Mathematics 4530

Assignment 4, due September 24, 2013

Read Sections 20 and 21 and the handout on the well-ordering theorem. Then do:

- p. 67: 6
- pp. 126–128: 1(a), 10
- pp. 133–134: 1, 4, 7

Additional problems:

1. Let X be a well-ordered set and let A be an order ideal in X. Prove that either $A = X$ or $A = X_{<x}$ for some $x \in X$. [This fact is needed at one point in the handout on the well-ordering theorem.]

2. (a) Prove that $|d(x, y) - d(x', y)| \leq d(x, x')$ for any three points x, x', y in a metric space X.
   (b) Deduce that $d(x, y)$ is continuous as a function of x for each fixed y.
   (c) Prove the stronger result that d is continuous as a function of two variables, i.e., that $d : X \times X \to \mathbb{R}$ is continuous.

3. (e.c.) There was a proof in class in which we started with a metric d and defined a new metric d' by
$$d'(x, y) := f(d(x, y)),$$
where $f(t) = \min\{t, \delta\}$ for $t \geq 0$. (Here δ is a positive constant.) Another function f that works for this purpose is $f(t) = t/(1 + t)$; see Exercise 11 on p. 129. Find a nice class of functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that this procedure will always work (i.e., so that $f \circ d$ will be a metric whenever d is a metric). This exercise is deliberately stated in an open-ended way, so there are many correct answers. The kind of answer that will make me happiest is one that makes it possible to tell whether f is good by glancing at the graph of f. [Hint: concavity.]

4. (e.c.) Let ℓ^2 be the space (called *Hilbert space*) defined in Exercise 10 on p. 128. The *Hilbert cube* is the set of all $\mathbf{x} = (x_i)_{i \geq 1} \in \ell^2$ such that $0 \leq x_i \leq 1/i$ for all i. Show that the Hilbert cube (with the ℓ^2 metric) is homeomorphic to the product of a countable number of copies of the unit interval $[0, 1]$.

5. (e.c.) In class (and in the handout) I proved an *induction principle* for an arbitrary well-ordered set X, generalizing the familiar method of proof by induction (which is the case $X = \mathbb{N}$). State and prove a *principle of recursive definition* for X. [See Theorem 8.4 on p. 54 for the case $X = \mathbb{N}$.]