Mathematics 7350

The Knuth–Bendix procedure: Example

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This example is adapted from one given by Holt in the documentation for his KBMAG package ("Knuth–Bendix for Monoids and Automatic Groups"). Let

$$G := \langle\, a, b \;;\; [b, a, b] = [b, a, a, a, a] = [b, a, a, a, b, a, a] = 1 \,\rangle\,.$$

Our convention for commutators here is that

$$[x, y] := x^{-1}y^{-1}xy,$$

and iterated commutators are computed from left to right. For example,

$$[x, y, z] := [[x, y], z].$$

To simplify working with the iterated commutators in the defining relators, we introduce five new generators c, d, e, f, g and five new relations:

$$\begin{aligned} c &= [b, a] \\ d &= [c, a] \\ e &= [d, a] \\ f &= [e, b] \\ g &= [f, a]. \end{aligned} \tag{1}$$

The original three relations can now be written as

$$\begin{aligned} cb &= bc \\ ea &= ae \\ ga &= ag. \end{aligned} \tag{2}$$

Finally, for reasons that we will explain at the end of this handout, it will be convenient to adjoin a new generator h and the relation

$$h = [g, b]. \tag{3}$$

Our goal is to find a complete rewriting system for G. To this end we write the relations in (1) and (3) as commutation relations:

$$\begin{aligned} ba &= abc \\ ca &= acd \\ da &= ade \\ eb &= bef \\ fa &= afg \\ gb &= bgh. \end{aligned} \tag{4}$$

And we introduce 8 new monoid generators $a^{-1}, b^{-1}, \dots, h^{-1}$ and the corresponding 16 relations $xx^{-1} = 1$ for $x \in \{a^{\pm 1}, b^{\pm 1}, \dots, h^{\pm 1}\}$. We now have a monoid with 16 generators and 25 relations, the latter being the relations (2) and (4) and the 16 relations $xx^{-1} = 1$.

This presentation is reminiscent of the one we gave for the Heisenberg group, for which a recursive (or wreath product) order was useful. So let's try the same thing here. We order the words in our 16 generators by the wreath product order, where the generators have priorities

$$a^{-1} > a > b^{-1} > b > \cdots > h^{-1} > h.$$

Note that the 25 relations we have defined are order decreasing if we take them as written and convert every "=" to "→".

We now let KBMAG go to work on this monoid presentation and try to find a complete rewriting system. It churns away for a few minutes and then gives up after producing 32,766 rewriting rules, which is the maximum it will produce (by default). We could try increasing the maximum number of equations and trying again, but it turns out that a better strategy is to tell KBMAG to only retain rules $u \to v$ in which u and v have length at most 10.

This time KBMAG terminates very quickly with a confluent system of 101 rewriting rules. It warns us that the resulting complete rewriting system might not give a presentation for the monoid G that we started with, because some rules have been discarded and hence the equivalence relation on words may have changed. But we can inspect the set of rules and see that it does indeed contain enough rules to deduce the defining relations, so we have succeeded. [Alternatively, we could simply adjoin rules for the defining relations and rerun the procedure, but this time without the limitation on length; it will terminate and report success.]

The resulting system turns out to be very similar to the one we gave for the Heisenberg group. In particular, we can see that G is a nilpotent group, which was by no means obvious a priori. The 101 rules can be organized as follows:

- There are 12 rules of the form $xx^{-1} \to 1$, for $x \in \{a^{\pm 1}, \ldots, f^{\pm 1}\}$.
- There are 4 rules that explain why we didn't need to include $g^{\pm 1}$ and $h^{\pm 1}$ in the first bunch of rules:

$$\begin{aligned} h^2 &\to 1 \\ h^{-1} &\to h \\ g^2 &\to h \\ g^{-1} &\to gh \end{aligned}$$

- There are 13 rules $hx \to xh$ for $x \in \{a^{\pm 1}, b^{\pm 1}, \ldots, f^{\pm 1}, g\}$, expressing the fact that h is central in G.
- There are 12 rules of the form $gx \to xgw$ for $x \in \{a^{\pm 1}, b^{\pm 1}, \ldots, f^{\pm 1}\}$, expressing the fact that g is central in $G/\langle h \rangle$:

$$\begin{aligned} g^{\pm 1}a &\to ag^{\pm 1} \\ gb^{\pm 1} &\to b^{\pm 1}gh \\ gc^{\pm 1} &\to c^{\pm 1}g \\ gd^{\pm 1} &\to d^{\pm 1}g \\ ge^{\pm 1} &\to e^{\pm 1}g \\ gf^{\pm 1} &\to f^{\pm 1}g \end{aligned}$$

- There are 20 rules of the form $f^{\pm 1}x \to xf^{\pm 1}w$ for $x \in \{a^{\pm 1}, b^{\pm 1}, \dots, e^{\pm 1}\}$, expressing the fact that f is central in $G/\langle g, h\rangle$:

$$fa \to afg$$
$$fa^{-1} \to a^{-1}fgh$$
$$f^{-1}a \to af^{-1}gh$$
$$f^{-1}a^{-1} \to a^{-1}f^{-1}g$$
$$f^{\pm 1}b^{\pm 1} \to b^{\pm 1}f^{\pm 1}h$$
$$f^{\pm 1}c^{\pm 1} \to c^{\pm 1}f^{\pm 1}h$$
$$f^{\pm 1}d^{\pm 1} \to d^{\pm 1}f^{\pm 1}$$
$$f^{\pm 1}e^{\pm 1} \to e^{\pm 1}f^{\pm 1}$$

- There are 16 rules of the form $e^{\pm 1}x \to xe^{\pm 1}w$ for $x \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}\}$, expressing the fact that e is central in $G/\langle f, g, h\rangle$:

$$e^{\pm 1}a^{\pm 1} \to a^{\pm 1}e^{\pm 1}$$
$$eb \to bef$$
$$eb^{-1} \to b^{-1}ef^{-1}h$$
$$e^{-1}b \to be^{-1}f^{-1}$$
$$e^{-1}b^{-1} \to b^{-1}e^{-1}fh$$
$$ec \to cegh$$
$$ec^{-1} \to c^{-1}eg$$
$$e^{-1}c \to ce^{-1}g$$
$$e^{-1}c^{-1} \to c^{-1}e^{-1}gh$$
$$e^{\pm 1}d^{\pm 1} \to d^{\pm 1}e^{\pm 1}$$

- There are 12 rules of the form $d^{\pm 1}x \to xd^{\pm 1}w$ for $x \in \{a^{\pm 1}, b^{\pm 1}, c^{\pm 1}\}$, expressing the fact that d is central in $G/\langle e, f, g, h\rangle$:

$$da \to ade$$
$$da^{-1} \to a^{-1}de^{-1}$$
$$d^{-1}a \to ad^{-1}e^{-1}$$
$$d^{-1}a^{-1} \to a^{-1}d^{-1}e$$
$$db \to bdf$$
$$db^{-1} \to b^{-1}df^{-1}h$$
$$d^{-1}b \to bd^{-1}f^{-1}$$
$$d^{-1}b^{-1} \to b^{-1}d^{-1}fh$$
$$dc \to cdf^{-1}h$$
$$dc^{-1} \to c^{-1}df$$
$$d^{-1}c \to cd^{-1}fh$$

$$d^{-1}c^{-1} \to c^{-1}d^{-1}f^{-1}$$

- There are 8 rules of the form $c^{\pm 1}x \to xc^{\pm 1}w$ for $x \in \{a^{\pm 1}, b^{\pm 1}\}$, expressing the fact that c is central in $G/\langle d, e, f, g, h\rangle$:

$$\begin{aligned} ca &\to acd \\ ca^{-1} &\to a^{-1}cd^{-1}e \\ c^{-1}a &\to ac^{-1}d^{-1}f^{-1} \\ c^{-1}a^{-1} &\to a^{-1}c^{-1}de^{-1}fgh \\ c^{\pm 1}b^{\pm 1} &\to b^{\pm 1}c^{\pm 1} \end{aligned}$$

- There are 4 rules of the form $b^{\pm 1}a^{\pm 1} \to a^{\pm 1}b^{\pm 1}w$, expressing the fact that b is central in $G/\langle c, d, e, f, g, h\rangle$:

$$\begin{aligned} ba &\to abc \\ ba^{-1} &\to a^{-1}bc^{-1}de^{-1}fgh \\ b^{-1}a &\to ab^{-1}c^{-1} \\ b^{-1}a^{-1} &\to a^{-1}b^{-1}cd^{-1}e \end{aligned}$$

Remark. It might seem like a miracle that we chose the right generators, in the right order, to show nilpotence. In fact, one can arrive at this choice systematically by using a "nilpotent quotient" algorithm to try to find a large nilpotent quotient of G. The algorithm produces the generators that we used, as well as the relations that we found via Knuth–Bendix. Of course, we didn't know until running Knuth–Bendix that this nilpotent quotient is actually G itself.