Mathematics 4340
A presentation for the symmetric group
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Here is a slightly different way to organize the solution to additional problem 3 on Assignment 9 than the one in the suggested outline. It may be a little clearer.

The crux of the matter is to prove that the group G defined by the given generators and relations has order $\leq n!$. I'll prove the following sharper statement: Given any word in the generators t_i, we can use the defining relations to rewrite it in "canonical form"

$$w_1 w_2 \cdots w_{n-1},$$

where the word w_i is either empty or is a decreasing run of consecutive generators, starting with t_i. Thus $w_1 = 1$ or t_1; $w_2 = 1$ or t_2 or t_2t_1; $w_3 = 1$ or t_3 or t_3t_2 or $t_3t_2t_1$; and so on. The rewriting process I have in mind involves three types of substitutions: (a) $t_i^2 \to 1$; (b) $t_it_j \to t_jt_i$ if $|i-j| \geq 2$; (c) $t_{i+1}t_it_{i+1} \to t_it_{i+1}t_i$. Note that none of these operations increases the length of the word. Note also that there are exactly $n!$ canonical forms, since there are $i+1$ choices for w_i. We will prove the assertion by induction on n.

Among all possible words w obtained by rewriting the given word, choose one of minimal length. Then, among all possible words w of minimal length, choose one that minimizes the number of occurences of t_{n-1}. I claim there will be at most one occurence of t_{n-1}. Suppose this is false. Then there is a subword of the form $t_{n-1}ut_{n-1}$, where u does not involve t_{n-1}. By the induction hypothesis we can assume u is in canonical form. Then t_{n-2} must occur, since otherwise we could shorten w by the rewrites

$$t_{n-1}ut_{n-1} \to ut_{n-1}^2 \to u.$$

And it must occur exactly once by the nature of the canonical forms. Our subword now has the form $t_{n-1}u_1t_{n-2}u_2t_{n-1}$, where u_1 and u_2 only involve t_i with $i < n-2$. We can therefore do the rewrites

$$t_{n-1}u_1t_{n-2}u_2t_{n-1} \to u_1t_{n-1}t_{n-2}t_{n-1}u_2 \to u_1t_{n-2}t_{n-1}t_{n-2}u_2;$$

this preserves the length of the word while reducing the number of occurrences of t_{n-1}. This contradiction proves the claim.

We're almost done. Choose w above so that t_{n-1}, if it occurs, is as far to the right as possible. If it doesn't occur, we're done by induction. If it does occur, then $w = ut_{n-1}v$, where u and v don't involve t_{n-1} and can be put in canonical form. Then v, if it is nontrivial, starts with t_{n-2}; otherwise we could move t_{n-1} to the right. Since v is in canonical form, it must be a decreasing run of consecutive generators starting with t_{n-2}. Then $t_{n-1}v$ is a decreasing run starting with t_{n-1}, and w is in canonical form.

Remark. Once the exercise is completely done, a counting argument shows that every element of S_n is represented by a *unique* canonical form.