Mathematics 4530

Assignment 1, due September 3, 2013

Browse through Sections 1–7 (mostly review, I hope). Read Sections 12–13, through Lemma 13.1. Then do:

- pp. 20–21: 1, 2, 3
- pp. 83–84: 3

The following additional problems involve the notion of "neighborhood," defined in class and on p. 96 of your text. For most purposes I prefer a slight variant of this notion, for which I will coin the term "mneighborhood" (pronounced with a silent "m") to avoid conflicting terminology: Let X be a topological space and $x \in X$ an arbitrary point. I will call a subset $N \subseteq X$ a *mneighborhood* of x if there is an open set U such that $x \in U \subseteq N$. By a *mneighborhood base* at x I mean a collection $\mathcal{N}$ of mneighborhoods of x such that every mneighborhood of x contains some element of $\mathcal{N}$. Intuitively, when you specify a mneighborhood base at x you're spelling out what it means to get close to x.

1. (a) If $X = \mathbb{R}^n$ with its usual topology, show that the collection of open balls centered at x forms a mneighborhood base at x, as does the collection of closed balls centered at x.
   (b) Give an example of a countable mneighborhood base at $x \in \mathbb{R}^n$.
   (c) In an arbitrary topological space X, show that a set is open if and only if it is a mneighborhood of each of its points.

2. (extra credit) This is an extra-credit problem not because of difficulty but because it uses concepts from algebra that are not part of the prerequisites for this course. Let G be a group and let $G_0 \geq G_1 \geq G_2 \cdots \geq$ be a descending chain of subgroups. Show that the collection of cosets xG_n ($x \in G$, $n \geq 0$) is a basis for a topology on G. Show further that the basic open sets xG_n are also closed sets (see p. 93 for the definition). Show, finally, that the collection $\{xG_n\}_{n \geq 0}$ for fixed x is a mneighborhood base at x. [If G is the additive group $\mathbb{Z}$ of integers and $G_n = p^n\mathbb{Z}$ for some fixed prime p, the resulting topology on $\mathbb{Z}$ is called the *p-adic topology*. It is used extensively in number theory. Intuitively, two integers are p-adically close if they are congruent modulo a high power of p.]