbols on modulation spaces [10]. In particular, we showed that symbols satisfying (6) map $L^2 \times L^2$ into a modulation-like space slightly larger than L^1. Further extensions were obtained in my joint work with A. Bényi [9]. Moreover, in a recent work, E. Cordero and I gave various examples of multilinear pseudodifferential operators with non-smooth symbols in $M^{\infty,1}$ [17]. In addition, we proved that for a large class of multilinear pseudodifferential operators, the condition that their symbols σ belong to $M^{\infty,1}$ is not only sufficient for their boundedness, but it is also a necessary condition.

Analysis on fractals

Some problems of classical analysis on Euclidean spaces such as differential equations and Fourier series expansions have been considered extensively on p.c.f. fractals, see [20]. In particular, and similarly to Fourier series on the circle, the solutions $\{u_j\}_{j=1}^{\infty}$ of the following eigenvalue problem

$$(7) \qquad \begin{cases} -\Delta u_j &= \lambda_j u_j \\ u_j|_{V_0} &= 0, \end{cases}$$

form an orthonormal basis of Dirichlet eigenfunctions with eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ of the Laplacian on K. Consequently, any $u \in L^2(K)$ has a Fourier series expansion. Moroever, the structure of the spectrum of the Laplacian on the Sierpinski gasket (SG) and other pcf fractals was completely described by Fukushima and Shima [32] (see also [60]) , who also proved that certain of these eigenvalues have very high multiplicity. Another example of a classical result of analysis which holds in the fractal setting is given in my joint work with R. S. Strichartz [50]. More precisely, we established a weak analogue of Heisenberg's Uncertainty Principle for functions defined on the Sierpinski gasket. Although the existence of localized eigenfunctions —these are eigenvalues which are highly localized in space and in frequency— on the Sierpinski gasket ([4, 32]) precludes an uncertainty principle in the vein of Heisenberg's inequality, we nonetheless proved that a function that is localized in space must have high energy, and hence have high frequency components. Moreover, in a joint work with A. Teplyaev [51], we proved that such weak uncertainty principle extends to a wide class of metric measure spaces, including p.c.f. and non p.c.f. fractals (such as the Sierpinski carpet), as well as some finite and infinite graphs. Furthermore, as a byproduct of our results we obtain a new proof of the classical Heisenberg's uncertainty principle.

Future research

The characterization of function spaces by coherent states such as Gabor or wavelet systems is essential in many applications. In particular, a lot of work has been done to understand the properties of the Besov and Triebel-Lizorkin spaces using wavelet analysis as well as the modulation spaces using Gabor analysis. However, it still unknown in general, what summability properties the Gabor coefficients of functions in a Besov or Triebel-Lizorkin space have, or in which sequence space lie the wavelet coefficients of functions in a modulation space. In ongoing work I have started what I think is the first step in answering such questions. More specifically, I am investigating a characterization of the Besov and Triebel-Lizorkin spaces by Gabor frames, as well as a description of the modulation spaces by wavelet bases.

I am also pursuing my work on the theory of finite frames. In particular, in ongoing work with B. D. Johnson, we are investigating a characterization of finite tight frames over finite groups by means of the frame potential. Such results will certainly have some applications for which frames with certain geometric properties (such as frames generated by a set of rotations matrices) are needed.

I also plan to work on issues related to the optimality of certain of the embedding results I proved in my previous work. The techniques I have used to obtain those results rely on the availability of various equivalent norms on the Triebel-Lizorkin and Besov spaces as well as the algebraic properties of the STFT. However, I think any optimal result in this direction must be stated in terms of sequence spaces associated to the function spaces under consideration. Therefore, I plan on investigating these embeddings at the sequence space level. This investigation would provide another comparison between time-frequency and time-scale analysis.

The increasingly important role that the modulation spaces have been playing in the analysis of (multilinear) pseudodifferential operators, suggests that one should investigate their role in the theory of PDEs. This is further exemplified by our ongoing work on the continuity of Fourier multipliers related to certain evolution PDEs on modulation spaces. I plan on pursuing this investigation with the goal of analyzing PDEs in the realm of the modulation spaces. Moreover, I seek to understand the coninuity properties of (multilinear) singular integral operators in this setting. Furthermore, in ongoing joint work with A. Bényi, K. Gröchenig and L. Rogers, we are investigating the continuity properties of the ball multiplier on modulation spaces. Our long term goal is to revisit the question of summation of spherical Fourier series in the context of modulation spaces.

Finally, I plan on pursuing my work on issues related to certain differential equations involving the Laplacian on fractals. I am currently investigating the possible use of some techniques from finite frame theory to construct frames with good "time-frequency" localization properties (i.e., frames whose grammian matrices have good off-diagonal decay) for the eigenspaces corresponding to eigenvalues of high multiplicity. Constructing such frames will lead to a better understand of sampling theory in the fractal setting. Additionally, in ongoing work R. S. Strichartz and I are studying the asymptotic behavior of the spectrum of certain Hamiltonian on the Sierpinski gasket. Finally, I plan to develop a theory of pseudodifferential operators in this fractal setting.

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