THE GALOIS CORRESPONDENCE BETWEEN
A RING AND ITS SPECTRUM

Let $A$ and $X$ be two sets equipped with a relation $R \subseteq A \times X$. The notation is intended to remind you of the case where $A$ is a commutative ring, $X$ is its spectrum $\text{Spec} \, A$, and $R$ is given by

$$(f, x) \in R \iff f(x) = 0 \quad \text{(i.e., } f \in p_x).$$

But mathematics is full of other examples, including Galois theory (which is where the phrase “Galois correspondence” in the title comes from).

For any subset $S \subseteq A$, define $S' \subseteq X$ by

$$S' := \{ x \in X \mid (f, x) \in R \text{ for all } f \in S \}.$$

Similarly, for any subset $Y \subseteq X$, define $Y' \subseteq A$ by

$$Y' := \{ f \in A \mid (f, x) \in R \text{ for all } x \in Y \}.$$

In our canonical example, $S' = V(S)$ (the “zero set” of $S$), and $Y' = I(Y) := \bigcap_{p \in Y} p$. In more intuitive language, $Y'$ is the ideal of functions that vanish on $Y$. These “prime” operations are order reversing:

$$S \subseteq T \implies T' \subseteq S',$$

and similarly for subsets of $X$.

Any subset of $X$ of the form $S'$ will be called closed. (This gives the Zariski topology in our canonical example.) Similarly, any subset of $A$ of the form $Y'$ will be called closed. (So the closed subsets of $A$ are the radical ideals in our canonical example.) Note that we always have

$$S \subseteq S'' \text{ and } Y \subseteq Y''$$

for $S \subseteq A$ and $Y \subseteq X$. I claim that equality holds for closed sets. Suppose, for example that $Y$ is closed, say $Y = S'$. Then we can prime the inclusion $S \subseteq S'$ to get $S' \supseteq S''$, which says precisely that $Y \supseteq Y''$, as claimed.

It follows that the prime operation gives a bijection between the closed subsets of $A$ and the closed subsets of $X$. In the canonical example, this is the bijection between radical ideals and Zariski-closed sets. Finally, I claim that double-prime is the closure operation. In other words, $Y''$ is the smallest closed set containing $Y$, and $S''$ is the smallest closed set containing $S$. Suppose, for instance, that $Y$ is contained in a closed set $Z$. Since double-prime is order preserving, we obtain $Z'' \supseteq Y''$, hence $Z \supseteq Y''$ since $Z = Z''$. This proves the claim.

Returning, finally, to our canonical example, the last observation has two parts. If we start with $S \subseteq A$, the assertion is that $I(V(S))$ is the radical of the ideal generated by $S$. I think I pointed this out in class, at least in the case where $S$ is an ideal. In the other direction, if we start with a subset $Y \subseteq X$, the assertion is that $V(I(Y))$ is the Zariski closure of $Y$. Intuitively, we look at all the functions that vanish on $Y$, and then we take all points where these functions vanish; this is the closure of $Y$. If $Y$ is a singleton $\{x\}$, this says that the closure is $V(p_x)$. That should look familiar to you.

For more information, google “Galois correspondence” or “Galois connection”. The latter will give you, in particular, a long Wikipedia article with many examples.