Recall that we gave an ad hoc proof in class that $A_5$ is simple. This proof is also in the book (p. 128), and there is a second ad hoc proof on p. 145. The book goes on in Section 4.6 to prove the simplicity of $A_n$ for $n > 5$. I'll give here a different proof, based on fairly general principles that are used in a lot of simplicity proofs.

The key idea is to play with the commutator $[g, h] := ghg^{-1}h^{-1}$. Notice that we can think of this either as $g$ times a conjugate of $g^{-1}$ or as a conjugate of $h$ times $h^{-1}$. We will exploit these two ways of thinking about the commutator in our proof. [You've actually seen this idea already in Exercise 3.1.42, where you proved that if two normal subgroups have trivial intersection, then the elements of one commute with those of the other.]

The first step is to find a set of generators for $A_n$ having small support (i.e., moving few points): I claim that $A_n$ is generated by 3-cycles; moreover, any two 3-cycles are conjugate if $n \geq 5$. For the first assertion, it suffices to observe that a product of two distinct transpositions is either a 3-cycle or a product of two 3-cycles. The typical cases are $(1 \, 2)(2 \, 3) = (1 \, 2 \, 3)$ and $(1 \, 2)(3 \, 4) = (1 \, 4 \, 3)(1 \, 2 \, 3)$. For the second assertion, recall first that any two 3-cycles $\sigma, \tau$ are conjugate in $S_n$, say $g\sigma g^{-1} = \tau$. If $g$ happens to be an odd permutation, replace it by $gh$, where $h$ is a transposition that centralizes $\sigma$. [This exists because $n \geq 5$.]

Now suppose $1 \neq H \trianglelefteq A_n$ ($n \geq 5$). In view of the claim that we've just proved, the simplicity of $A_n$ will follow if we can show that $H$ contains a 3-cycle. So our strategy will be to try to find an element $h \in H$ with the simplest possible cycle structure, until we eventually find a 3-cycle in $H$.

Start with any nontrivial $h \in H$, say $h(1) = 2$. Let $g$ be any 3-cycle that doesn't involve 2, e.g., $g = (1 \, 3 \, 4)$, and consider the commutator $[g, h]$. Thinking of the commutator as $kh^{-1}$, where $k$ is a conjugate of $h$, we see that it is in $H$. But we can also view $[g, h]$ as $g(h^{-1}h^{-1})$, so it is a product of two 3-cycles, $(1 \, 3 \, 4)(2 \, h(4) \, h(3))$. The presence of 2 in only one spot shows that $[g, h] \neq 1$. Replacing $h$ by this commutator, then, we now have a nontrivial $h \in H$ that is a product of two 3-cycles! Let's try to simplify it further.

Suppose first that the two 3-cycles are disjoint, so that $h$ has a cycle decomposition of the type $h = (1 \, 2 \, 3)(4 \, 5 \, 6)$. All permutations of this type are conjugate in $A_n$, since $(1 \, 2 \, 3)(4 \, 5 \, 6)$ is centralized by the odd permutation $(1 \, 4)(2 \, 5)(3 \, 6)$. So $H$ contains all of them. In particular, $H$ contains $k := (1 \, 2 \, 3)(4 \, 5 \, 6)^{-1}$, and the product $kh$ is a 3-cycle in $H$, as desired.

Suppose now that the two 3-cycles are not disjoint. Then $h$ moves at most 5 points, so $h$ is contained in an isomorphic copy of $A_5$ inside $A_n$. Thus $H' := H \cap A_5$ is a nontrivial normal subgroup of $A_5$ and hence is all of $A_5$, since we already know that the latter is simple. In particular, $H'$ contains a 3-cycle, and the proof is complete.

What made this proof work, aside from some ad hoc arguments once we got down to $A_5$? I see two general principles: (a) If $H \trianglelefteq G$, then $H$ is closed under the operation of forming commutators with arbitrary elements of $G$. (b) If $g$ has “small support”, then $[g, h]$ is often simpler than $h$ [because it is the product of $g$ and something that resembles $g$].