THE GALOIS CORRESPONDENCE BETWEEN A RING AND ITS SPECTRUM

Let A and X be two sets equipped with a relation $R \subseteq A \times X$. The notation is intended to remind you of the case where A is a commutative ring, X is its spectrum $\operatorname{Spec} A$, and R is given by

$$(f,x) \in R \iff f(x) = 0 \quad \text{(i.e., } f \in \mathfrak{p}_x).$$

But mathematics is full of other examples, including Galois theory (which is where the phrase "Galois correspondence" in the title comes from).

For any subset $S \subseteq A$, define $S' \subseteq X$ by

$$S' := \{x \in X \mid (f,x) \in R \text{ for all } f \in S\}.$$

Similarly, for any subset $Y \subseteq X$, define $Y' \subseteq A$ by

$$Y' := \{f \in A \mid (f,x) \in R \text{ for all } x \in Y\}.$$

In our canonical example, $S' = V(S)$ (the "zero set" of S), and $Y' = I(Y) := \bigcap_{y \in Y} \mathfrak{p}_y$. In more intuitive language, Y' is the ideal of functions that vanish on Y. These "prime" operations are order reversing:

$$S \subseteq T \implies T' \subseteq S',$$

and similarly for subsets of X.

Any subset of X of the form S' will be called *closed*. (This gives the Zariski topology in our canonical example.) Similarly, any subset of A of the form Y' will be called *closed*. (So the closed subsets of A are the radical ideals in our canonical example.) Note that we always have

$$S \subseteq S'' \text{ and } Y \subseteq Y''$$

for $S \subseteq A$ and $Y \subseteq X$. I claim that equality holds for closed sets. Suppose, for example that Y is closed, say $Y = S'$. Then we can prime the inclusion $S \subseteq S''$ to get $S' \supseteq S'''$, which says precisely that $Y \supseteq Y''$, as claimed.

It follows that the prime operation gives a bijection between the closed subsets of A and the closed subsets of X. In the canonical example, this is the bijection between radical ideals and Zariski-closed sets. Finally, I claim that double-prime is the closure operation. In other words, Y'' is the smallest closed set containing Y, and S'' is the smallest closed set containing S. Suppose, for instance, that Y is contained in a closed set Z. Since double-prime is order preserving, we obtain $Z'' \supseteq Y''$, hence $Z \supseteq Y''$ since $Z = Z''$. This proves the claim.

Returning, finally, to our canonical example, the last observation has two parts. If we start with $S \subseteq A$, the assertion is that $I(V(S))$ is the radical of the ideal generated by S. I think I pointed this out in class, at least in the case where S is an ideal. In the other direction, if we start with a subset $Y \subseteq X$, the assertion is that $V(I(Y))$ is the Zariski closure of Y. Intuitively, we look at all the functions that vanish on Y, and then we take *all* points where these functions vanish; this is the closure of Y. If Y is a singleton $\{x\}$, this says that the closure is $V(\mathfrak{p}_x)$. That should look familiar to you.

For more information, google "Galois correspondence" or "Galois connection". The latter will give you, in particular, a long Wikipedia article with many examples.