**Mathematics 6310**
**The ordinal numbers**
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## 1. Natural numbers

All of modern mathematics is based on set theory. In the beginning, there is one primitive undefined term ("set"), and one primitive undefined relation ("membership"). There are then axioms that (we hope) conform with our intuition about what sets are and how one can construct them. All mathematical objects have to be built using these axioms. In particular, natural numbers have to be built in this way.

Let's start with 0. If 0 is to be a set, there is one and only one sensible definition:

$$0 := \emptyset.$$

Now how should we define 1? It has to be a set, and in order for it to be a reasonable candidate for 1, it should be a singleton. Again, there is an obvious candidate:

$$1 := \{0\}.$$

We can similarly define

$$2 := \{0, 1\}, \quad 3 := \{0, 1, 2\},$$

and so on. Each number $n$ that we construct in this way is a set containing (intuitively) exactly $n$ elements. The axioms of set theory enable us make the phrase "and so on" precise, and we obtain the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \dots\},$$

with all the usual properties. Having done this, we can say that a natural number $n$ is a specific set with exactly $n$ elements, those elements being precisely the predecessors of $n$:

$$n = \{0, 1, \dots, n-1\}.$$

## 2. Properties of natural numbers

Numbers are sets. (*Everything* is a set!) If $n$ is a natural number, then all previous natural numbers are elements of $n$. In fact, $n$ *is* the set of previous natural numbers. Moreover, every previous natural number is a subset of $n$. Thus $n$ is a set with the property that each of its elements is simultaneously an element of $n$ and a subset of $n$. We can therefore use set-theoretic membership or set-theoretic inclusion to characterize the standard order relation on $\mathbb{N}$:

$$m < n \iff m \in n \iff m \subset n.$$

[Here and throughout this handout I use $\subset$ for *strict* inclusion.] Note next that there is a simple set-theoretic description of the successor operation $n \mapsto n + 1$:

$$n + 1 = n \cup \{n\}.$$

In words, we get the set $n + 1$ from the set $n$ by adjoining a single new element, that element being $n$ itself. (If you think about it, this is the only sensible thing we could add that isn't already there.)

Finally, recall that the ordering on $\mathbb{N}$ is a well-ordering, so that one can give proofs and definitions by induction.

## 3. Ordinal numbers

By analogy with what we've done above, we wish to define ordinals in such a way that they are well-ordered, with an order characterized by

$$\alpha < \beta \iff \alpha \in \beta \iff \alpha \subset \beta. \tag{1}$$

Moreover, each ordinal $\alpha$ should have a successor $\alpha + 1$, such that

$$\alpha + 1 = \alpha \cup \{\alpha\}\,. \tag{2}$$

We expect (again by analogy with the natural numbers) that every element of an ordinal is again an ordinal. The content of the first equivalence in (1) is then that every ordinal *is* the set of its predecessors, just as for natural numbers. These considerations motivate the following definition.

**Definition.** An *ordinal number* is a well-ordered set $\alpha$ such that every element of $\alpha$ is the set of its predecessors.

Note that the well-ordering on $\alpha$ does not have to be specified, since it is necessarily given by the membership relation: Given $x, y \in \alpha$, we have

$$x < y \iff x \in y.$$

The finite ordinals (i.e., the ordinals that are finite sets) are precisely the natural numbers. The first infinite ordinal is the set $\mathbb{N}$. It is customary to use lowercase Greek letters for ordinals, and we usually write $\omega$ for $\mathbb{N}$ when we want to think of $\mathbb{N}$ as an ordinal.

It is not hard to verify that there is a well-ordering of the ordinals characterized by (1). Moreover, every ordinal is equal to the set of its predecessor ordinals, and every ordinal has a successor ordinal $\alpha + 1$ defined by (2). Finally, nonempty collections of ordinals have greatest lower bounds and least upper bounds, obtained by taking intersections and unions. [Ordinals are sets, so it makes sense to take the intersection and union of a collection of ordinals; one checks that these are again ordinals.] All of these assertions are proved in virtually every book on set theory. See, for example, Jech, *Set theory*, for a clear, concise presentation in just a few pages. For a more leisurely treatment, see Halmos, *Naïve set theory*.

One last remark: The totality of all ordinals is *not* a set. (In some treatments of set theory it is called a *class*.) If it were a set, it would have a least upper bound, which would then be a largest ordinal. But that's absurd, since every ordinal has a successor.

## 4. Ordinals and the well-ordering theorem

As an illustration of the use of ordinals, we outline a very short proof of the well-ordering theorem, which asserts that every set $X$ admits a well-ordering. Since every ordinal is a well-ordered set, it suffices to show that $X$ is in 1–1 correspondence with some ordinal. To this end we define by induction a 1–1 transfinite sequence $x_\alpha \in X$, indexed by an initial segment of the ordinals. Assume inductively that $\alpha$ is an ordinal such that $x_\beta$ has been defined for $\beta < \alpha$. If $\{x_\beta\}$ is a proper subset of $X$, choose $x_\alpha$ arbitrarily in $X \smallsetminus \{x_\beta\}$ [using the axiom of choice]. Since the ordinals do not form a set, the sequence cannot be defined for all ordinals. Let $\alpha$ be the first ordinal such that $x_\alpha$ is not defined. Then we have a bijection between $X$ and $\{\beta \mid \beta < \alpha\} = \alpha$.