Mathematics 4530

Assignment 10, due November 12, 2013

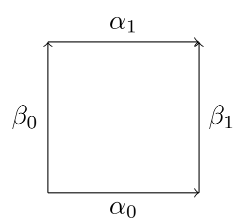
Read Sections 52 and 53. Then do:

- pp. 334–335: 4, 5
- p. 341: 2, 3, 6(b)

Additional problems:

1. An inclusion $i\colon A \hookrightarrow X$ induces a homomorphism $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ for any basepoint $a_0 \in A$. Show that i_* is 1–1 if A is a retract of X (cf. Exercise 4 on p. 335).

2. (e.c.) Let $F\colon I \times I \to X$ be a continuous map. Let $\alpha_0, \alpha_1, \beta_0, \beta_1$ be the paths obtained by restricting F to the four sides of the square, as indicated in the following picture:



   Thus $\alpha_0(s) = F(s, 0)$, $\beta_1(s) = F(1, s)$, and so on. Show that $\alpha_0 * \beta_1 \simeq_p \beta_0 * \alpha_1$. [Hint: Think about the universal example.]

3. Suppose we are given two maps $h_0, h_1\colon (X, x_0) \to (Y, y_0)$.
   (a) If $h_0 \simeq h_1$ relative to the basepoint, show that the induced homomorphisms $\phi_0 := (h_0)_*$ and $\phi_1 := (h_1)_*$ on $\pi_1(X, x_0)$ are equal.
   (b) (e.c.) If h_0 and h_1 are only assumed to be "freely" homotopic (i.e., homotopic but not necessarily relative to the basepoint), show that ϕ_0 and ϕ_1 are *conjugate*, in the sense that there is an element $c \in \pi_1(Y, y_0)$ such that $\phi_0(g) = c\phi_1(g)c^{-1}$ for all $g \in \pi_1(X, x_0)$. [Hint: Problem 2 is relevant.

4. (e.c.) Prove that every contractible space is simply connected.

5. Recall that every covering map is an open map. (See the last paragraph on p. 336.) Prove that every finite covering map is a closed map. Here a covering map is said to be *finite* if every fiber is a finite set.

6. (a) Let $p\colon \widetilde{X} \to X$ be a covering map of degree 2 (i.e., each fiber has exactly two points). Prove that there is a fixed-point-free involution of $\widetilde{X}$ whose orbits are the fibers. [Ask in class if you don't know some of the terminology.]
   (b) (e.c.) Conversely, suppose $\widetilde{X}$ is a Hausdorff space with a fixed-point-free involution h. Construct a space X and a covering map $p\colon \widetilde{X} \to X$ whose fibers are the orbits of h.