Mathematics 4530

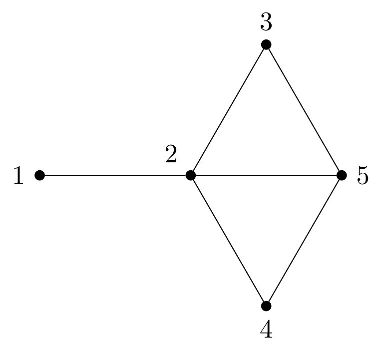
Assignment 5, due October 1, 2013

Read Sections 22 and 23. Then do:

- pp. 144–145: 2, 3, 4
- p. 152: 1, 2, 3, [5], 10, 11

Additional problems:

1. The quotient construction is often used to build spaces by gluing together simpler spaces. We saw several examples of this in class. This exercise gives another example, involving graphs, in which the building blocks are points and line segments. A *graph*, for our purposes, is a pair $G = (V, E)$, where V is a finite set and E is a set of unordered pairs of elements of V. We call V the set of *vertices* and E the set of *edges*. We often describe a graph by drawing a picture. For example, the picture



represents the graph with

$$V = \{1, 2, 3, 4, 5\} \quad \text{and} \quad E = \{12, 23, 24, 25, 35, 45\}.$$

Here ij denotes the unordered pair $\{i, j\}$. A picture of a graph G is actually a picture of a certain topological space $|G|$, called the *geometric realization* of G, which is defined as follows: Let $\widetilde{E}$ be the set of ordered pairs (v, w) such that vw is an edge. [An element of $\widetilde{E}$ is sometimes called a *directed edge*.] Form the disjoint union $Z = V \amalg (\widetilde{E} \times I)$; here V and $\widetilde{E}$ have the discrete topology, I is the unit interval $[0, 1]$, and the disjoint union is topologized by declaring a set open if its intersections with V and $\widetilde{E} \times I$ are open. Let $\sim$ be the smallest equivalence relation on Z such that if $e = (v, w)$ is a directed edge and $\bar{e}$ is the opposite edge (w, v), then $(e, 0) \sim v$, $(e, 1) \sim w$, and $(e, t) \sim (\bar{e}, 1 - t)$ for all $t \in I$. Then $|G|$ is defined to be the quotient space $Z/\sim$.

   (a) Explicitly describe the equivalence classes. (No proof required.)

   (b) Draw a picture of Z and the quotient map $Z \twoheadrightarrow |G|$ for the graph pictured above. [Drawing a picture of the map is most easily done by using labels and arrowheads. If you prefer, you can use colors instead of labels.]

   (c) Show for any graph G that the quotient map is a closed map. Give an example to show that it is not always an open map.

   (d) (e.c.) Show that $|G|$ is metrizable.

2. (e.c.) If X is a topological space, the *cone* on X, denoted CX, is the quotient space obtained from the "cylinder" $X \times [0,1]$ by collapsing $X \times 1$ to a single point c (called the cone point). This was illustrated in class with $X = S^1$.
   (a) Show that $B^n \cong CS^{n-1}$.
   (b) Show that the homeomorphism in (a) may be chosen so that any given interior point of B^n corresponds to the cone point c. [It's not too hard to do this by picture drawing and hand waving; the work required to do it rigorously is what accounts for this being an extra credit problem.]