Brief solutions to Prelim 1

1. A ball B of radius r is bounded because $d(x, y) \leq 2r$ for all $x, y \in B$ by the triangle inequality. So it suffices to show that a finite union of bounded sets is bounded. A straightforward induction reduces this to the case of two bounded sets A_1, A_2. Let M_1 and M_2 be bounds for A_1 and A_2, respectively. We may assume that A_1 and A_2 are both nonempty, in which case we can choose points $a_i \in A_i$. I claim that $M := M_1 + d(a_1, a_2) + M_2$ is a bound for $A_1 \cup A_2$: Given $x, y \in A_1 \cup A_2$, if they are both in A_1, then $d(x, y) \leq M_1 \leq M$, so the claim holds in this case. Similarly, the claim holds if they are both in A_2. Finally, if they are in different sets, say $x \in A_1$ and $y \in A_2$, then two applications of the triangle inequality give

$$\begin{aligned} d(x, y) &\leq d(x, a_1) + d(a_1, y) \\ &\leq d(x, a_1) + d(a_1, a_2) + d(a_2, y) \\ &\leq M_1 + d(a_1, a_2) + M_2 \\ &= M. \end{aligned}$$

2. See Munkres.

3. This was a homework problem, but here's the solution: The given topology on B has $\{(U \cap A) \cap B\}$ as its open sets, where U ranges over the open sets of X. Since $(U \cap A) \cap B = U \cap (A \cap B) = U \cap B$, this is the same as the subspace topology that B inherits from X.

4. Yes, f is continuous because each component function is continuous.

5. (a) This is true; see Munkres, Theorem 23.3.

(b) This is false. A counterexample is provided by the circle, which is the union of two closed arcs intersecting in two points.

6. Recall that compactness for a subspace A of a space X can be formulated as follows (see Munkres, Lemma 26.1): If $\mathcal{U}$ is a family of open subsets of X that covers A, then a finite subset of $\mathcal{U}$ covers A.

(a) This is true. If $\mathcal{U}$ is a family of open subsets of X that covers $A \cup B$, then finitely many cover A and finitely many cover B; hence finitely many cover $A \cup B$.

(b) This is also true. A and B are closed in X because X is Hausdorff, so $A \cap B$ is closed. Therefore $A \cap B$ is compact, being a closed subset of the compact space A.