Mathematics 4530 **Final Exam** **December 18, 2013**

Do any 5 of the following 8 problems, and indicate clearly which 5 you have chosen. You may use any result proved in class, in the assigned reading, or in the homework, unless the problem asks you to prove such a result.

1. True or false. Justify each answer with a proof or counterexample.

(a) If X is a simply connected space and A is a path connected subspace, then A is simply connected.

(b) If $\alpha\colon I \to X$ is a path in a Hausdorff space X, then the image of α is a closed subset of X.

(c) Let X be a topological space and A a subspace, and let $i\colon A \hookrightarrow X$ be the inclusion map. Then the induced homomorphism $i_*\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is injective for any choice of basepoint $a_0 \in A$.

(d) If U and V are path connected open subsets of a space X, then their intersection is path connected.

2. Let $p\colon \widetilde{X} \to X$ be a covering map. Let $\tilde{x}_0$ be a basepoint in $\widetilde{X}$, and let $x_0 := p(\tilde{x}_0)$. Suppose the following condition holds: For any closed curve α in X at x_0 such that $[\alpha]$ is nontrivial in $\pi_1(X, x_0)$, its lift $\tilde{\alpha}$ starting at $\tilde{x}_0$ is not closed. Show that the group $\pi_1(\widetilde{X}, \tilde{x}_0)$ is trivial.

3. Given a topological space X, we define the *connected components* of X to be the equivalence classes determined by the following relation: $x \sim y$ if and only if there is a connected subset of X containing both x and y.

(a) Prove that this relation is in fact an equivalence relation.

(b) Prove that every connected component is connected.

4. Let $q\colon X \to Y$ be a quotient map. State and prove a universal mapping property that characterizes continuous maps from Y to an arbitrary space Z.

5. Let X be a locally compact Hausdorff space whose one-point compactification is metrizable. Show that X is the union of an increasing sequence of compact subsets $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$.

6. (a) What does it mean to say that a topological space is *regular*?

(b) What does it mean to say that a topological space is *normal*?

(c) State Urysohn's lemma and Tietze's extension theorem.

(d) Let X be a metric space and let $(x_n)_{n\geq 1}$ be an infinite sequence in X such that $d(x_i, x_j) \geq 1$ for $i \neq j$. Prove that there is a continuous map $f\colon X \to \mathbb{R}$ such that $f(x_n) = n$ for all n.

7. (a) Define *covering map*.

(b) Suppose $p\colon \widetilde{X} \to X$ is a covering map with $\widetilde{X}$ and X path connected. If X is simply connected, show that p is a homeomorphism.

8. (a) State the Seifert–van Kampen theorem, with hypotheses and conclusion spelled out precisely.

(b) Give an example of a space whose fundamental group is a free group on 3 generators. Justify your answer as completely as you can; in particular, spell out explicitly three closed curves that represent the 3 generators.