WEAK UNCERTAINTY PRINCIPLES ON FRACTALS

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ABSTRACT. We use the analytic tools such as the energy, and the Laplacians defined by Kigami for a class of post-critically finite (pcf) fractals which includes the Sierpinski gasket (SG), to establish some uncertainty relations for functions defined on these fractals. Although the existence of localized eigenfunctions on some of these fractals precludes an uncertainty principle in the vein of Heisenberg's inequality, we prove in this paper that a function that is localized in space must have high energy, and hence have high frequency components. We also extend our result to functions defined on products of pcf fractals, thereby obtaining an uncertainty principle on a particular type of non-pcf fractal.

1. INTRODUCTION

The uncertainty principle in harmonic analysis can be seen as a manifestation of the fact that it is impossible for a nonzero function and its Fourier transform to both be sharply localized. More precisely, defining the Fourier transform of a function $f \in L^1(\mathbb{R})$ (with the usual extension to $L^2(\mathbb{R})$ functions) by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx,$$

a quantitative statement of the above fact is the classical Heisenberg's inequality which asserts that for all $f \in L^2(\mathbb{R})$

$$\inf_{a\in\mathbb{R}}\left(\int_{\mathbb{R}} (x-a)^2\, |f(x)|^2\, dx\right)\, \inf_{b\in\mathbb{R}}\left(\int_{\mathbb{R}} (\xi-b)^2\, |\hat{f}(\xi)|^2\, d\xi\right) \geq \frac{\|f\|_2^4}{16\pi^2}. \tag{1}$$

Moreover, equality holds in (1) if and only if $f(x) = C\, e^{2\pi i b x}\, e^{-\gamma(x-a)^2}$ for some constants $C \in \mathbb{C}$ and $\gamma > 0$. We refer to [4] and the references therein for a survey of uncertainty principles and related results. To see how the above inequality relates to localization properties of functions, note that if $f \in L^2(\mathbb{R})$ with $\|f\|_2 = 1$ then by Plancherel's equality $|f(x)|^2\, dx$ and $|\hat{f}(\xi)|^2\, d\xi$ may be thought as probability measures on $\mathbb{R}$. Thus, $Var(|f|^2) = \int_{\mathbb{R}}(x-\mu_f)^2\, |f(x)|^2\, dx$, where μ_f is the mean of the probability measure $|f(x)|^2\, dx$, achieves the first infimum on the left-hand side of (1), and measures the concentration of f around its mean. Similarly, $Var(|\hat{f}|^2)$ achieves the second infimum on the left-hand side of (1) and measures the concentration of $\hat{f}$

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around its mean. Therefore, it is clear that a function and its Fourier transform cannot both be highly concentrated around any points, since (1) prevents $Var(|f|^2)$ and $Var(|\hat{f}|^2)$ to both be arbitrarily small. Other variant of the Heisenberg's inequality have been obtained by different authors, see [20, 11, 12].

A non symmetric version of (1) exists on the interval $[0, 2\pi]$, where the role of the Fourier transform is replaced by the Fourier series. More precisely, let $u \in L^2([0, 2\pi])$ such that $\|u\|_2 = 1$. Then u can be expand into its L^2 convergent Fourier series, i.e.,

$$u(x) = \sum_{-\infty}^{\infty} a_k \, e^{ikx} \qquad \text{where} \qquad a_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) \, e^{-ikx} \, dx,$$

and moreover, $\sum_{k=-\infty}^{\infty} |a_k|^2 = \|u\|_2^2$.

Let $u \in L^2([0, 2\pi])$ with $\|u\|_2 = 1$. Following [8] define the first trigonometric moment of u by

$$\tau(u) := \frac{1}{2\pi} \int_0^{2\pi} e^{ix} \, |u(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} a_k \bar{a}_{k+1},$$

its angle variance by

$$Var_A(u) := \frac{1 - |\tau(u)|^2}{|\tau(u)|^2} = \frac{1}{\left(|\sum_{k=-\infty}^{\infty} a_k \, \bar{a}_{k+1}|\right)^2} - 1,$$

and its frequency variance by

$$Var_F(u) := \|u'\|_2^2 + \langle u', u\rangle^2 = \sum_{k=-\infty}^{\infty} k^2 \, |a_k|^2 - \Big(\sum_{k=-\infty}^{\infty} k \, |a_k|^2\Big)^2,$$

where we assume that u is smooth enough for the last definition to make sense. In particular, this will be the case if we assume that $u \in L^2([0, 2\pi])$ is absolutely continuous and that $u' \in L^2([0, 2\pi])$. The following uncertainty principle for periodic functions first appeared in [8]: For all $u, u' \in L^2([0, 2\pi])$ such that $\|u\|_2 = 1$ and such that u is absolutely continuous,

$$Var_A(u) \, Var_F(u) > \frac{1}{4}. \tag{2}$$

Moreover, it was observed in [10] that the lower bound in (2) is not attained by any function, but is the best possible. We refer to [3] for a physical interpretation of the variances appearing in (2) and to [10, 8, 9, 13, 14] for more background on this periodic uncertainty principle.

It is clear that the two versions of the uncertainty principle we have just stated bear some resemblances despite the fact that the underlying spaces on which the functions are defined are different.

In view of the recent interest that analysis on fractals has been drawing, it is quite natural to ask how much of the "classical" analysis on Euclidean spaces can be extended to the fractal setting. Our goal in this paper is to show to what extent an uncertainty principle of the above type holds for functions defined on certain fractals. However, Fukushima and Shima [5], and Barlow and Kigami [2] have shown that on

SG and other pcf fractals, there exist localized eigenfunctions, i.e., eigenfunctions that are completely localized in frequency and highly localized in space. This clearly indicates that inequalities analogous to (1) and (2) cannot hold in this fractal setting. Nevertheless, we prove some uncertainty relations for functions defined on pcf fractals —of which the Sierpinski gasket will be our model— on which analytic tools such as the energy, the Laplacian, and harmonic functions have been defined and extensively studied. In particular, we will show that a slightly weaker uncertainty estimate holds: if a function defined on a pcf fractal is localized in space then there must be very high frequencies involved. We call this phenomenon a *weak uncertainty principle.*

We briefly describe here the pcf fractals that will be considered in the sequel, and we refer to [1, 6, 19] and the references there for more background on analysis on fractals.

The fractal K to be considered below, is a pcf self-similar set with boundary V_0, self-similar measure μ, and self-similar energy $\mathcal{E}$. Moreover, assuming that K is embedded in some Euclidean space, K can be shown to be the invariant set for a contractive linear iterated function system (ifs) $\{F_i\}_{i=1}^{n_0}$. Let q_i denote the fixed point of F_i and let $V_0 = \{q_i\}_{i=1}^{n_1}$ for some $n_1 \leq n_0$. The pcf condition is that K is connected and

$$F_i K \cap F_j K \subset F_i V_0 \cap F_j V_0 \qquad \text{for} \qquad i \neq j.$$

$F_i K$ is referred to as a cell of level 1. If we let $\omega = (\omega_1, \ldots, \omega_m)$ be a word of length $|\omega| = m$, and $F_\omega = F_{\omega_m} \circ \ldots \circ F_{\omega_1}$, then $F_\omega K$ is said to be a cell of level m. We also note that the self-similar probability measure μ is determined by the choice of probability weight $\{\mu_i\}_{i=1}^{n_0}$ by

$$\mu = \sum_{i=1}^{n_0} \mu_i \, \mu \circ F_i^{-1} \qquad \text{or equivalently} \qquad \int_K f \, d\mu = \sum_{i=1}^{n_0} \mu_i \int_K f \circ F_i \, d\mu.$$

We can also define K as the limit of graphs Γ_m with vertices in V_m and edge relation $x \sim_m y$ defined inductively as follows: Γ_0 is the complete graph V_0, and $V_m = \cup_{i=1}^{n_0} F_i V_{m-1}$ and $x \sim_m y$ if and only if x and y belong to the same cell $F_\omega K$ of level m.

For $i = 0, 1, \ldots, n_0$, let $r_i > 0$ be given, and denote $r_\omega = r_{\omega_1} r_{\omega_2} \ldots r_{\omega_m}$ for a word ω of length m. Assume that for a function u defined on V_0 we are given the energy form

$$\mathcal{E}_0(u, u) = \sum_{i<j} c_{ij} \left(u(q_i) - u(q_j)\right)^2, \tag{3}$$

where the coefficients c_{ij} are nonnegative, and are such that $\mathcal{E}_0(u, u) = 0$ if and only if u is constant on V_0. We can now define an energy form for functions u defined on V_m (graph energy at level m) by

$$\mathcal{E}_m(u, u) = \sum_{|\omega|=m} r_\omega^{-1} \, \mathcal{E}_0(u \circ F_\omega, u \circ F_\omega). \tag{4}$$

If u is defined on V_0, we call $\tilde{u}$ *the harmonic extension* to V_m if it minimizes $\mathcal{E}_m$. The main assumption is that $\mathcal{E}_m(\tilde{u}, \tilde{u}) = \mathcal{E}_0(u, u)$, which holds once we check it for $m = 1$. In this case, $(\mathcal{E}_0, \{r_i\})$ is called a *harmonic structure.* Moreover, if $0 < r_i < 1$ for all i then $(\mathcal{E}_0, \{r_i\})$ is called a *regular harmonic structure*. See [6] for more details. Consequently, it can be easily seen that the sequence $\{\mathcal{E}_m(u,u)\}_m$ is an increasing sequence of nonnegative real numbers for any u defined on K. Thus

$$\mathcal{E}(u,u) = \lim_{m\to\infty} \mathcal{E}_m(u,u) \tag{5}$$

exists as an extended real number. We define $dom\mathcal{E}$ to be the set of all continuous functions on K for which the above limit is finite. It can be shown that $dom\mathcal{E}$ modulo the constant functions is a Hilbert spaces with norm $\mathcal{E}(u,u)^{1/2}$, and with corresponding inner product given by $\mathcal{E}(u,v) = \lim_{m\to\infty} \mathcal{E}_m(u,v)$, for $u, v \in dom\mathcal{E}$. The energy also satisfies the following self-similarity relation

$$\mathcal{E}(u,u) = \sum_{|\omega|=m} r_\omega^{-1}\, \mathcal{E}_u(u\circ F_\omega, u\circ F_\omega), \tag{6}$$

where the sum is taken over all words of length m. It should be noted that the existence of regular harmonic structures is a difficult problem. Nevertheless there are a lot of known examples.

With the energy comes a distance, which, in a sense, is intrinsic to the fractal, called *the effective resistance metric* on K, and defined by

$$d_R(x,y) = \Big(\min\{\mathcal{E}(u,u) : u(x) = 0 \,\text{and}\, u(y) = 1\}\Big)^{-1} \quad x, y \in K; \tag{7}$$

we refer again to [6] for more details about this metric. It is worth noticing that in the effective resistance metric, the cell $F_\omega K$ has diameter equivalent to r_ω.

Harmonic functions form an n_1-dimensional space $\mathcal{H}_0$, and are obtained by assigning values in V_0, and taking the minimum energy extension at each level, and extending from $V_* = \cup_{m=0}^{\infty} V_m$ to K by continuity. Moreover, $\mathcal{E}_m(u,u)$ is independent of m if u is harmonic.

Finally we can define a Laplacian Δ with domain $dom\Delta$ as follows: $u \in dom\Delta$ with $\Delta u = f$ for $u \in dom\mathcal{E}$ and $f \in \mathcal{C}(K)$ if

$$-\mathcal{E}(u,v) = \int_K f\, v\, d\mu \quad \text{for all} \quad v \in dom_0\mathcal{E},$$

where $dom_0\mathcal{E}$ denotes the subspace of $dom\mathcal{E}$ of functions vanishing on the boundary V_0. Equivalently, the Laplacian Δ can be defined pointwise as follows

$$\Delta u = \lim_{m\to\infty} \Delta_m u, \quad \text{on} \quad V_* = \cup_{m\geq 0} V_m$$

where the limit is uniform in a precise sense, and where Δ_m is a graph Laplacian defined by

$$\Delta_m u(x) = \left(\int_K \psi_x^{(m)}\, d\mu\right)^{-1} \sum_{x\sim_m y} r_\omega^{-1}\,(u(x) - u(y)),$$

where $\psi_x^{(m)}$ is the piecewise harmonic function of level m equal to 1 at x and 0 at all other vertices in V_m.

Our paper is organized as follows. In Section 2 we introduce all the measures of localization that will be used to state and prove our results. Moreover, we will prove a lemma which suggests why a weak uncertainty principle must hold on pcf fractals. In Section 3 we prove our first main result which establishes a "weak" uncertainty principle on (finite) pcf fractals. Additionally, we will extend this result to finite and infinite blowups of pcf fractals. Finally, in Section 4, we use the results of the previous sections to formulate and prove an uncertainty principle on products of pcf fractals. Since the product is a self-similar fractal which is not pcf, our result in this setting may be seen as a first attempt to generalize our weak uncertainty principles to non-pcf fractal. However, it remains to be seen whether our results generalized to "genuine" non-pcf fractals such as the Sierpinski carpet.

2. Preliminaries and motivations

The following expression of the energy function will play a key role in stating and proving our uncertainty principles. Remember that V_0 is the boundary of our pcf fractal K. Let $u \in L^2(K)$ with $\|u\|_2 = 1$ and assume that u vanishes on V_0. Let $\{u_j\}_{j=1}^{\infty}$ be a complete set of orthonormal Dirichlet eigenfunctions with (strictly positive) eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ arranged in an increasing order. More precisely,

$$\begin{cases} -\Delta u_j &= \lambda_j u_j \\ u_j|_{V_0} &= 0. \end{cases}$$

Then

$$u = \sum_{j=1}^{\infty} a_j(u)\, u_j \qquad \text{with} \qquad a_j(u) = \int_K u\, u_j\, d\mu.$$

Moreover, $\sum_{j=1}^{\infty} |a_j(u)|^2 = 1$ and in [15, Theorem 3.7.d] it was shown that

$$\mathcal{E}(u,u) = \sum_{j=1}^{\infty} \lambda_j |a_j(u)|^2. \tag{8}$$

Note that by considering $\{|a_j(u)|^2\}_{j=1}^{\infty}$ as a probability measure on $\mathbb{Z}^+$, (8) is exactly the average frequency in the expansion of u. Thus, if $\mathcal{E}(u,u)$ is large this implies that u has significant high frequency components.

We now propose a measure of spatial concentration for functions defined on a pcf fractal K. Observe that if $d\mu$ is a probability measure on $\mathbb{R}$, then its variance is

$$\begin{aligned} Var(|\mu|^2) &= \int_{\mathbb{R}} \left(x - \int_{\mathbb{R}} y\, d\mu(y) \right)^2 d\mu(x) \\ &= \int_{\mathbb{R}} x^2\, d\mu(x) - \left(\int_{\mathbb{R}} x\, d\mu(x) \right)^2 \\ &= \frac{1}{2} \iint_{\mathbb{R}\times\mathbb{R}} (x-y)^2\, d\mu(x)\, d\mu(y). \end{aligned}$$

Since $(x-y)^2 = |x-y|^2$ is the Euclidean distance between two random points x and y, we are tempted to use the last equality to define the measure of spatial localization for a function on K, with the Euclidean metric replaced by the effective resistance metric d_R. However, we also need to adjust the power on the distance as shown in

Lemma 1 below. Thus we define the variance of a function $u \in L^2(K)$ with $\|u\|_2 = 1$ by

$$Var_d(|u|^2) = \iint_{K\times K} d_R(x,y)^{d+1}|u(x)|^2\,|u(y)|^2\,d\mu(x)\,d\mu(y), \tag{9}$$

where d is the Hausdorff dimension of K with respect to the resistance metric (note that this is not the same as the Hausdorff dimension with respect to the Euclidean metric).

Remark 1. Note that it easy to see that for all $u \in L^2(K)$ with $\|u\|_2 = 1$,

$$\inf_{y\in K}\int_K d_R(x,y)^{d+1}\,|u(x)|^2\,dx \le Var_d(|u|^2).$$

Moreover, let $m = \inf_{y\in K}\int_K d_R(x,y)^{d+1}\,|u(x)|^2\,dx$. Then for all $\epsilon > 0$ there exists $y_0 \in K$ such that

$$\int_K d_R(x,y_0)^{d+1}\,|u(x)|^2\,dx \le m+\epsilon.$$

Note also that there exists $C > 0$ such that

$$d_R(x,y)^{d+1} \le C\big(d_R(x,y_0)^{d+1} + d_R(y_0,y)^{d+1}\big) \quad \forall\, x,y \in K.$$

Consequently, it is straightforward to show that

$$Var_d(|u|^2) \le Cm = C\inf_{y\in K}\int_K d_R(x,y)^{d+1}\,|u(x)|^2\,dx.$$

Thus, for $u \in L^2(K)$ with $\|u\|_2 = 1$

$$Var_d(|u|^2) \sim \inf_{y\in K}\int_K d_R(x,y)^{d+1}\,|u(x)|^2\,dx,$$

and we will use either of these expressions as measure of spatial localization. This will be particularly useful when proving the uncertainty principle on products of fractals.

Remark 2. Note that if $u \in L^2(K)$ with $\|u\|_2 = 1$, then

$$Var_d(|u|^2) \le diam(K),$$

where $diam(K)$ is the diameter of K in the resistance metric.

The next lemma can be seen as a first quantitative formulation of the fact that if a function is highly localized in space, then there must be very high frequencies involved.

Lemma 1. a. *If u is a function supported in a cell $F_\omega K$ such that $\|u\|_2 = 1$ then*

$$r_\omega\mu_\omega\mathcal{E}(u,u) \ge \lambda_1,$$

with equality if and only if $u = \pm u_1$, and where λ_1 is the first Dirichlet eigenvalue of the Laplacian.

b. *If u is a function supported on two adjacent cells $F_{\omega_1}K$ and $F_{\omega_2}K$ such that $\|u\|_2 = 1$ then*

$$r_\omega \mu_\omega \mathcal{E}(u,u) \geq \lambda_1,$$

with equality if and only if $u = \pm u$, and where λ_1 is the first Dirichlet eigenvalue of the Laplacian on the union of two copies of K glued together at a boundary point.

Proof. a. Assume that $u|_{V_0} = 0$, then $\mathcal{E}(u,u) = \sum_{k=1}^{\infty} \lambda_k |a_k|^2$ where the sequence a_k is given by the expansion of u in the orthonormal basis of eigenvalues of the Dirichlet Laplacian on K, i.e., $u = \sum_{k=1}^{\infty} a_k u_k$, and thus by (8)

$$\mathcal{E}(u,u) \geq \lambda_1 \sum_{k=1}^{\infty} |a_k|^2 = \lambda_1.$$

Assume now that u is supported on a cell $F_\omega K$, then $u \circ F_\omega$ vanishes on V_0, and thus by the above arguments $\mathcal{E}(u \circ F_\omega, u \circ F_\omega) \geq \lambda_1 \|u \circ F_\omega\|_2^2$. Using the self-similarity of the energy function and of the measure μ, the last inequality becomes

$$r_\omega \mathcal{E}(u,u) \geq \frac{\lambda_1}{\mu_\omega} \|u\|^2 = \frac{\lambda_1}{\mu_\omega},$$

which concludes the proof of a.

b. The proof is similar to the previous one and relies on the fact that the eigenvalues of the Dirichlet Laplacian on the fractafolds obtained by gluing together two copies of K at a boundary point, see [17], are strictly positive. □

Remark 3. It is natural to relate the resistance contraction factors r_i and the measure contraction factors μ_i by the identity $\mu_i = r_i^d$ where d is the unique number determined by the condition $\sum_{i=1}^{n_0} \mu_i = \sum_{i=1}^{n_0} r_i^d = 1$; in fact d is just the Hausdorff dimension K. Assuming this, then $r_\omega \mu_\omega = r_\omega^{d+1}$. Consequently, if u is supported on $F_\omega K$, Lemma 1 suggests that the variance of u must be $r_\omega \mu_\omega = r_\omega^{d+1}$, and this justifies the power $d+1$ in (9).

From now on, we will assume that the resistance contraction factors r_i and the measure contraction factors are related through the identity $\mu_i = r_i^d$ where d is defined as above.

The next result will also play a key role in our proof. To get some intuition about its meaning, we consider the case of the unit interval $I = [0,1]$; harmonic functions are just linear functions, and what the result says is that if f is a (continuous) function on I with $f(0) = a$, and if $|f(x)| \leq \frac{|a|}{2}$ for some $x \in I$, then the energy in f is greater than or equal to the energy of the harmonic function which assumes the same values as f at 0 and x.

Lemma 2. *Suppose u is a finite energy function on K with $u(x) = a$ and such that $|u(y)| \leq \delta |a|$, for some $x, y \in K$ and $\delta \in (0,1)$. Then there exists a constant $C = C(\delta) > 0$ such that $\mathcal{E}(u,u) \geq C\, a^2$.*

Proof. Let $b = u(y)$, and note that $\mathcal{E}(u,u) = \mathcal{E}(u-b, u-b)$. Moreover, $|b-a| \geq (1-\delta)|a|$. Let $v = \frac{u-b}{a-b}$. Then

$$\mathcal{E}(v,v) = \frac{1}{(a-b)^2}\,\mathcal{E}(u,u) \leq \frac{1}{(1-\delta)^2}\,\frac{\mathcal{E}(v,v)}{a^2}.$$

However, because $v(x) = 1$ and $v(y) = 0$, it follows from the definition of the resistance metric d_R that $\mathcal{E}(v,v) \geq (d_R(x,y))^{-1}$. Consequently,

$$\mathcal{E}(u,u) \geq (1-\delta)^2\,(d_R(x,y))^{-1}\,a^2.$$

Because K has a finite diameter in the effective resistance metric, we conclude that there exists $C > 0\,(C = \text{diam}(K)^{-1}(1-\delta)^2)$, such that $\mathcal{E}(u,u) \geq C\,a^2$. □

Remark 4. In the sequel, and for simplicity, we will use Lemma 2 with $\delta = 1/2$.

3. Weak uncertainty principle on pcf fractals

3.1. **Main result.** Armed with the above lemmas, we can now state our first main result, which we only prove for the Sierpinski gasket $K = SG$, as this will make the proofs more transparent and thus less technical. In particular, we will choose $r_i = \frac{3}{5}$ and $\mu_i = \frac{1}{3}$, for $i = 1, 2, 3$. Note that in this case $d = \log 3/\log(5/3)$. However, nothing in the proofs is peculiar to SG and so they extend to all pcf fractals with obvious modifications.

Since our goal is to show that if a function has a "small" concentration in space, then there must be some large frequencies, we will assume that the variance of functions under considerations are "small" enough. This will be made very precise below.

Theorem 1. *There exists a positive constant C such that for all $u \in dom\mathcal{E}$ with $\|u\|_{L^2(K)} = 1$ and $Var_d(|u|^2) \leq \frac{1}{2}$ we have*

$$Var_d(|u|^2)\,\mathcal{E}(u,u) \geq C. \tag{10}$$

Proof. Let m be the smallest integer such that

$$\frac{3}{5}\,Var_d(|u|^2) < r_\omega^{d+1} \leq Var_d(|u|^2), \tag{11}$$

where ω is a word of length m. The above inequalities are derived by decomposing K into union of level 1 cells, i.e., $F = \cup_{i=1}^{3} F_i K$, and checking if (11) holds. If the inequalities fail we iterate the decomposition until the inequalities hold. For general pcf fractal we can use a "stopping time" argument to arrive at the result; however, the lengths of ω in the decomposition vary from cell to cell. Note that (11) implies that $Var_d(|u|^2) \leq \frac{5}{3}\,r_\omega^{d+1}$.

Note that decomposition (11) yields a decomposition of the fractal K into disjoint cells of level m, i.e., $K = \cup_{\omega \in W_m} F_\omega K$, where the union is taken over all words of length m. This yields,

$$1 = \|u\|_2^2 = \sum_{\omega'} \int_{F_{\omega'}K} |u(x)|^2\,dx = \sum_{\omega'} I_{\omega'},$$

where $I_\omega = \int_{F_\omega K} |u(x)|^2\,dx$. Let $\eta = \max_{\omega'} I_{\omega'}$. We want to show that (11) imposes a lower bound on η.

Note that for each word ω of length m there exists a unique cell $F_{\omega_{m-1}}K$ of level $m-1$ containing $F_\omega K$. For words ω and ω' of length m, write $\omega' \simeq \omega$ if $F_{\omega'}K$ belongs to the cell $F_{\omega_{m-1}}K$ of level $m-1$ containing $F_\omega K$, or if it belongs to one of the three cells of level $m-1$ that intersect $F_{\omega_{m-1}}K$. For a fixed ω_0 there are at most 12 choices of ω' such that $\omega_0 \simeq \omega'$. Thus

$$\sum_{\omega' \not\simeq \omega_0} I_{\omega'} \geq 1 - 12\eta,$$

and moreover, for $\omega' \not\simeq \omega_0$, $x \in F_{\omega_0}K$ and $y \in F_{\omega'}K$ we have $d_R(x,y) \geq A\, r_\omega$, where $A = 5/3$. This estimate follows from the definition of the relation $\simeq$, which implies that there is at least one cell of diameter $r_{\tilde\omega}$ separating $F_{\omega_0}K$ and $F_{\omega'}K$, where $\tilde\omega$ is a word of length $m-1$. Consequently

$$\begin{aligned}
\frac{5}{3}\, r_\omega^{d+1} &\geq Var_d(|u|^2) \\
&= \sum_\omega \sum_{\omega'} \int_{F_\omega K} \int_{F_{\omega'} K} d_R(x,y)^{d+1}\, |u(x)|^2\, |u(y)|^2\, dx\, dy \\
&\geq A^{d+1} \sum_{\omega' \not\simeq \omega} r_\omega^{d+1}\, I_\omega\, I_{\omega'}.
\end{aligned}$$

So $A^{d+1} \sum_{\omega' \not\simeq \omega} I_\omega\, I_{\omega'} \leq \frac{5}{3}$.

Fixing ω and summing over ω' with $\omega' \not\simeq \omega$ gives

$$\sum_{\omega' \not\simeq \omega} I_{\omega'}\, I_\omega \geq (1 - 12\eta) I_\omega.$$

Summing now over all ω and recalling that $\sum_\omega I_\omega = 1$ yields

$$\frac{5}{3} \geq A^{d+1} \sum_\omega \sum_{\omega' \not\simeq \omega} I_\omega\, I_{\omega'} \geq A^{d+1}\, (1 - 12\eta),$$

hence $\eta \geq \frac{1}{12}\,(1 - \frac{5}{3}\, A^{-(d+1)}) = \frac{1}{12}\,(1 - (\frac{3}{5})^d) > 0$. Therefore, one can find a word ω of length m such that

$$\eta = I_\omega = \int_{F_\omega K} |u(x)|^2\, dx \geq C_1 = \frac{1}{12}\,(1 - (\frac{3}{5})^d). \tag{12}$$

Now let u_1 be the function defined as follow:

$$\begin{cases} u_1|_{F_\omega K} &= \; u|_{F_\omega K}, \\ u_1|_{F_{\omega_i} K} &= \; \text{the harmonic function with boundary values}\, (a_i, 0, 0), \\ u_1 &= \; 0 \quad \text{elsewhere}, \end{cases}$$

where $\{a_i\}$ are the boundary values of u on $F_\omega K$.

Note that u_1 is supported on a cell of level $m-2$, or on the union on two such cells, but at any rate we can use Lemma 1 to get

$$\mathcal{E}(u_1, u_1) \geq \frac{C_2}{r_\omega^{d+1}}\, \|u_1\|_2^2, \tag{13}$$

where $C_2 = \frac{\lambda_1}{25}$ and λ_1 is a Dirichlet eigenvalue of the Laplacian operator.

Using the definition of the energy along with inequalities (12) and (13) we obtain

$$\frac{1}{r_\omega}\mathcal{E}(u\circ F_\omega, u\circ F_\omega) + \frac{2}{r_\omega}\sum_{i=1}^{3} a_i^2 \geq \frac{C_1\, C_2}{r_\omega^{d+1}}. \tag{14}$$

Observe that we really need to control the second term of the left-hand side in the last inequality to conclude our proof. We will do this by examining different cases.

- Case I:
  If $\frac{1}{r_\omega}\sum_{i=1}^3 a_i^2 \leq \frac{C_1\, C_2}{4\, r_\omega^{d+1}}$, then (14) immediately implies that
  $$\frac{1}{r_\omega}\mathcal{E}(u\circ F_\omega, u\circ F_\omega) \geq \frac{C_1\, C_2}{2\, r_\omega^{d+1}}.$$
  Hence, by (11) we obtain that $\mathcal{E}(u,u) \geq \frac{C}{Var_d(|u|^2)}$ which concludes the proof in this case.
- Case II:
  If $\frac{1}{r_\omega}\sum_{i=1}^3 a_i^2 > \frac{C_1\, C_2}{4\, r_\omega^{d+1}}$, or equivalently $\mu_\omega \sum_{i=1}^3 a_i^2 > \frac{C_1\, C_2}{4}$ (we have used the fact that $r_\omega^{d+1} = \mu_\omega\, r_\omega$), then because harmonic functions minimize the energy among all functions with the same boundary value, we have the following estimate
  $$\mathcal{E}(u\circ F_\omega, u\circ F_\omega) \geq (a_1 - a_2)^2 + (a_1 - a_3)^2 + (a_2 - a_3)^2.$$
  If $\frac{\min|a_i|}{\max|a_i|} \leq 1/2$ (note that $1/2$ can be replaced with any $\alpha \in (0,1)$ without affecting the arguments given below, except for an adjustment of the constants), then there exits $C_3 > 0$ such that
  $$\sum_{i=1}^{3} a_i^2 \leq C_3\left((a_1-a_2)^2 + (a_1-a_3)^2 + (a_2-a_3)^2\right) \leq C_3\, \mathcal{E}(u\circ F_\omega, u\circ F_\omega).$$
  Using this last inequality in (14) immediately yields the proof of our result in this case.
  If instead $\frac{\min|a_i|}{\max|a_i|} \geq 1/2$ then $|a_i|$ are of comparable size for $i = 1,2,3$. More precisely the following estimates hold
  $$\frac{1}{12}\sum_{i=1}^{3} a_i^2 \leq a_j^2 \leq \frac{4}{3}\sum_{i=1}^{3} a_i^2 \qquad \forall \quad j = 1,2,3. \tag{15}$$
  Our goal now is to estimate a_i^2 (or equivalently $\sum_{i=1}^3 a_i^2$ by the previous observation) in terms of the energy of u on the cells $F_{\omega_i}K$. We achieve this using Lemma 2. More precisely, if there exists $i\,(i = 1,2,3)$ such that $u\circ F_{\omega_i}$ takes on a value less than or equal to $|a_i|/2$ at some point $y \in K$, then Lemma 2 implies that $\mathcal{E}(u\circ F_{\omega_i}, u\circ F_{\omega_i}) \geq C_4 a_i^2$. Combining this last estimate, with (15), and (14) yield the result.
  However, if for all i, and for all $x \in K$ we have $|u\circ F_{\omega_i}(x)| \geq |a_i|/2$ then Lemma 2 is no longer applicable. We will show that by involving more

neighboring cells we can derive a contradiction to the initial estimate on the variance (11). More precisely, if $|u \circ F_{\omega_i}(x)| \geq |a_i|/2 \quad \forall x \in K$ then

$$\mu_\omega \|u \circ F_{\omega_i}\|_2^2 \geq \mu_\omega \frac{a_i^2}{4} \geq \frac{C_1 C_2}{4} \frac{1}{48},$$

where we have used (15) in the last inequality. Choose two cells that have a positive separation, i.e., choose i, j such that $F_{\omega_i}K \cap F_{\omega_j}K = \phi$, and such that $d(x,y) \geq r_\omega$ for all $x \in F_{\omega_i}K$, $y \in F_{\omega_j}K$. We can estimate the variance of u as follow

$$\begin{aligned} Var_d(|u|^2) &= \int_K \int_K d_R(x,y)^{d+1} \, |u(x)|^2 \, |u(y)|^2 \, dx \, dy \\ &\geq \int_{F_{\omega_i}K} \int_{F_{\omega_j}K} d_R(x,y)^{d+1} \, |u(x)|^2 \, |u(y)|^2 \, dx \, dy \\ &\geq r_\omega^{d+1} \left(\frac{C_1 C_2}{48}\right)^2 \frac{1}{4^2}. \end{aligned}$$

We can now continue in this fashion to get the desired contradiction. In particular, it is easy to see that if $|u \circ F_{\omega_i}(x)| \geq |a_i|/2 \quad \forall i = 1,2,3, \quad x \in K$, then $|u| \geq |a_i|/2$ at the three corners (boundary points) of the cell of level $m-2$ containing $F_\omega K$; otherwise we may appeal to Lemma 2 to conclude the proof. Consider now the cell of level $m-2$ containing $F_\omega K$ and denote it by $F_{\tilde{\omega}}K$, see Figure 1 below. Let the boundary values of u on that cell be $b_i, i = 1,2,3$ where $|b_i| \geq |a_i|/2$, and let us look at the values of u at the neighboring cells (of level $m-2$) as in the last case.

If for some $i = 1,2,3$ $|u \circ F_{\tilde{\omega}_i}(x)| \leq |b_i|/2$ for some $x \in K$, then Lemma 2 applies again to give us the desired result. Otherwise, if $|u \circ F_{\tilde{\omega}_i}| \geq |b_i|/2$ on K and for all i, then we can again estimate the variance of u to obtain

$$Var_d(|u|^2) \geq r_{\tilde{\omega}}^{d+1} \left(\frac{C_1 C_2}{48}\right)^2 \frac{3^4}{2^8}.$$

By proceeding in this way, either we encounter a favorable case (i.e., a case where Lemma 2 can be applied), or we go a few steps further using the above process to obtain a lower bound on $Var_d(|u|^2)$. More precisely, one can get to a cell $F_{\tilde{\omega}}K$ of level $m-n$, where $n = 2k$ such that

(16) $$Var_d(|u|^2) \geq r_{\tilde{\omega}}^{d+1} \frac{3^{2n}}{2^{2n+8}} \left(\frac{C_1 C_2}{48}\right)^2.$$

We can now use (16) and (11) to write

$$\frac{3^{2n}}{2^{2n+8}} \left(\frac{C_1 C_2}{48}\right)^2 r_{\tilde{\omega}}^{d+1} \leq Var_d(|u|^2) \leq \frac{5}{3} r_\omega^{d+1}.$$

From this last inequality we obtain that

$$n < \frac{\ln\left(\frac{2^8 5}{3}\left(\frac{48}{C_1 C_2}\right)^2\right)}{\ln 5} = N_0.$$

Hence, the assumption of Lemma 2 is satisfied for $n \geq N_0$, from which the result follows. Note that we have implicitly assumed that $m > N_0$, which is the case since for $Var_d(|u|^2)$ small, we can actually show that the integer m that appears in (11) is large enough so that $m > N_0$.

□

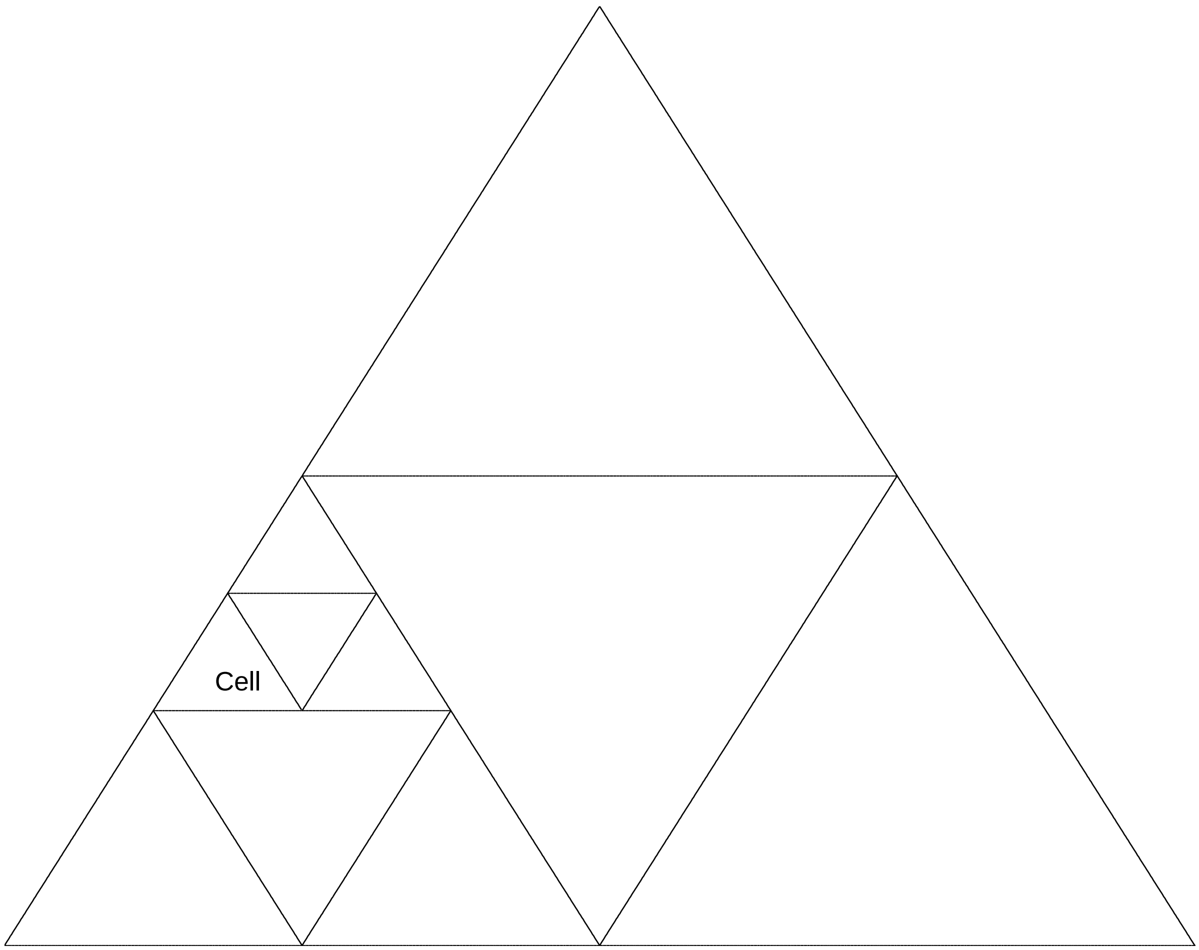

FIGURE 1. Cell = level m cell of the Sierpinski Gasket on which u is mostly concentrated, i.e., Cell $= F_\omega K$ with $I_\omega = \eta$.

Remark 5. a. In the hypothesis $Var_d(|u|^2) \leq \frac{1}{2}$ of Theorem 1, we could have replaced $1/2$ by any number $\alpha \in (0, 1)$ without affecting the proof.

b. Obviously, (10) cannot hold for constant functions defined on K. However, the conditions $\|u\|_2 = 1$ and $Var_d(|u|^2) \leq \frac{1}{2}$ imply that u is not constant.

c. If in the hypotheses of Theorem 1 we remove the restriction that $Var_d(|u|^2) \leq \frac{1}{2}$, we can get a weaker estimate. More precisely, there exists a positive constant C such that for all $u \in L^2(K)$ with $\|u\|_{L^2(K)} = 1$ and $\mathcal{E}(u, u) < \infty$, the following inequality holds

$$Var_d(|u|^2)\,\big(\mathcal{E}(u,u)+1\big) \geq C. \tag{17}$$

d. More generally, if $u \in dom\mathcal{E}$, with $Var_d(|u|^2) \leq \frac{1}{2}\|u\|_2^4$ then

$$Var_d(|u|^2)\,\mathcal{E}(u,u) \geq C\|u\|_2^6.$$

3.2. **Weak uncertainty principle on blowups of pcf fractals.** Let K be a pcf fractal which is the invariant set of the ifs $\{F_i\}_{i=1}^{n_0}$ of similarity transformations with similarity ratios $c_i \in (0,1)$. Given any sequence $F_{\omega_1}, F_{\omega_2} \ldots$ from the ifs, set

$$K_j = F_{\omega_1}^{-1} \circ F_{\omega_2}^{-1} \circ \ldots \circ F_{\omega_j}^{-1} K.$$

Then it is immediate to see that $F_0 = F \subset F_1 \subset F_2 \subset \ldots$, and we say that $\mathcal{K}$ is a blowup of K if $\mathcal{K}$ is the union of K_j. Moreover, for any integer N let F_N be defined by $F_N = F_{\omega_N} F_{\omega_{N-1}} \ldots F_{\omega_1}$, and let $K_N = F_N^{-1}K = \cup_{i=1}^N K_i$. We will call K_N a finite blowup of K. We refer to [18, 21] for more about blowups of fractals.

For a function u defined on K_N let $\sigma_N u$ be the function defined on K by

$$\sigma_N u(x) = u(F_N^{-1}x), \quad x \in K.$$

Note that

$$\int_{K_N} u(x)\,\mu(dx) = \mu_N^{-1}\int_K \sigma_N u(x)\,\mu(dx),$$

where μ is the Hausdorff measure on $\mathcal{K}$ normalized such that, $\mu(K) = 1$, and $\mu_N = \mu_\omega = \mu_{\omega_1}\mu_{\omega_2}\ldots\mu_{\omega_N}$, for a word ω of length N. In fact K_N is just a rescaled version of K.

Using Theorem 1 we can prove a similar uncertainty principle for functions define of finite blowup K_N of pcf fractals. In particular, the following result holds.

Corollary 1. *Let $r_N = r_{\omega_1} r_{\omega_2} \ldots r_{\omega_N}$. There exists a positive constant C, such that for all $u \in dom\mathcal{E}_{K_N}$ with $\|u\|_{L^2(K_N)} = 1$ and $Var_{d,N}(|u|^2) \leq \frac{1}{2}\frac{1}{r_N^{3d+1}}$ we have*

$$Var_{d,N}(|u|^2)\,\mathcal{E}_{K_N}(u,u) \geq C.$$

Moreover, C is independent of N.

Proof. For $u \in L^2(K_N)$ and using the notations adopted at the beginning of this subsection we easily obtain that

$$\sigma_N u \in L^2(K) \quad \text{with} \quad \|u\|^2_{L^2(K_N)} = \mu_N^{-1}\|\sigma_N u\|^2_{L^2(K)},$$

and that $\mathcal{E}_{K_N}(u,u) = r_N\,\mathcal{E}(\sigma_N u, \sigma_N u)$, where $r_N = r_\omega = r_{\omega_1} r_{\omega_2} \dots r_{\omega_N}$ for a word ω of length N. Moreover,

$$\begin{aligned} Var_d(|\sigma_N u|^2) &= \iint_{K\times K} d_R(x,y)^{d+1}\,|\sigma_N u(x)|^2\,|\sigma_N u(y)|^2\,dx\,dy \\ &= \mu_N^2 \iint_{K_N\times K_N} d_R(F_N x, F_N y)^{d+1}\,|u(x)|^2\,|u(y)|^2\,dx\,dy \\ &\le \mu_N^2\, r_N^{d+1} \iint_{K_N\times K_N} d_R(x,y)^{d+1}\,|u(x)|^2\,|u(y)|^2\,dx\,dy \\ &= \mu_N^2\, r_N^{d+1}\, Var_{d,N}(|u|^2), \end{aligned}$$

where we have used the fact that $d_R(F_N x, F_N y) \le r_N\, d_R(x,y)$. Thus the theorem follows from the following estimates:

$$\begin{aligned} Var_{d,N}(|u|^2)\,\mathcal{E}_{K_N}(u,u) &\ge \mu_N^{-2}\, r_N\, \frac{1}{r_N^{d+1}}\, Var_d(|\sigma_N u|^2)\mathcal{E}(\sigma_N u, \sigma_N u) \\ &\ge C\,\mu_N^{-2}\, r_N\, \frac{1}{r_N^{d+1}}\, \|\sigma_N u\|_{L^2(K)}^6 \\ &\ge C\,\mu_N^{-2}\, r_N\, \frac{1}{r_N^{d+1}}\, \mu_N^3\, \|u\|_{L^2(K_N)}^6 \\ &\ge C, \end{aligned}$$

where we have used again the fact that $r_N^d = \mu_N$ in the last step. □

Remark 6. Similarly to Remark 5 c., by removing the fact that $Var_{d,N}(|u|^2) \le \frac{1}{2}\frac{1}{r_N^{3d+1}}$, and just assuming that $\|u\|_{L^2(K_N)} = 1$, and that $\mathcal{E}_{K_N}(u,u) < \infty$, it can be shown that there exists a positive constant C such that

$$Var_{d,N}(|u|^2)\,\big(\mathcal{E}_{K_N}(u,u) + 1\big) \ge C. \tag{18}$$

Since the constant C appearing in Corollary 1 is independent of N we immediately have the following result, which yields a weak uncertainty principle for function defined of the (infinite) blowup $\mathcal{K}$ of K. Note that the condition $Var_{d,N}(|u|^2) \le \frac{1}{2}\frac{1}{r_N^{3d+1}}$ disappears in the limit. In fact, on $\mathcal{K}$ the variance may be infinite, but we are only interested in the finite variance case.

Theorem 2. *There exists a positive constant C such that for all $u \in dom\mathcal{E}_{\mathcal{K}}$, with $\|u\|_{L^2(\mathcal{K})} = 1$ we have*

$$Var_{d,\infty}(|u|^2)\,\mathcal{E}_{K_\infty}(u,u) \ge C.$$

Remark 7. It is likely that weak uncertainty principles similar to Corollary 1 and Theorem 2 can be stated on other self-similar fractals, e.g., the blowup fractals considered in [7].

4. Weak uncertainty principle on product of fractals

Let $u(x', x'')$ be a function defined on the product K of two pcf fractals K', K''. We wish in this section to establish a weak uncertainty principle of the same flavor as the one we derived in the previous section. This is a small step toward formulating an weak uncertainty principle on non-pcf fractals. We adopt the same notation as in [16], where more background on analysis on product of fractals can be found. In particular, we use $'$ for variable defined on the first factor of the product $K = K' \times K''$, and $''$ for any variable related to the second factor. Moreover, we will assume in the sequel that $d' = d'' = d$, or make the stronger assumption that $K' = K''$.

We begin with the following estimate obtained by applying the result of the last sections to the functions obtained by freezing one of the variables x' or x'' of the function u defined on $K = K' \times K''$.

$$\left(\int_{K'} |u(x', x'')|^2 \, dx'\right)^3 \leq C\, \mathcal{E}(u(\cdot, x''), u(\cdot, x''))\times$$
$$\iint_{K' \times K'} d'_R(x', y')^{d+1} \, |u(x', x'')|^2 \, |u(y', x'')|^2 \, dx' \, dy'.$$

By raising to power $1/3$, integrating over x'', and applying Hölder's inequality with $p = 3/2$ and $p' = 3$ we obtain

$$\|u\|_2^2 \leq C \left(\int_{K''} \left(\iint_{K' \times K'} d'_R(x', y')^{d+1} \, |u(x', x'')|^2 \, |u(y', x'')|^2 \, dx' \, dy'\right)^{1/2} dx''\right)^{2/3} \times$$
$$\left(\int_{K''} \mathcal{E}'(u(\cdot, x''), u(\cdot, x''))dx''\right)^{1/3},$$

or equivalently (after raising both sides to the power 3) we have that

$$\|u\|_2^6 \leq C \left(\int_{K''} \mathcal{E}'(u(\cdot, x''), u(\cdot, x''))dx''\right) \times$$
$$\left(\int_{K''} \left(\iint_{K' \times K'} d'_R(x', y')^{d+1} \, |u(x', x'')|^2 \, |u(y', x'')|^2 \, dx' \, dy'\right)^{1/2} dx''\right)^2.$$

A similar estimate can be obtained by freezing x' instead of x''. Now the energy on K can be defined as

$$\mathcal{E}(u, u) = \int_{K'} \mathcal{E}'(u(x', \cdot), u(x', \cdot)) \, dx' + \int_{K''} \mathcal{E}''(u(\cdot, x''), u(\cdot, x'')) \, dx'';$$

see [16] for more details about analysis on product of fractals. We also define the variance of a function $u \in L^2(K)$ with $\|u\|_2 = 1$ as follow

$$Var_d(|u|^2) = \iiiint (d'_R(x', y')^{d+1} + d''_R(x'', y'')^{d+1}) |u(x', x'')|^2 |u(y', y'')|^2 \, dx' \, dx'' \, dy' \, dy''.$$

However, by Remark 1 one can easily see that an equivalent definition of the variance in the product setting is given by

$$(19)\qquad Var_d(|u|^2) = \inf_{(y',y'')\in K} \iint_K \left(d'_R(x',y')^{d+1} + d''_R(x'',y'')^{d+1}\right)|u(x',x'')|^2\,dx'\,dx''.$$

All we now need to prove the weak uncertainty principle on K is to establish the following lemma:

Lemma 3. *There exists a constant $C > 0$ such that for all $u \in L^2(K)$ with $\|u\|_2 = 1$, then the following estimate holds*

$$Var_d(|u|^2) \geq C\left(\int_{K''}\left(\iint_{K'\times K'} d'_R(x',y')^{d+1}\,|u(x',x'')|^2\,|u(y',x'')|^2\,dx'\,dy'\right)^{1/2} dx''\right)^2 +$$
$$\left(\int_{K'}\left(\iint_{K''\times K''} d''_R(x',y')^{d+1}\,|u(x',x'')|^2\,|u(x',y'')|^2\,dx''\,dy''\right)^{1/2} dx'\right)^2.$$

Proof. One can prove directly the above estimate, by showing that it holds for the class of piecewise pluri-harmonic functions (PPH), see [16], which are dense in $L^2(K)$. However, we prefer giving a shorter proof based on the equivalent definition of the variance given by (19). Let

$$m = \inf_{(y',y'')\in K} \iint_K \left(d'_R(x',y')^{d+1} + d''_R(x'',y'')^{d+1}\right)|u(x',x'')|^2\,dx'\,dx'',$$

$$m_1 = \left(\int_{K''}\left(\inf_{y'\in K'}\int_{K'} d'_R(x',y')^{d+1}\,|u(x',x'')|^2\,dx'\right)^{1/2} dx''\right)^2$$

and

$$m_2 = \left(\int_{K'}\left(\inf_{y''\in K''}\int_{K''} d''_R(y'',x'')^{d+1}\,|u(x',x'')|^2\,dx''\right)^{1/2} dx'\right)^2.$$

Then by the preceding remarks $m_1 + m_2$ is equivalent to the right hand side of the estimate in Lemma 3 while m is equivalent to $Var_d(|u|^2)$. Thus we just have to show that $m_1 + m_2 \leq C Var_d(|u|^2)$ for some positive constant C to complete the proof.

Let $\epsilon > 0$, there exists $(y'_0, y''_0) \in K$ such that

$$\iint_K (d'_R(x',y'_0)^{d+1} + d''_R(x'',y''_0)^{d+1})\,|u(x',x'')|^2\,dx'\,dx'' \leq m + \epsilon.$$

Note also that for all $x' \in K'$ and for all $x'' \in K''$, we have

$$\inf_{y'\in K'}\int_{K'} d'_R(x',y')^{d+1}\,|u(x',x'')|^2\,dx' \leq \int_{K'} d'_R(x',y'_0)^{d+1}\,|u(x',x'')|^2\,dx',$$

and

$$\inf_{y''\in K''}\int_{K''} d''_R(x'',y'')^{d+1}\,|u(x',x'')|^2\,dx'' \leq \int_{K''} d''_R(x'',y''_0)^{d+1}\,|u(x',x'')|^2\,dx''.$$

Applying Hölder's inequality immediately yields $m_1 + m_2 \leq m \leq C Var_d(|u|^2)$ which completes the proof.

□

Using the Lemma 3 and the observations made at the beginning of this section we can now prove the weak uncertainty principle on $K = K' \times K''$.

Theorem 3. *There exists* $C > 0$ *such that for all* $u \in dom\mathcal{E}$ *with* $\|u\|_2 = 1$ *and* $Var_d(|u|^2) \leq \frac{1}{2}$ *we have:*

$$Var_d(|u|^2)\mathcal{E}(u,u) \geq C.$$

Proof. Follows easily from Lemma 3 and the observations made earlier. □

References

[1] M. T. Barlow, Diffusion on Fractals, in: Lectures Notes in Mathematics, Vol. 1690, Springer, Berlin, 1998.

[2] M. T. Barlow and J. Kigami, *Localized eigenfunctions of the Laplacian on p.c.f self-similar sets,* J. London Math. Soc., **56**(2):320–332, 1997.

[3] E. Breitenberger, *Uncertainty measures and uncertainty relations for angle observables,* Found. Phys., **15**:353–364.

[4] G. B. Folland and A. Sitaram, *The uncertainty principle:A mathematical survey,* J. Fourier Anal. Appl., **3**(3):207–238, 1997.

[5] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpinski gasket,* Potential Anal., **1**:1–35, 1992.

[6] J. Kigami, "Analysis on Fractals," Cambridge University Press, New York, 2001.

[7] B. Krön and E. Teufl, *Asymptotics of the transition probabilities of the simple random walk on self-similar graph,* Trans. Amer. Math. Soc., **356**(1):393–414, 2003.

[8] F. J. Narcowich and J. D. Ward, *Wavelets associated with periodic basis functions,* Appl. Comput. Harmon. Anal., **3**:40–56, 1996.

[9] J. Prestin, E. Quak, H. Rauhunt and K. Selig, *On the connection of the uncertainty principles for functions on the circle and on the real line,* J. Fourier Anal. Appl., **9**(4):387–409, 2003.

[10] J. Prestin and E. Quak, *Optimal functions for a periodic uncertainty principle and multiresolution analysis,* Proc. Edinb. Math. Soc., **42**:225–241, 1999.

[11] J. F. Price and P. C. Racki, *Local uncertainty inequalities for Fourier series,* Proc. Amer. Math. Soc., **93**:245–251, 1985.

[12] J. F. Price, *Inequalities and local uncertainty principles,* J. Math. Phys., **24**:1711–1714, 1983.

[13] K. Selig, *Trigonometric wavelets and uncertainty principles,* in "Approximation Theory," M. W. Müller, M. Felten, D. H. Mache, Eds, Akedemie Verlag, 293–304, 1995.

[14] K. Selig, *Uncertainty principles revisited,* Electron. Trans. Num. Anal., **14**:165–177, 2002.

[15] R. S. Strichartz, *Function spaces on fractals,* J. Funct. Anal., **198**:43–83, 2003.

[16] R. S. Strichartz, *Analysis on products of fractals,* Trans. Amer. Math. Soc., to appear.

[17] R. S. Strichartz, *Fractafolds based on the Sierpinski gasket and their spectra,* Trans. Amer. Math. Soc., **355**(10):4019–4043, 2003.

[18] R. S. Strichartz, *Fractals in large,* Can. J. Math., **50**(3):638–657, 1996.

[19] R. S. Strichartz, *Analysis on fractals,* Notices Amer. Math. Soc., **46**:1199–1208, 1999.

[20] R. S. Strichartz, *Uncertainty principles in harmonic analysis,* J. Funct. Anal., **84**:97–114, 1989.

[21] A. Teplyaev, *Spectral analysis on infinite Sierpiński gaskets,* J. Funct. Anal., **159**:537–567, 1998.

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