Homework 4. Solutions.

It's really not possible to type the solutions to all problems because of the number of formulas. So I only give the hints and sketches of the solutions.

Problem 2.4. Hint. Use the fact that the integral of an odd function over an even interval iz equal to 0.

Problem 2.5. Hint. Note that the integral you want to compute is the area of a certain rectangle with heigth 1; its width is equal to x, if $0 \leq x \leq 1$, and to $2-x$, if $1 \leq x \leq 2$.

Problem 2. In this problem, the main step is to compute the derivative of $F(\lambda) = \int_{-\infty}^{+\infty} e^{-x^2} e^{-i\lambda x} dx$. To do this, one would like to change the order of the differentiation and integration; one may use the following

Theorem. Suppose g(x,y) is a continous function on R^2, and $g'_x(x,y)$ is also a continuous function. Assume that the integrals $G(x) = \int_{-\infty}^{+\infty} g(x,y)dy$ and $D(x) = \int_{-\infty}^{+\infty} g'_x(x,y)dy$ converge *uniformly* in x. Then $G'(x) = D(x)$.

The condition of uniform convergence of the improper integrals means that if $R(x,r) = \int_{-\infty}^{-r} g(x,y)dy + \int_{r}^{+\infty} g(x,y)dy$, $R_1(x,r) = \int_{-\infty}^{-r} g'_x(x,y)dy + \int_{r}^{+\infty} g'_x(x,y)dy$, then for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) > 0$, such that $R(x,r) < \varepsilon$, $R_1(x,r) < \varepsilon$ for *any* x and any $r > r_0$ ($uniformity$ means that $r_0(\varepsilon)$ can be chosen one and the same for all x).

The conditions of the Theorem above should be checked for the function $g(x,y) = e^{-y^2} e^{-ixy}$. As soon as it is done (use the fact that $|e^{-y^2} e^{-ixy}| = |e^{-y^2}|$), one can claim that $F'(\lambda) = \int_{-\infty}^{+\infty} [e^{-x^2} e^{-i\lambda x}]'_\lambda dx = \int_{-\infty}^{+\infty} (-ix) e^{-x^2} e^{-i\lambda x}$. The latter integral could be computed, using change of varibles $u = x + i\lambda/2$ and the fact that ue^{-u^2} has a primitive function $(-1/2)e^{-u^2}$. Everything else in this problem is just simple computation.

Remark. The solutions of this problem I have seen either didn't explain why derivative could be taken inside the integral, or gave a very unsatisfying explanation. I think it's important to understand what theorem you are using here and what are the conditions for it to work.

Problem 3. Part b) and c) are mostly the application of the facts about the Fourier transform given in the textbook.

Part a). I'll show $f_2 \in L^1(\mathbb{R})$, $||f_2||_{L^1} = ||f||_{L_1}$ (solution for other functions is similar). We have $\int_{-\infty}^{+\infty} f_2 dx = lim_{r_1, r_2 \to \infty} \int_{-r_1}^{r_2} f_2(x)dx = lim_{r_1, r_2 \to \infty}, \int_{-r_1-b}^{r_2+b} f(x)dx = \int_{-\infty}^{+\infty} f(x)dx - lim_{r_1, r_2 \to \infty} [\int_{-\infty}^{-r_1-b} f_2 dx + \int_{r_2-b}^{+\infty} f_2 dx]$. But the latter limit is equal to 0, since f is in $L^1(\mathbb{R})$. So $\int_{-\infty}^{+\infty} f_2 dx = \int_{-\infty}^{+\infty} f dx$, and we are done.