Mathematics 4340

On the relation $bab^{-1} = a^2$

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In 1962 G. Baumslag and D. Solitar introduced an interesting family of groups, each presented by two generators a, b and one relation of the form $ba^m b^{-1} = a^n$. Each particular choice of the pair m, n yields a specific group. If $(m, n) = (1, 1)$, for example, the relation just says that $ab = ba$, and we get the *free abelian group on two generators*, isomorphic to $\mathbb{Z} \times \mathbb{Z}$. (Compare Exercise 6.3.11 on p. 221.) Another easy case is $(m, n) = (1, -1)$. This is isomorphic to the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}$ with the (unique) nontrivial action of $\mathbb{Z}$ on $\mathbb{Z}$; it is well-known to topologists as the fundamental group of the Klein bottle. Here we will treat the case $(m, n) = (1, 2)$ that occurred in the extra-credit problem 4 on Assignment 9. The relation in this case says $bab^{-1} = a^2$.

Let $G := \left\langle a, b \mid bab^{-1} = a^2 \right\rangle$. Consider the conjugates $a_n := b^{-n}ab^n$:

$$\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots.$$

Note that conjugation by b^{-1} shifts every element in this list one place to the right (i.e., $b^{-1}a_n b = a_{n+1}$), and conjugation by b shifts left. The defining relation, with this notation, becomes $a_{-1} = a_0^2$. Conjugating by positive and negative powers of b, we deduce that $a_n^2 = a_{n-1}$ for all n. Thus each element in the list is a square root of the preceding one. [Intuitively, then, we might think of a_n as $a^{1/2^n}$.] Since $a_{n-1} \in \langle a_n \rangle$, we have a chain of cyclic subgroups

$$\cdots \langle a_{-1} \rangle \leq \langle a_0 \rangle \leq \langle a_1 \rangle \leq \langle a_2 \rangle \cdots .$$

Let A be the union. It is an abelian subgroup of G, in which each element can be written in the form a_n^m. [Intuitively, this is a^r, where $r = m/2^n$.] Note also that $ba_n^m b^{-1} = a_{n-1}^m = a_n^{2m}$. [Intuitively, $ba^r b^{-1} = a^{2r}$.]

The next observation is that $G = AB$, where $B = \langle b \rangle$. To see this, use the relations $ba_n^m = a_n^{2m} b$ and $b^{-1}a_n^m = a_{n+1}^m b^{-1}$ to move all powers of b to the right in any $\left\{a^{\pm 1}, b^{\pm 1}\right\}$-word. Initially, of course, the only a_n that occurs is $a = a_0$; but the first time we move a b^{-1} past an a, we introduce a_1, and then higher subscripts can creep in as we continue.

It should now seem quite plausible that A is isomorphic to the additive group $\mathbb{Z}[1/2]$ of dyadic rationals, that B is infinite cyclic, and that G is the semidirect product of A and B. In other words, we are guessing that

$$G \cong \mathbb{Z}[1/2] \rtimes \langle 2 \rangle < \mathbb{Q} \rtimes \mathbb{Q}^{\times},$$

where the multiplicative group acts on the additive group by multiplication. Recall that this semidirect product consists of pairs $(r, 2^i)$ with $r \in \mathbb{Z}[1/2]$, with group law

$$(r, 2^i)(s, 2^j) = (r + 2^i s, 2^{i+j}).$$

This group contains $\mathbb{Z}[1/2]$ as an abelian normal subgroup, embedded by $r \mapsto (r, 2^0)$, and the element $\beta := (0, 2^1)$ acts on this subgroup by

$$\beta(r, 2^0)\beta^{-1} = (2r, 2^0) = (r, 2^0)^2.$$

In particular, if $\alpha := (1, 2^0)$, then $\beta^{-n}\alpha\beta^n = (1/2^n, 2^0)$ for all $n \in \mathbb{Z}$.

We can now prove the guess. Use the universal mapping property associated to the presentation of G to construct $\phi\colon G \to \mathbb{Z}[1/2] \rtimes \langle 2 \rangle$ with $\phi(a) = \alpha$ and $\phi(b) = \beta$. We know that any $g \in G$ can be written as $g = a_n^m b^i$, and the formulas in the previous paragraph show that $\phi(g) = (r, 2^i)$, where $r = m/2^n$. It follows at once that ϕ is 1–1 and onto.