

Mathematics 4530
Prelim 2, due December 5, 2013

What follows is a generalization of the calculation of $\pi_1(S^1)$. It is essentially a handout, but with all the proofs removed. Your task is to fill in the proofs in Sections 1–6. [You may read Section 7 for fun if you want, but there is nothing in it for you to write up.] I have given a clear indication of the places where you are expected to fill in a proof. I have also labeled some proofs as “extra credit” to keep the exam from being too long; they are not necessarily more difficult than the other proofs.

You may use any result proved in class, in the assigned reading, or in your homework. You may not use any sources other than your textbook and class notes. You may not use results from unassigned homework problems or from sections of the book that you have not been told to read. Andrew and I will be glad to clear up any ambiguities; **you may not discuss the exam with anyone else.**

1. INTRODUCTION

A crucial feature of the exponential covering $p: \mathbb{R} \rightarrow S^1$ is that two points in the same fiber differ by an integral multiple of 2π . This was used in the definition of the degree homomorphism $\pi_1(S^1) \rightarrow \mathbb{Z}$, which was then proved to be an isomorphism. We wish to generalize this to other covering maps, the so-called *regular* ones. As a consequence, one can calculate many fundamental groups with no more effort than was required for S^1 .

2. GROUP ACTIONS

We have defined a group to be a set G together with a binary operation $*$ satisfying certain axioms. This may give you the wrong impression of group theory, in that it gives you no clue as to how groups arise in nature. The way groups arise in nature is that they *act* on things, thereby exhibiting the symmetry that those things have.

Definition 1. Let G be a group and X a topological space. By an *action* of G on X we mean a function $G \times X \rightarrow X$, denoted $(g, x) \mapsto g * x$, satisfying:

- (1) $g * (h * x) = (g * h) * x$ for all $g, h \in G$ and $x \in X$.
- (2) $1 * x = x$ for all $x \in X$, where 1 is the identity element of G .
- (3) For each $g \in G$, the map $x \mapsto g * x$ is continuous.

It follows that the map $x \mapsto g * x$, which we will call the *action* of g , is actually a homeomorphism. Its inverse is given by the action of g^{-1} . Group actions are studied extensively in algebra courses, such as Math 4340. For our purposes, however, it suffices to have some examples.

Example 1. The additive group \mathbb{Z} acts on \mathbb{R} by $(n, x) \mapsto 2\pi n + x$ for $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. (We say “ \mathbb{Z} acts on \mathbb{R} by translation.”) This is the action that’s implicit in our work on the exponential covering map, which led to the calculation $\pi_1(S^1) \cong \mathbb{Z}$.

Example 2. The multiplicative group $\{\pm 1\}$ acts on S^2 by scalar multiplication. This action is related to the double covering $S^2 \rightarrow P^2$ mentioned in class, where P^2 is the projective plane.

Example 3. More generally, whenever we have a 2-fold covering map $p: X \rightarrow Y$, there is an associated action of a group of order 2 on X . This is essentially what you proved in additional problem 6 on Assignment 10.

Example 4. Let \mathbb{Z}_n be the additive group of integers mod n . There is an action of \mathbb{Z}_n on S^1 , in which the integer k mod n acts as rotation through $2\pi k/n$ radians. Alternatively, if we replace \mathbb{Z}_n by the isomorphic group consisting of the n th roots of unity in \mathbb{C} , then we can describe the action using multiplication of complex numbers.

Example 5. The concept of “Cayley graph,” which I will explain in Section 7, leads to many examples of group actions. For example, the snowflake that covers the figure 8 is the Cayley graph of the free group F_2 ; as a result, there is an action of F_2 on the snowflake.

To simplify the notation in what follows, we will always use multiplicative notation for our groups (i.e., we will write gh instead of $g * h$), and similarly we will write $(g, x) \mapsto gx$ for group actions.

Every one of the examples above has the property that it is *fixed-point free*, i.e., the action of any $g \neq 1$ has no points x such that $gx = x$. The significance of this for us is that if two points x, x' are in the same G -orbit, i.e., if there is a $g \in G$ such that $gx = x'$, then the element g is unique. [If $gx = x'$ and $hx = x'$, then $gx = hx \implies h^{-1}gx = x \implies h^{-1}g = 1 \implies g = h$.] We can think of g as something like the “difference” between x and x' . For the case of \mathbb{Z} acting on \mathbb{R} as in Example 1, it really is the difference (divided by 2π).

3. REGULAR G -COVERS

Definition 2. Let G be a group. A *regular G -cover* is a covering map $p: X \rightarrow Y$ together with a fixed-point-free action of G on X whose orbits are the fibers of p .

Let’s spell this out explicitly:

Lemma 1. Let $p: X \rightarrow Y$ be a covering map with an action of a group G on X . Then p is a regular G -cover if and only if it satisfies the following two conditions:

- (a) $p(gx) = p(x)$ for all $g \in G, x \in X$.
- (b) If $p(x) = p(x')$ then there is a (unique) $g \in G$ such that $x' = gx$.

Proof. *** Your proof goes here. ***

□

Remark 1. The group action is useful in connection with path liftings. Namely, if α is a path in Y and $\tilde{\alpha}$ is the lift starting at some point in the fiber over $\alpha(0)$, then we can construct all the other liftings of α by applying the group action.

*** Your explanation goes here. ***

We have already seen many examples of regular G -covers. Indeed, every example in the previous section has an associated regular G -cover, which was mentioned explicitly except in Example 4. [You can ask me if you can’t figure out the cover that goes with that example.] On the other hand, there are many covering maps that are not regular because they do not have enough symmetry. Indeed, conditions (a) and (b) imply, intuitively, that a regular G -cover looks the same no matter where we stand in a given fiber. For example:

Lemma 2. *Let $p: X \rightarrow Y$ be a regular G -cover for some group G , and let α be a closed curve in Y that admits a closed lift $\tilde{\alpha}$. Then every lift of α is closed.*

Proof. *** Your proof goes here. *** □

One can deduce from this that not all covering maps are regular.

Example 6. *** Your example goes here. ***

[To make it interesting, make sure that X and Y are both connected.]

4. THE GENERALIZED DEGREE HOMOMORPHISM

Recall that we defined the degree of a closed curve α in S^1 by lifting it to \mathbb{R} and measuring the difference between the starting and ending points of the lift. We can do something similar for any regular cover.

Let $p: X \rightarrow Y$ be a regular G -cover. Pick a basepoint $y \in Y$ and a basepoint $x \in p^{-1}(y)$. Given a path homotopy class $[\alpha] \in \pi_1(Y, y)$, let $\tilde{\alpha}$ be the lift of α starting at x . Then $\tilde{\alpha}$ ends in the fiber $p^{-1}(y)$, so there is a unique $g \in G$ such that $\tilde{\alpha}(1) = gx$. We set

$$\delta([\alpha]) := g.$$

In case p is the exponential covering of S^1 , this is precisely the degree defined in class. Imitating the four steps that occurred in our calculation of $\pi_1(S^1)$, we will prove:

Theorem 1. *Suppose that X is simply connected. Then*

$$\delta: \pi_1(Y, y) \rightarrow G$$

is an isomorphism.

The four steps are given in the following four lemmas, valid for any regular G -covering.

Lemma 3. *δ is well defined.*

Proof. *** Your proof goes here. *** □

Lemma 4. *δ is a group homomorphism.*

Proof. *** Your proof goes here. *** □

Lemma 5. *If X is path connected, then δ is surjective.*

Proof. *** Your proof goes here. *** □

Lemma 6. *If X is simply connected, then δ is bijective.*

Proof. *** Your proof goes here. *** □

5. EXAMPLES

Example 7. Combining Theorem 1 and Example 2, we obtain:

*** Your conclusion goes here, with full justification. ***

Example 8. Applying Theorem 1 to a suitable covering of the torus, we obtain:

*** Your conclusion goes here, with full justification. ***

Example 9. I promised that I would prove that the fundamental group of the figure 8 is the free group F_2 on two generators. One way to do this is to apply Theorem 1 to the snowflake cover of the figure 8. I will sketch how this can be done in Section 7. A different proof will be given in class.

6. EXISTENCE OF REGULAR G -COVERS

It is natural to ask how widely applicable Theorem 1 is. In this section we will see that, in principle, Theorem 1 can be used to calculate the fundamental group of *every* reasonable space. Here “reasonable” has a somewhat technical definition:

Definition 3. A space Y is said to be *semilocally simply connected* (SLSC) if every point $y \in Y$ has a neighborhood U such that the inclusion $U \hookrightarrow Y$ induces the trivial homomorphism $\pi_1(U, y) \rightarrow \pi_1(Y, y)$.

This condition may seem strange; but in fact, most spaces that one meets “in nature” satisfy the even stronger condition of being locally contractible. [Feel free to ask for examples if you have trouble seeing this.] Here’s more evidence that the SLSC condition is reasonable:

Proposition 1. *If Y admits a simply connected covering space, then Y is SLSC.*

Proof. *** Your proof goes here. *** □

Our goal in this section is to prove the following theorem:

Theorem 2. *Suppose Y is path connected, locally path connected, and SLSC. Then there exists a group G and a regular G -cover $p: X \rightarrow Y$ such that X is simply connected.*

[Necessarily, G will be isomorphic to $\pi_1(Y)$.]

The first step is to figure out what X has to look like.

Lemma 7. (a) *If U satisfies the condition in Definition 3, then every smaller neighborhood of y also satisfies the condition.*

(b) *Suppose Y is SLSC and locally path connected. Let \mathcal{B} be the collection of path-connected open sets U such that $\pi_1(U) \rightarrow \pi_1(Y)$ is the trivial homomorphism for some (or every) basepoint in U . Then \mathcal{B} is a basis for Y .*

(c) *For any $U \in \mathcal{B}$ let β_1, β_2 be paths in U with $\beta_1(0) = \beta_2(0)$ and $\beta_1(1) = \beta_2(1)$. Then $[\beta_1] = [\beta_2]$ in the fundamental groupoid of Y .*

Proof. *** Extra credit. *** □

Let Y be a path-connected space with basepoint y_0 , let \mathcal{G} be its fundamental groupoid, and let $\mathcal{G}_0 \subseteq \mathcal{G}$ be the subset consisting of path classes starting at y_0 . Thus \mathcal{G}_0 is a set whose elements are equivalence classes $a = [\alpha]$, where α is a path in Y with $\alpha(0) = y_0$. Let $p: X \rightarrow Y$ be a covering map, and let $x_0 \in p^{-1}(y_0)$ be a

basepoint. Then one can use path lifting to define a function $h: \mathcal{G}_0 \rightarrow X$. Namely, given $[\alpha] \in \mathcal{G}_0$, we lift α to a path $\tilde{\alpha}$ in X starting at x_0 and set

$$h([\alpha]) = \tilde{\alpha}(1).$$

Lemma 8. (a) h is well defined.

(b) h is surjective if X is path connected and bijective if X is simply connected.

Proof. *** Your proof goes here. *** □

From now on we assume that Y is path connected, locally path connected, and SLSC. We have just seen that if Y admits a simply connected covering space X , then the points of X must be in 1–1 correspondence with the elements of the set \mathcal{G}_0 . With this as motivation, we will prove Theorem 2 by starting with \mathcal{G}_0 and putting a suitable topology on it. Let \mathcal{B} be the basis for Y given in Lemma 7(b).

Let $a = [\alpha]$ be an element of \mathcal{G}_0 . For any $U \in \mathcal{B}$ with $\alpha(1) \in U$, let $\tilde{U}_a \subseteq \mathcal{G}_0$ be the set of path classes of the form $a * b$, where $b = [\beta]$ for some path β in U with $\beta(0) = \alpha(1)$. We will call \tilde{U}_a a *basic neighborhood* of a , and we will call a subset of \mathcal{G}_0 *open* if it contains a basic neighborhood of each of its points.

Lemma 9. The open sets just defined form a topology on \mathcal{G}_0 , with the sets \tilde{U}_a as a basis.

Proof. *** Extra credit. *** □

[Hint: It might help to show that if $a_1 \in \tilde{U}_a$, then $\tilde{U}_{a_1} = \tilde{U}_a$.]

As an aid to the intuition and a reminder of what we're trying to do, we will denote by X the set \mathcal{G}_0 with the topology that we have just defined. Let $p: X \rightarrow Y$ be defined by $p([\alpha]) = \alpha(1)$.

Lemma 10. p is continuous and open and maps \tilde{U}_a homeomorphically onto U .

Proof. *** Extra credit. *** □

Lemma 11. Suppose a_1 and a_2 are classes of paths starting at y_0 and having a common endpoint y . Let U be a member of \mathcal{B} such that $y \in U$. If $a_1 \neq a_2$, then \tilde{U}_{a_1} and \tilde{U}_{a_2} are disjoint.

Proof. *** Extra credit. *** □

Lemma 12. p is a covering map.

Proof. *** Extra credit. *** □

Given a path α starting at y_0 and given $t \in [0, 1]$, let α_t be the path given by $\alpha_t(s) = \alpha(ts)$ for $0 \leq s \leq 1$. You may find it helpful to describe by means of words and/or pictures the path $t \mapsto [\alpha_t]$ in X . [You don't need to write this down.]

Lemma 13. The path $t \mapsto [\alpha_t]$ in X is indeed a path, i.e. it is continuous.

Proof. *** Extra credit. *** □

Note that this lemma shows you how to lift any path α in Y starting at y_0 to a path $\tilde{\alpha}$ in X starting at $x_0 := [e_{y_0}]$.

Lemma 14. X is path connected.

Proof. *** Extra credit. *** □

Lemma 15. *X is simply connected.*

Proof. *** Your proof goes here. *** □

[Hint: If $[\alpha] \in \pi_1(Y, y_0)$ is nontrivial, show that its lift $\tilde{\alpha}$ is not a closed curve; deduce the result from this by covering space theory.]

Lemma 16. *There is an action of $\pi_1(Y, y_0)$ on X that makes p a regular $\pi_1(Y, y_0)$ -cover.*

Proof. *** Extra credit. *** □

This completes the proof of Theorem 2.

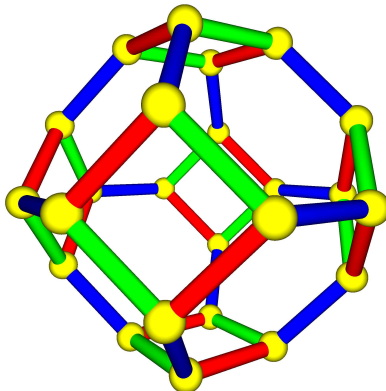
Remark 2. It is possible to show that X covers every other path-connected covering space of Y . [One can use Lemma 8.] For this reason X is often called the *universal cover* of Y .

This is the end of the exam unless you want to read the next section for fun.

7. CAYLEY GRAPHS AND THE FIGURE 8

This (optional) section assumes a little more knowledge of group theory than I have assumed previously.

Let G be a group with a generating set S . The *Cayley graph* of (G, S) is the graph $\Gamma = \Gamma(G, S)$ with G as its vertex set and with an edge joining g and gs for each $g \in G$ and $s \in S$. For example, the Cayley graph of the free group F_2 with its standard generating set is the snowflake. A second example is shown in the following picture; see if you can guess what G and S are. [Hint 1: The number of vertices is the order of the group. Hint 2: Each generator $s \in S$ is assigned a color; so the number of colors of edges is the number of generators.]



The left-translation action of G on itself induces an action of G on the graph Γ and hence an action of G on the geometric realization $X := |\Gamma|$. This action is fixed-point free and makes X a regular G -cover of $Y := X/G$. If $G = F_2$ with its standard generating set, X is a tree and hence is contractible. In particular, it is simply connected. And Y is the figure 8 in this case. Theorem 1 therefore implies that the fundamental group of the figure 8 is free of rank 2, as claimed.