Forest Diagrams for Elements of Thompson's Group F

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Abstract

We introduce *forest diagrams* to represent elements of Thompson's group F. These diagrams relate to a certain action of F on the real line in the same way that tree diagrams relate to the standard action of F on the unit interval. Using forest diagrams, we give a conceptually simple length formula for elements of F with respect to the $\{x_0, x_1\}$ generating set, and we discuss the construction of minimum-length words for positive elements. Finally, we use forest diagrams and the length formula to examine the structure of the Cayley graph of F.

1 Introduction

Thompson's group F is defined by the following infinite presentation:

$$F = \langle x_0, x_1, x_2, \ldots \mid x_n x_k = x_k x_{n+1} \text{ for } n > k \rangle$$

It is isomorphic to the group $\mathrm{PL}_2(I)$ of all piecewise-linear, orientation-preserving homeomorphisms of the unit interval satisfying the following conditions:

1. All slopes are integral powers of 2, and

2. All breakpoints have dyadic rational coordinates.

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The group F was first studied by Richard J. Thompson in the 1960s. The standard introduction to F is [CFP].

This paper is organized as follows:

- In **Section 2**, we give the necessary background regarding F. In particular, we review how elements of $\mathrm{PL}_2(I)$ can be described by tree diagrams.
- In **Section 3**, we introduce a group $\mathrm{PL}_2(\mathbb{R})$ of piecewise-linear homeomorphisms of the real line that is isomorphic with F. We then show how to represent elements of $\mathrm{PL}(\mathbb{R})$ by *forest diagrams*.
- In **Section 4**, we use forest diagrams to examine the lengths of elements of Thompson's group with respect to the $\{x_0, x_1\}$ generating set. We begin by studying positive elements, where the situation is quite simple, and then move on to the general length formula.
- In **Section 5**, we give some further applications of forest diagrams and the length formula.

Note. Throughout this paper, we will use the following convention for composition of functions:

$$(f \circ g)(x) = f(g(x))$$

This disagrees with Thompson's original notation, but it agrees with the notation in [CFP].

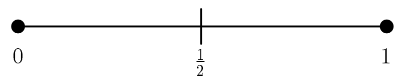
We are grateful to the referee for reading the original version of this paper so carefully and making many helpful suggestions and comments.

2 Background on F

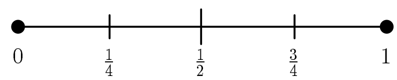
Most of the results in this section are stated without proof. Details can be found in [CFP].

2.1 Tree Diagrams

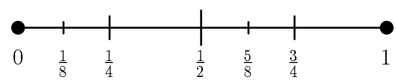
Suppose we take the interval $[0, 1]$ and cut it in half, like this:



We then cut each of the resulting intervals in half:



and then cut some of the new intervals in half:



to get a certain subdivision of $[0, 1]$. Any subdivision of $[0, 1]$ obtained in this manner (i.e. by repeatedly cutting intervals in half) is called a *dyadic subdivision*.

The intervals of a dyadic subdivision are all of the form:

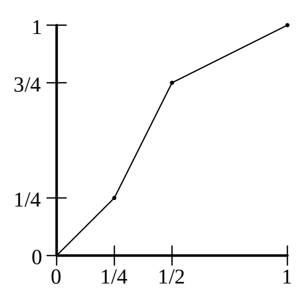
$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \qquad k, n \in \mathbb{N}$$

These are the *standard dyadic intervals*. We could alternatively define a dyadic subdivision as any partition of $[0, 1]$ into standard dyadic intervals.

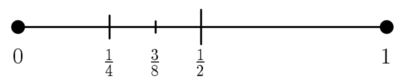
Each element of $\mathrm{PL}_2(I)$ can be described by a pair of dyadic subdivisions:

Proposition 2.1.1. *Let $f \in \mathrm{PL}_2(I)$. Then there exist dyadic subdivisions $\mathcal{D}, \mathcal{R}$ of $[0, 1]$ such that f maps each interval of $\mathcal{D}$ linearly onto an interval of $\mathcal{R}$.* $\square$

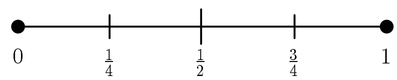
Example 2.1.2. Consider the element $f \in \mathrm{PL}_2(I)$ with graph:



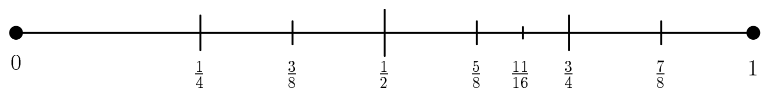
Then f maps intervals of the subdivision:



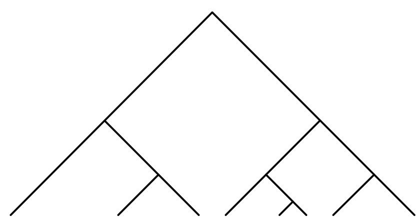
linearly onto intervals of the subdivision:



We can represent dyadic subdivisions of $[0, 1]$ by finite binary trees. For example, the subdivision:



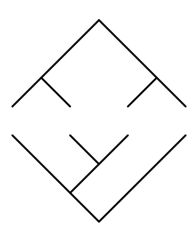
corresponds to the binary tree:



Each leaf of this tree represents an interval of the subdivision, and the root represents the interval $[0, 1]$. The other nodes represent standard dyadic intervals from intermediate stages of the dyadic subdivision.

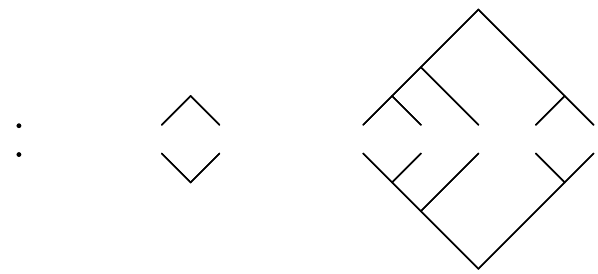
Combining this observation with proposition 2.1.1, we see that any element $f \in \mathrm{PL}_2(I)$ can be described by a pair of binary trees. This is called a *tree diagram* for f.

Example 2.1.3. Let f be the element of $\mathrm{PL}_2(I)$ from example 2.1.2. Then f has tree diagram:

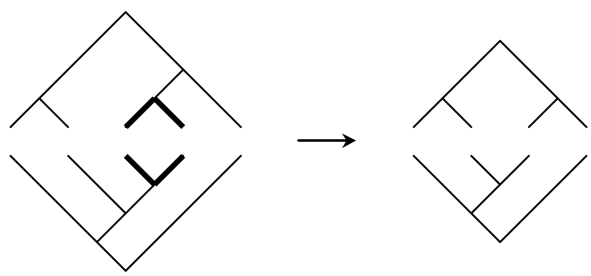


We have aligned the two trees vertically so that corresponding leaves match up. By convention, the *domain tree* appears on the *bottom*, and the *range tree* appears on the *top*.

The tree diagram for an element $f \in \mathrm{PL}_2(I)$ is not unique. For example, all of the following are tree diagrams for the identity:



In general, a *reduction* of a tree diagram consists of removing an opposing pair of carets, like this:



Performing a reduction does not change the element of $\mathrm{PL}_2(I)$ described by a tree diagram: it merely corresponds to removing an unnecessary "cut" from the subdivisions of the domain and range.

Definition 2.1.4. A tree diagram is *reduced* if it has no opposing pairs of carets.

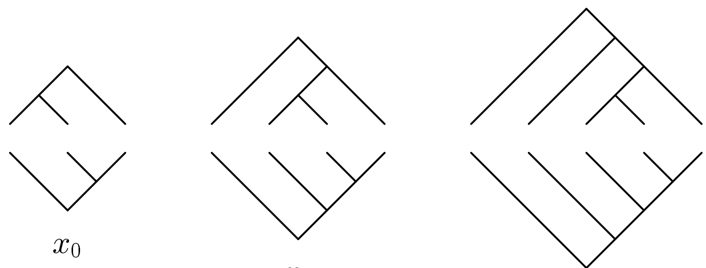
Proposition 2.1.5. *Every element of* $\mathrm{PL}_2(I)$ *has a unique reduced tree diagram.* $\square$

2.2 Positive Elements and Normal Form

Recall that F has presentation:

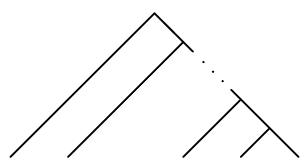
$$F = \langle x_0, x_1, x_2, \ldots \mid x_n x_k = x_k x_{n+1} \text{ for } k < n \rangle$$

We have previously asserted that F is isomorphic with $\mathrm{PL}_2(I)$. One such isomorphism is defined as follows:



x_0 x_1 x_2 $\cdots$

Note that the domain trees of the x_i's all have the property that no caret has a left child. Such a tree is called a *right vine*:



Definition 2.2.1. An element of F is *positive* if it lies in the submonoid generated by $\{x_0, x_1, x_2, \ldots\}$.

Proposition 2.2.2. *An element of F is positive if and only if the bottom tree of its reduced tree diagram is a right vine.* $\square$

It turns out that F is the group of fractions of its positive monoid, in the sense that any element of F can be written as pq^{-1} for some positive p and q. More precisely:

Proposition 2.2.3 (Normal Form). *Every element of F can be expressed uniquely in the form:*

$$x_0^{a_0} \cdots x_n^{a_n} x_n^{-b_n} \cdots x_0^{-b_0}$$

where $a_0, \ldots, a_n, b_0, \ldots, b_n \in \mathbb{N}$ and:

1. *Either* $a_n > 0$ *or* $b_n > 0$*, but not both.*

2. *If both* $a_i > 0$ *and* $b_i > 0$*, then either* $a_{i+1} > 0$ *or* $b_{i+1} > 0$. $\square$

The first half of the normal form is called the *positive part* of an element, and the second half is called the *negative part*. These halves correspond to the two halves of the tree diagram:

Proposition 2.2.4. *Let* $\begin{bmatrix} T_+ \\ T_- \end{bmatrix}$ *be the reduced tree diagram for an element* $f \in F$*, and let* V *be a right vine with the same number of leaves as* T_+ *and* T_-*. Then* $\begin{bmatrix} T_+ \\ V \end{bmatrix}$ *is a tree diagram for the positive part of* f*, and* $\begin{bmatrix} V \\ T_- \end{bmatrix}$ *is a tree diagram for the negative part of* f. $\square$

3 Forest Diagrams

It is immediate from the presentation of F that:

$$x_n = x_0^{1-n} x_1 x_0^{n-1}$$

for all $n \geq 1$. Therefore, F is generated by the two elements $\{x_0, x_1\}$.

In this section, we describe a group $\mathrm{PL}_2(\mathbb{R})$ of self-homeomorphisms of the real line that is isomorphic to F, and develop *forest diagrams* in analogy with the development of tree diagrams in the previous section. These forest diagrams seem to interact particularly nicely with the $\{x_0, x_1\}$-generating set.

The existence of forest diagrams was noted by K. Brown in [Bro], but the pictures themselves have not previously appeared in the literature. They are similar to the "diagrams" of V. Guba and M. Sapir (see [GuSa] and [Guba]).

3.1 The Group $\mathrm{PL}_2(\mathbb{R})$

Let $\mathrm{PL}_2(\mathbb{R})$ be the group of all piecewise-linear, orientation-preserving self-homeomorphisms f of $\mathbb{R}$ satisfying the following conditions:

1. Each linear segment of f has slope a power of 2.

2. f has only finitely many breakpoints, each of which has dyadic rational coordinates.

3. The leftmost linear segment of f is of the form:

$$f(t) = t - m$$

   and the rightmost segment is of the form:

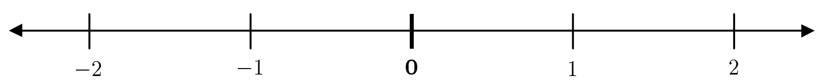
$$f(t) = t - n$$

   for some integers m, n.

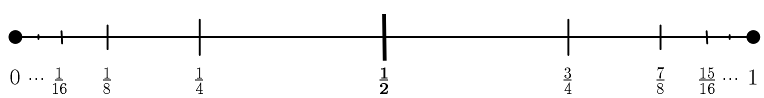
The following is well-known:

Proposition 3.1.1. $\mathrm{PL}_2(\mathbb{R})$ *is isomorphic with* $\mathrm{PL}_2(I)$*.*

Proof. Let $\psi \colon \mathbb{R} \to (0,1)$ be the piecewise-linear homeomorphism that maps the intervals:



linearly onto the intervals:



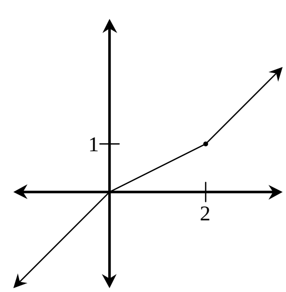
Then $f \mapsto \psi^{-1} f \psi$ is the desired isomorphism $\mathrm{PL}_2(I) \to \mathrm{PL}_2(\mathbb{R})$. $\square$

Corollary 3.1.2. $\mathrm{PL}_2(\mathbb{R})$ *is isomorphic with F. The generators $\{x_0, x_1\}$ of F map to the functions:*

$$x_0(t) = t - 1$$

and:

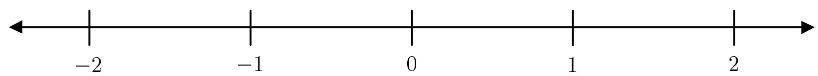
$$x_1(t) = \begin{cases} t & t \leq 0 \\ \frac{1}{2}t & 0 \leq t \leq 2 \\ t - 1 & t \geq 2 \end{cases}$$



x_1

3.2 Forest Diagrams for Elements of $\mathrm{PL}_2(\mathbb{R})$

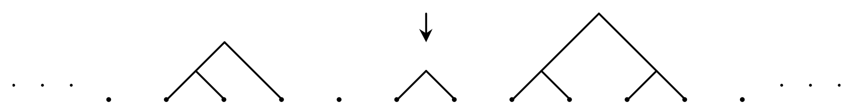
We think of the real line as being pre-subdivided as follows:



A *dyadic subdivision* of $\mathbb{R}$ is a subdivision obtained by cutting finitely many of these intervals in half, and then cutting finitely many of the resulting intervals in half, etc.

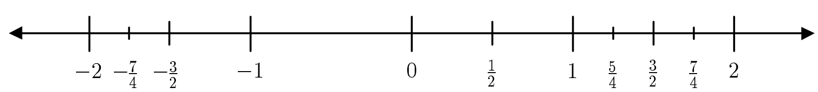
Proposition 3.2.1. *Let $f \in \mathrm{PL}_2(\mathbb{R})$. Then there exist dyadic subdivisions $\mathcal{D}, \mathcal{R}$ of $\mathbb{R}$ such that f maps each interval of $\mathcal{D}$ linearly onto an interval of $\mathcal{R}$.* $\square$

A *binary forest* is a sequence $(\ldots, T_{-1}, T_0, T_1, \ldots)$ of finite binary trees. We depict such a forest as a line of binary trees together with a pointer at T_0:



A binary forest is *bounded* if only finitely many of the trees T_i are nontrivial.

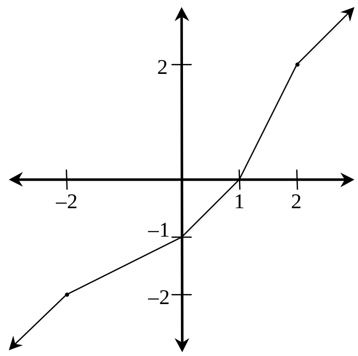
Every bounded binary forest corresponds to some dyadic subdivision of the real line. For example, the forest above corresponds to the subdivision:



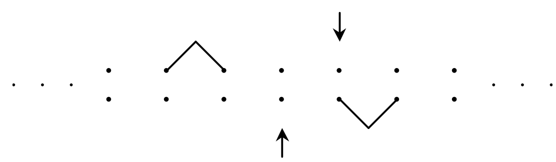
Each tree T_i represents an interval $[i, i+1]$, and each leaf represents an interval of the subdivision.

Combining this with proposition 3.2.1, we see that any $f \in \mathrm{PL}_2(\mathbb{R})$ can be represented by a pair of bounded binary forests, together with an order-preserving bijection of their leaves. This is called a *forest diagram* for f.

Example 3.2.2. Let f be the element of $\mathrm{PL}_2(\mathbb{R})$ with graph:

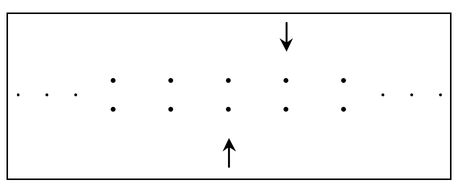


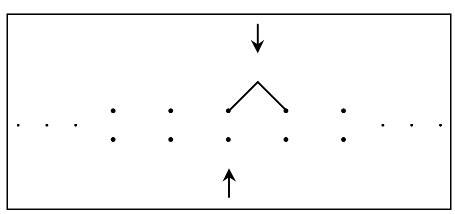
Then f has forest diagram:



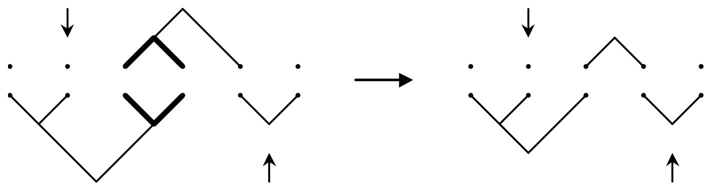
Again, we have aligned the two forests vertically so that corresponding leaves match up. By convention, the *domain* tree appears on the *bottom*, and the *range* tree appears on the *top*.

Example 3.2.3. Here are the forest diagrams for x_0 and x_1:

x_0: 

x_1: 

Of course, there are several forest diagrams for each element of $\mathrm{PL}_2(\mathbb{R})$. In particular, it is possible to delete an opposing pair of carets:

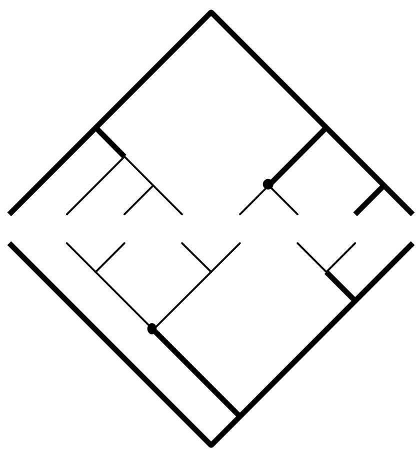


without changing the resulting homeomorphism. This is called a *reduction* of a forest diagram. A forest diagram is *reduced* if it does not have any opposing pairs of carets.

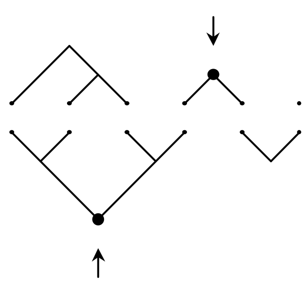
Proposition 3.2.4. *Every element of* $\mathrm{PL}_2(\mathbb{R})$ *has a unique reduced forest diagram.* $\square$

Remark 3.2.5. From this point forward, we will only draw the *support* of the forest diagram (i.e. the minimum interval containing both pointers and all nontrivial trees), and we will omit the "$\cdots$" indicators.

Remark 3.2.6. It is fairly easy to translate between tree diagrams and forest diagrams. Given a tree diagram:



we simply remove the outer layer of each tree to get the corresponding forest diagram:



The pointers of the forest diagram point to the first trees hanging to the right of the roots in the original tree diagram.

3.3 The Action of $\{x_0, x_1\}$

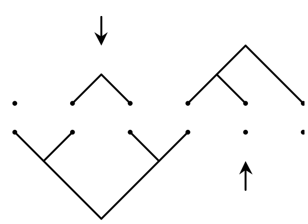
The action of $\{x_0, x_1\}$ on forest diagrams is particularly nice:

Proposition 3.3.1. *Let $\mathfrak{f}$ be a forest diagram for some $f \in F$. Then:*

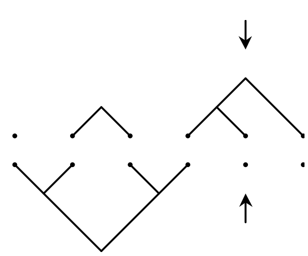
1. *A forest diagram for $x_0 f$ can be obtained by moving the top pointer of $\mathfrak{f}$ one tree to the right.*
2. *A forest diagram for $x_1 f$ can be obtained by attaching a caret to the roots of the 0-tree and 1-tree in the top forest of $\mathfrak{f}$. Afterwards, the top pointer points to the new, combined tree.* $\square$

If $\mathfrak{f}$ is reduced, then the given forest diagram for $x_0 f$ will always be reduced. The forest diagram given for $x_1 f$ will not be reduced, however, if the caret that was created opposes a caret from the bottom tree. In this case, left-multiplication by x_1 effectively "cancels" the bottom caret.

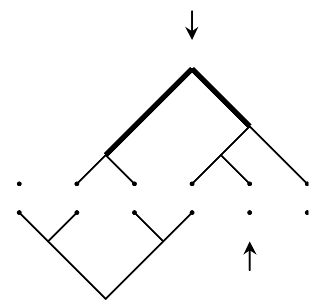
Example 3.3.2. Let $f \in F$ have forest diagram:



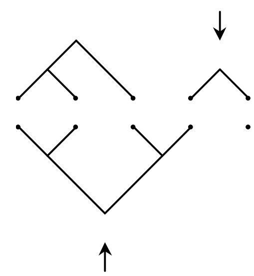
Then $x_0 f$ has forest diagram:



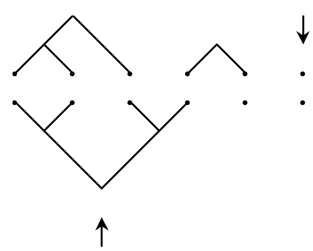
and $x_1 f$ has forest diagram:



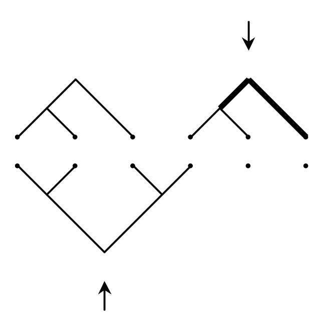
Example 3.3.3. Let $f \in F$ have forest diagram:



Then $x_0 f$ has forest diagram:

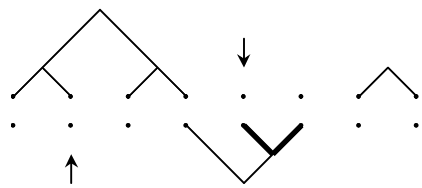


and $x_1 f$ has forest diagram:

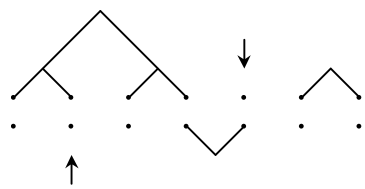


Note that the forest diagrams for $x_0 f$ and $x_1 f$ both have larger support than the forest diagram for f.

Example 3.3.4. Let $f \in F$ have forest diagram:



Then $x_1 f$ has forest diagram:



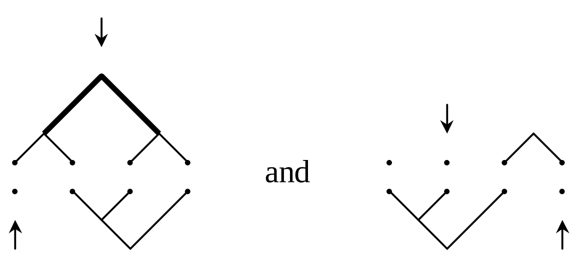
Note that left-multiplication by x_1 canceled the highlighted bottom caret.

Proposition 3.3.5. *Let $\mathfrak{f}$ be a forest diagram for some $f \in F$. Then:*

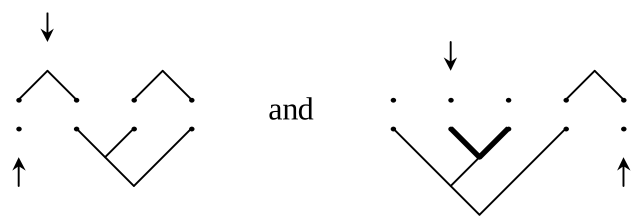
1. *A forest diagram for $x_0^{-1} f$ can be obtained by moving the top pointer of $\mathfrak{f}$ one tree to the left.*

2. *A forest diagram for $x_1^{-1} f$ can be obtained by "dropping a negative caret" at the current position of the top pointer. If the current tree is nontrivial, the negative caret cancels with the top caret of the current tree, and the pointer moves to the resulting left child. If the current tree is trivial, the negative caret "falls through" to the bottom forest, attaching to the specified leaf.* □

Example 3.3.6. Let f and g be the elements of F with forest diagrams:



Then $x_1^{-1}f$ and $x_1^{-1}g$ have forest diagrams:



In the first case, the x_1^{-1} simply removed a caret from the top tree. In the second case, there was no caret on top to remove, so a new caret was attached to the leaf on the bottom. Note that this creates a new column immediately to the right of the pointer.

3.4 Positive Elements and Normal Form

There is a close relationship between the normal form of an element and its forest diagram. It hinges on the following proposition:

Proposition 3.4.1. *Let $\mathfrak{f}$ be the forest diagram for some $f \in F$, and let $n > 1$. Then a forest diagram for $x_n f$ can be obtained by attaching a caret to the roots of T_{n-1} and T_n in the top forest of $\mathfrak{f}$.*

Proof. For $n > 1$, $x_n = x_0^{1-n} x_1 x_0^{n-1}$. □

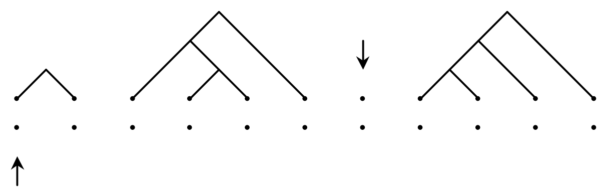
Corollary 3.4.2. *Let $f \in F$, and let $\mathfrak{f}$ be its reduced forest diagram. Then f is positive if and only if:*

1. *The entire bottom forest of $\mathfrak{f}$ is trivial, and*

2. *The bottom pointer is at the left end of the support of $\mathfrak{f}$.*

Using proposition 3.4.1, it is easy to construct the forest diagram for any positive element. It is also possible to find the normal form when given the forest diagram:

Example 3.4.3. Suppose $f \in F$ has forest diagram:



Then:

$$f = x_0^2 x_1 x_3^2 x_4 x_8^3$$

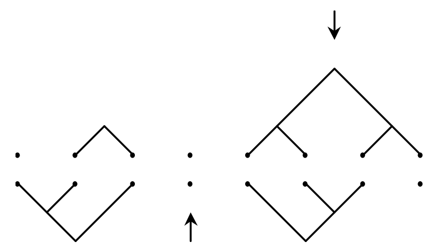
Since the top pointer of f is two trees from the left, the normal form of f has an x_0^2. The powers of the other generators are determined by the number of carets built upon the corresponding leaf. Note that the carets are constructed from right to left.

It is not much harder to deal with mixed (non-positive) elements:

Example 3.4.4. The element:

$$x_0^3 x_2 x_5^2 x_7 x_6^{-1} x_5^{-1} x_1^{-2} x_0^{-1}$$

has forest diagram:



4 Lengths in F

In this section, we derive a formula for the lengths of elements of F with respect to the $\{x_0, x_1\}$-generating set. This formula uses the forest diagrams introduced in section 3.

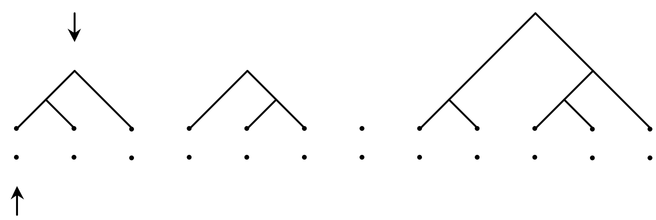
Lengths in F were first studied by S. B. Fordham in his 1995 thesis (recently published, see [Ford]). Fordham gave a formula for the length of an element of F based on its tree diagram. Our length formula can be viewed as a simplification of Fordham's work.

V. Guba has recently obtained another length formula for F using the "diagrams" of Guba and Sapir. See [Guba] for details.

4.1 Lengths of Strongly Positive Elements

We shall begin by investigating the lengths of strongly positive elements. The goal is to develop some intuition for lengths before the statement of the general length formula in section 4.2.

An element $f \in F$ is *strongly positive* if it lies in the submonoid generated by $\{x_1, x_2, \ldots\}$. Here is a forest diagram for a typical strongly positive element:



Note that the entire bottom forest is trivial, and that both pointers are at the left end of the support of f.

Logically, the results of this section depend on the general length formula. In particular, we need the following lemma:

Lemma 4.1.1. *Let $f \in F$ be strongly positive. Then there exists a minimum-length word for f with no appearances of x_1^{-1}.*

This lemma is intuitively obvious: there should be no reason to ever create bottom carets, or to delete top carets, when constructing a strongly positive element. Unfortunately, it would be rather tricky to supply a proof of this fact. Instead we refer the reader to corollary 4.3.8, from which the lemma follows immediately.

From this lemma, we see that any strongly positive element $f \in F$ has a minimum-length word of the form:

$$x_0^{a_n} x_1 \cdots x_0^{a_1} x_1 x_0^{a_0}$$

where $a_0, \ldots, a_n \in \mathbb{Z}$. Since f is strongly positive, we have:

$$a_0 + \cdots + a_n = 0$$

and

$$a_0 + \cdots + a_i \geq 0 \qquad (\text{for } i = 0, \ldots, n-1)$$

Such words can be represented by words in $\{x_1, x_2, \ldots\}$ via the identifications $x_n = x_0^{1-n} x_1 x_0^{n-1}$. For example, the word:

$$x_0^{-5}\, x_1\, x_0^{-2}\, x_1\, x_0^{4}\, x_1\, x_0^{-3}\, x_1\, x_0^{6}$$

can be represented by:

$$x_6\, x_8\, x_4\, x_7$$

More generally:

Notation 4.1.2. We will use the word:

$$x_{i_n} \cdots x_{i_2} x_{i_1}$$

in $\{x_1, x_2, \ldots\}$ to represent the word:

$$x_0^{1-i_n}\, x_1\, \cdots\, x_0^{i_3-i_2}\, x_1\, x_0^{i_2-i_1}\, x_1\, x_0^{i_1-1}$$

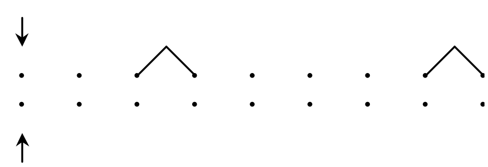
in $\{x_0, x_1\}$.

Note then that $x_{i_n} \cdots x_{i_2} x_{i_1}$ represents a word with length:

$$(|1 - i_n| + \cdots + |i_3 - i_2| + |i_2 - i_1| + |i_1 - 1|) + n$$

We now proceed to some examples, from which we will derive a general theorem.

Example 4.1.3. Let $f \in F$ be the element with forest diagram:



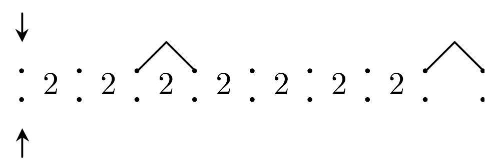
There are only two candidate minimum-length words for f: $x_3 x_8$ and $x_7 x_3$. Their lengths are:

$$\begin{array}{lll} & (2+5+7)+2 = 16 & \text{for the word } x_3x_8 \\ \text{and} & (6+4+2)+2 = 14 & \text{for the word } x_7x_3. \end{array}$$

Let's see if we can explain this. The word $x_3 x_8 = x_0^{-2} x_1 x_0^{-5} x_1 x_0^{7}$ corresponds to the following construction of f:

1. Starting at the identity, move right seven times and construct the right caret.
2. Next move left five times, and construct the left caret.
3. Finally, move left twice to position of the bottom pointer.

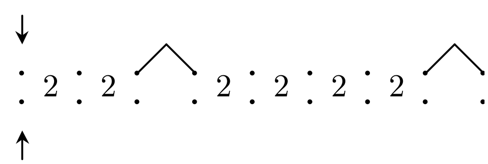
This word makes a total of fourteen moves, crossing twice over each of seven spaces:



On the other hand, the word $x_7 x_3 = x_0^{-6} x_1 x_0^4 x_1 x_0^2$ corresponds to the following construction:

1. Starting at the identity, move right twice and construct the left caret.
2. Next move right four more times, and construct the right caret.
3. Finally, move left six times to the position of the bottom pointer.

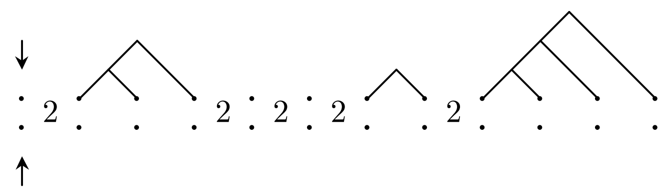
This word makes only twelve moves:



In particular, this word never moves across the space under the left caret. It avoids this by *building the left caret early.* Once the left caret is built, the word can simply pass over the space under the left caret without spending time to move across it.

Terminology 4.1.4. We call a space in a forest *interior* if it lies under a tree (or over a tree, if the forest is upside-down) and *exterior* if it lies between two trees.

Example 4.1.5. Let $f \in F$ be the element with forest diagram:



Clearly, each of the five exterior spaces in the support of f must be crossed twice during construction. Furthermore, it is possible to avoid crossing any of the interior spaces by *constructing carets from left to right*. In particular:

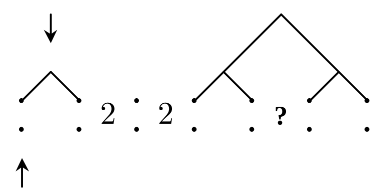
$$x_6^3\, x_5\, x_2^2$$

is a minimum-length word for f. Therefore, f has length:

$$(5+1+3+1)+6=16$$

It is not always possible to avoid crossing all the interior spaces:

Example 4.1.6. Let $f \in F$ be the element with forest diagram:



Clearly, each of the two exterior spaces in the support of f must be crossed twice during construction. However, the space marked (?) must also be crossed twice, since we must create the caret immediately to its right before we can create the caret above it.

It turns out that these are the only spaces which must be crossed. For example, the word:

$$x_3\, x_4\, x_3\, x_1$$

crosses only these spaces. Therefore, f has length:

$$(2+1+1+2+0)+4=10$$

In this last example, we learned that it is not always possible to construct carets from left to right. However, if one always constructs the *leftmost possible caret first*, then it is never necessary to move more than one space to the left in the middle of the construction. This is the content of the following theorem:

Theorem 4.1.7 (Anti-Normal Form). *Let $f \in F$ be strongly positive. Then f can be expressed uniquely in the form:*

$$x_{i_n} \cdots x_{i_2} x_{i_1}$$

where $i_{k+1} \geq i_k - 1$ for all k. $\square$

We say that a word:

$$x_{i_n} \cdots x_{i_2} x_{i_1}$$

is in *anti-normal form* if $i_{k+1} \geq i_k - 1$ for each k. On the forest diagram, anti-normal form corresponds to constructing the *leftmost possible caret* at each stage.

In contrast, the normal form for an element satisfies $i_{k+1} \leq i_k$ for each k, and corresponds to constructing the rightmost possible caret at each stage. This explains our terminology.

The anti-normal form for a strongly positive element of F is clearly minimum-length, since it crosses only those spaces in the forest diagram that must be crossed. We can give an explicit length formula by counting these spaces:

Theorem 4.1.8. *Let $f \in F$ be strongly positive. Then the length of f is:*

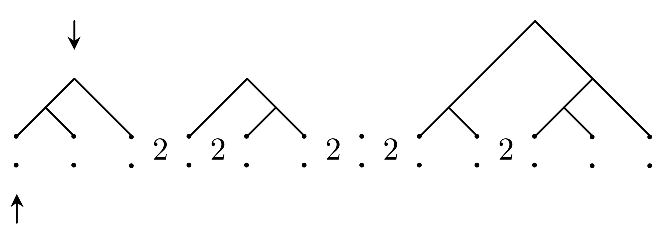
$$2\,n(f) + c(f)$$

where $n(f)$ and $c(f)$ are defined as follows. Let $\mathfrak{f}$ be the reduced forest diagram for f. Then:

1. *$n(f)$ is the number of spaces in the support of $\mathfrak{f}$ that are either exterior or lie immediately to the left of some caret, and*

2. *$c(f)$ is the number of carets in $\mathfrak{f}$.* $\square$

Example 4.1.9. Let $f \in F$ be the element with forest diagram:



Then $c(f) = 8$ and $n(f) = 5$, so f has length 18. The anti-normal form for f is:

$$x_4 x_5^2 x_4 x_2 x_3 x_1^2$$

Therefore, a minimum-length $\{x_0, x_1\}$-word for f is:

$$x_0^{-3} x_1 x_0^{-1} x_1^2 x_0 x_1 x_0^2 x_1 x_0^{-1} x_1 x_0^2 x_1^2$$

Currently, our only algorithm to find the anti-normal form for a strongly positive element involves drawing the forest diagram. It is interesting to note that an entirely algebraic algorithm is available:

Theorem 4.1.10. *Let $f \in F$ be strongly positive, and let w be an expression for f as a product of $\{x_1, x_2, \ldots\}$. Suppose we repeatedly apply operations of the form:*

$$x_k x_n \longmapsto x_{n-1} x_k \qquad (k < n-1)$$

to w. Then we reach the anti-normal form for f after at most $\binom{c\,(f)}{2}$ steps.

Proof. Let $\mathcal{C}$ be the set of carets in the reduced forest diagram for f. Suppose that:

$$w = x_{i_m} \cdots x_{i_2} x_{i_1}$$

Each generator x_{i_k} appearing in w corresponds to the construction of some caret c_k of the forest diagram for f. Let $<$ denote the order in which these carets are created:

$$c_1 < c_2 < \cdots < c_m$$

Now, the anti-normal form for f is just another word for f in the generators $\{x_1, x_2, \ldots\}$. Let $<_{\text{AN}}$ denote the resulting order on $\mathcal{C}$. Note that:

$$c_k <_{\text{AN}} c_{k+1} \qquad \Longleftrightarrow \qquad i_k - 1 \leq i_{k+1}$$

Therefore, any operation of the form:

$$x_{i_{k+1}} x_{i_k} \longmapsto x_{i_k - 1} x_{i_{k+1}} \qquad (i_{k+1} < i_k - 1)$$

reduces the number:

$$\left| \{(c, c') : c <_{\text{AN}} c' \text{ but } c > c'\} \right|$$

by exactly one. When this number reaches zero, f is in anti-normal form.

Finally, note that the number in question is bounded by $\binom{|\mathcal{C}|}{2}$. □

Example 4.1.11. Let's find the length of the element:

$$x_1\, x_3^3\, x_6\, x_7\, x_{10}$$

We put the word into anti-normal form:

$$\begin{aligned} & x_1\, x_3^3\, x_6\, x_7\, x_{10} \\ &= x_4\, x_1\, x_3^3\, x_6\, x_7 && (x_{10} \text{ moved left}) \\ &= x_4\, x_2^3\, x_5\, x_6\, x_1 && (x_1 \text{ moved right}) \\ &= x_4\, x_2\, x_3\, x_4\, x_2^2\, x_1 && (x_2^2 \text{ moved right}) \end{aligned}$$

Hence, the length is:

$$(3+2+1+1+2+1+0)+7 = 17$$

4.2 The Length Formula

We now give the length formula for a general element of F. Afterwards, we will give several examples to illustrate intuitively why the formula works. We defer the proof to section 4.3.

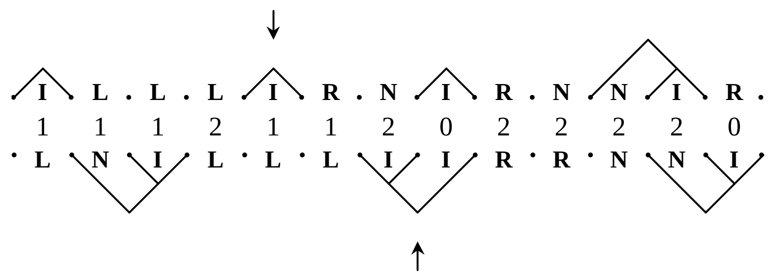
Let $f \in F$, and let $\mathfrak{f}$ be its reduced forest diagram. We label the spaces of each forest of $\mathfrak{f}$ as follows:

1. Label a space **L** (for *left*) if it exterior and to the left of the pointer.
2. Label a space **N** (for *necessary*) if it lies immediately to the left of some caret (and is not already labeled **L**).
3. Label a space **R** (for *right*) if it exterior and to the right of the pointer (and not already labeled **N**).
4. Label a space **I** (for *interior*) if it interior (and not already labeled **N**).

We assign a *weight* to each space in the support of $\mathfrak{f}$ according to its labels:

top label \ bottom label	**L**	**N**	**R**	**I**
L	2	1	1	1
N	1	2	2	2
R	1	2	2	0
I	1	2	0	0

Example 4.2.1. Here are the labels and weights for a typical forest diagram:



Theorem 4.2.2 (The Length Formula). *Let $f \in F$, and let $\mathfrak{f}$ be its reduced forest diagram. Then the $\{x_0, x_1\}$-length of f is:*

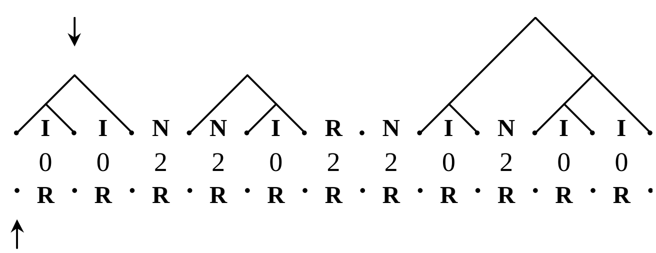
$$\ell(f) = \ell_0(f) + \ell_1(f)$$

where:

1. *$\ell_0(f)$ is the sum of the weights of all spaces in the support of $\mathfrak{f}$, and*
2. *$\ell_1(f)$ is the total number of carets in $\mathfrak{f}$.*

Remark 4.2.3. Intuitively, the weight of a space is just the number of times it must be crossed during the construction of f. Hence, there ought to exist a minimum-length word for f with $\ell_0(f)$ appearances of x_0 or x_0^{-1} and $\ell_1(f)$ appearances of x_1 or x_1^{-1}. This will be established at the end of the next section.

Example 4.2.4. Let $f \in F$ be the element from example 4.1.9:



Then $\ell_0(f) = 10$ and $\ell_1(f) = 8$, so f has length 18.

In general, suppose $f \in F$ is strongly positive, and let $\mathfrak{f}$ be its reduced forest diagram. Then every space of $\mathfrak{f}$ is labeled $\begin{bmatrix}\mathbf{N}\\\mathbf{R}\end{bmatrix}$, $\begin{bmatrix}\mathbf{R}\\\mathbf{R}\end{bmatrix}$, or $\begin{bmatrix}\mathbf{I}\\\mathbf{R}\end{bmatrix}$. Each $\begin{bmatrix}\mathbf{I}\\\mathbf{R}\end{bmatrix}$ space has weight 0, and each $\begin{bmatrix}\mathbf{N}\\\mathbf{R}\end{bmatrix}$ or $\begin{bmatrix}\mathbf{R}\\\mathbf{R}\end{bmatrix}$ space has weight 2, so that:

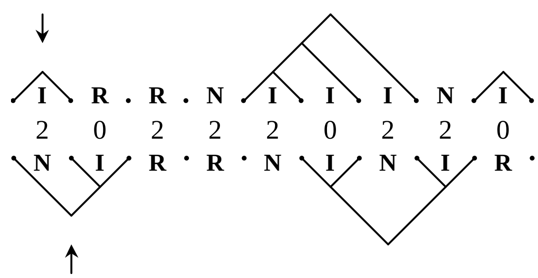
$$\ell_0(f) = 2\,n(f)$$

and hence:

$$\ell_0(f) + \ell_1(f) = 2\,n(f) + c(f)$$

Therefore, the length formula of theorem 4.2.2 reduces to theorem 4.1.8 for strongly positive elements.

Example 4.2.5. Let $f \in F$ be the element with forest diagram:



Then $\ell_0(f) = 12$ and $\ell_1(f) = 10$, so f has length 22. One minimum-length word for f is:

$$x_1 x_0^{-1} x_1^{-1} x_0 x_1^{-1} x_0^{-3} x_1 x_0 x_1^{3} x_0^{-1} x_1^{-1} x_0^{-1} x_1^{-1} x_0 x_1^{-1} x_0^{3}$$

In general, an element $f \in F$ is *right-sided* if it lies in the subgroup generated by $\{x_1, x_2, \ldots\}$. Equivalently, f is right-sided if and only if both pointers in the forest diagram for f are at the left edge of the support. Note then that every space of a right-sided element is labeled either **N**, **R**, or **I**. The weight table for such spaces is:

		bottom label		
		N	**R**	**I**
top label	**N**	2	2	2
	R	2	2	0
	I	2	0	0

Observe that a space has weight 2 if and only if:

1. It is exterior on both the top and the bottom, or
2. It lies immediately to the left of some caret, on either the top or the bottom.

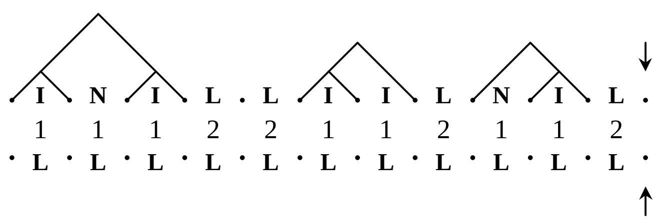
This can be viewed as a generalization of the length formula for strongly positive elements. Specifically, if f is right-sided, then:

$$\ell(f) = 2\,n(f) + c(f)$$

where $n(f)$ is the number of spaces satisfying condition (1) or (2), and $c(f)$ is the number of carets of f.

As with strongly positive elements, it is intuitively obvious that this is a lower bound for the length. Unfortunately, we have not been able to find an analogue of the "anti-normal form" argument to show that it is an upper bound.

Example 4.2.6. Let $f \in F$ be the element with forest diagram:



Then $\ell_0(f) = 15$ and $\ell_1(f) = 7$, so f has length 22.

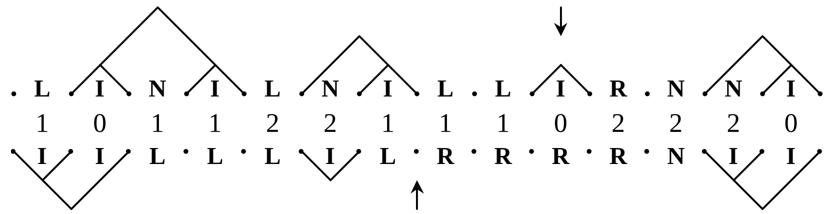
It is interesting to note that every interior space of f has weight 1: for trees to the left of the pointer, one cannot avoid crossing interior spaces at least once. Specifically, each caret is created from its *left* leaf, and we must move to this leaf somehow.

One minimum-length word for f is

$$x_0^4 x_1^2 x_0^{-2} x_1 x_0^{-3} x_1^2 x_0^{-3} x_1 x_0^{-1} x_1 x_0^{-2}$$

Note that this word creates carets *right to left*.

Example 4.2.7. Let $f \in F$ be the element with forest diagram:



Then $\ell_0(f) = 16$ and $\ell_1(f) = 13$, so f has length 29. One minimum-length word for f is:

$$x_0^{-2}x_1x_0^{-1}x_1x_0x_1^{-2}x_0^2x_1x_0^3x_1^2x_0x_1^{-2}x_0^{-1}x_1x_0^{-2}x_1x_0^{-1}x_1x_0x_1^{-1}x_0^{-1}$$

This is our first example with $\begin{bmatrix}\mathbf{L}\\ \mathbf{R}\end{bmatrix}$ pairs: note that they only need to be crossed once. Also note how it affects the length to have bottom trees to the left of the pointer. In particular, observe that the $\begin{bmatrix}\mathbf{N}\\ \mathbf{I}\end{bmatrix}$ pair to the left of the pointers must crossed twice.

4.3 The Proof of the Length Formula

We prove the length formula using the same technique as Fordham [Ford]:

Theorem 4.3.1. *Let G be a group with generating set S, and let $\ell\colon G \to \mathbb{N}$ be a function. Then ℓ is the length function for G with respect to S if and only if:*

1. *$\ell(e) = 0$, where e is the identity of G.*
2. *$|\ell(sg) - \ell(g)| \leq 1$ for all $g \in G$ and $s \in S$.*
3. *If $g \in G \setminus \{e\}$, there exists an $s \in S \cup S^{-1}$ such that $\ell(sg) < \ell(g)$.* □

Proof. Conditions (1) and (2) show that ℓ is a lower bound for the length, and condition (3) shows that ℓ is an upper bound for the length. □

Let ℓ denote the function defined on F specified by Theorem 4.2.2. Clearly ℓ satisfies condition (1). To show that ℓ satisfies conditions (2) and (3), we need only gather information about how left-multiplication by generators affects the function ℓ.

Terminology 4.3.2. If $f \in F$, the *current tree* of f is the tree in forest diagram indicated by the top pointer. The *right space* of f is the space immediately to the right of the current tree, and the *left space* of f is the space immediately to the left of the current tree.

Note that, if the top pointer is at the right edge of the support of f, then the right space of f has no label. Similarly, if the top pointer is at the left edge of the support, then the left space of f has no label.

Proposition 4.3.3. *If $f \in F$, then $\ell(x_0 f) = \ell(f) \pm 1$. Specifically, $\ell(x_0 f) = \ell(f) - 1$ unless one of the following conditions holds:*

1. *$x_0 f$ has larger support than f.*
2. *The right space of f has bottom label* **L***, and left-multiplication by x_0 does not remove this space from the support.*
3. *The right space of f is labeled $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$.*

Proof. Clearly $\ell_1(x_0 f) = \ell_1(f)$. As for ℓ_0, note that the only space whose label changes is the right space of f.

Case 1: Suppose $x_0 f$ has larger support than f. Then the right space of f is unlabeled, and has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{R}\end{bmatrix}$ in $x_0 f$. Hence $\ell_0(x_0 f) = \ell_0(f) + 1$.

Case 2: Suppose $x_0 f$ has smaller support than f. Then the right space of f has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{L}\end{bmatrix}$, but becomes unlabeled in $x_0 f$. Hence $\ell_0(x_0 f) = \ell_0(f) - 1$.

Case 3: Suppose $x_0 f$ has the same support as f. Then the right space of f has top label **N** or **R**, but top label **L** in $x_0 f$. The relevant rows of the weight table are:

		bottom label			
		L	**N**	**R**	**I**
top label	**L**	**2**	1	1	**1**
	N	**1**	2	2	2
	R	**1**	2	2	**0**

Each entry of the **N** and **R** rows differs from the corresponding entry of the **L** row by exactly one. In particular, moving from an **R** or **N** row to an **L** row only increases the weight when in the **L** column or when starting at $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$. $\square$

Corollary 4.3.4. *Let $f \in F$. Then $\ell(x_0^{-1} f) < \ell(f)$ if and only if one of the following conditions holds:*

1. *$x_0^{-1} f$ has smaller support than f.*
2. *The left space of f has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{L}\end{bmatrix}$.*
3. *The left space of f has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{I}\end{bmatrix}$, and the current tree is trivial.*

Proposition 4.3.5. *Let $f \in F$. If left-multiplying f by x_1 cancels a caret from the bottom forest, then $\ell(x_1 f) = \ell(f) - 1$.*

Proof. Clearly $\ell_1(x_1 f) = \ell_1(f) - 1$. We must show that ℓ_0 remains unchanged.

Note first that the right space of f is destroyed. This space has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$, and hence has weight 0. Therefore, its destruction does not affect ℓ_0.

The only other space affected is the left space of f. If this space is not in the support of f, it remains unlabeled throughout. Otherwise, observe that it must have top label **L** in both f and $x_1 f$. The relevant row of the weight table is:

	L	**N**	**R**	**I**
L	2	1	1	1

In particular, the only important property of the bottom label is whether or not it is an **L**. This property is unaffected by the deletion of the caret. ☐

Proposition 4.3.6. *Let $f \in F$, and suppose that left-multiplying f by x_1 creates a caret in the top forest. Then $\ell(x_1 f) = \ell(f) \pm 1$. Specifically, $\ell(x_1 f) = \ell(f) - 1$ if and only if the right space of f has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$.*

Proof. Clearly $\ell_1(x_1 f) = \ell_1(f) + 1$. As for ℓ_0, observe that the only space whose label could change is the right space of f.

Case 1: Suppose $x_1 f$ has larger support than f. Then the right space of f is unlabeled, but has label $\begin{bmatrix}\mathbf{I}\\ \mathbf{R}\end{bmatrix}$ in $x_1 f$. This does not affect the value of ℓ_0.

Case 2: Otherwise, note that the right space of f has top label **N** or **R**. If the top label is an **N**, it remains and **N** in $x_1 f$. If it is an **R**, then it changes to an **I**. The relevant rows of the weight table are:

		bottom label			
		L	**N**	**R**	**I**
top label	**R**	1	2	**2**	0
	I	1	2	**0**	0

Observe that the weight decreases by two if the bottom label is an **R**, and stays the same otherwise. $\square$

We have now verified condition (2). Also, we have gathered enough information to verify condition (3):

Theorem 4.3.7. *Let $f \in F$ be a nonidentity element.*

1. *If current tree of f is nontrivial, then either $\ell(x_1^{-1}f) < \ell(f)$, or $\ell(x_0 f) < \ell(f)$.*
2. *If left-multiplication by x_1 would remove a caret from the bottom tree, then $\ell(x_1 f) < \ell(f)$.*
3. *Otherwise, either $\ell(x_0 f) < \ell(f)$ or $\ell(x_0^{-1} f) < \ell(f)$.*

Proof.

Statement 1: If $\ell(x_1^{-1}f) > \ell(f)$, then the right space of $x_1^{-1}f$ has type $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$. The right space of f therefore has type $\begin{bmatrix}\mathbf{R} \text{ or } \mathbf{N}\\ \mathbf{R} \text{ or } \mathbf{N}\end{bmatrix}$, so that $\ell(x_0 f) < \ell(f)$.

Statement 2: See proposition 4.3.5.

Statement 3: Suppose $\ell(x_0 f) > \ell(f)$. There are three cases:

Case 1: The right space of f is not in the support of f. Then the left space of f has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{R}\end{bmatrix}$, $\begin{bmatrix}\mathbf{L}\\ \mathbf{L}\end{bmatrix}$, or $\begin{bmatrix}\mathbf{L}\\ \mathbf{I}\end{bmatrix}$. In all three cases, $\ell(x_0^{-1}f) < \ell(f)$.

Case 2: The right space of f has bottom label **L**, and right-multiplication by x_0 does not remove this space from the support. Then the left space of f must have label $\begin{bmatrix}\mathbf{L}\\ \mathbf{L}\end{bmatrix}$ or $\begin{bmatrix}\mathbf{L}\\ \mathbf{I}\end{bmatrix}$, and hence $\ell(x_0^{-1}f) < \ell(f)$.

Case 3: The right space of f has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$. Then the tree immediately to the right of the top pointer is trivial, and the bottom leaf under it is a right leaf. If the bottom leaf under the top pointer were a left leaf, then left-multiplying f by x_1 would cancel a caret. Hence, it is also a right leaf, so the left space of f has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{I}\end{bmatrix}$. We conclude that $\ell(x_0^{-1}f) < \ell(f)$. $\square$

Corollary 4.3.8. *Let $f \in F$, and let $\mathfrak{f}$ be the reduced forest diagram for f. Then there exists a minimum-length word w for f with the following properties:*

1. *Each instance of x_1 in w creates a top caret of $\mathfrak{f}$.*

2. *Each instance of x_1^{-1} in w creates a bottom caret of $\mathfrak{f}$.*

In particular, w has $\ell_1(f)$ instances of x_1 or x_1^{-1}, and $\ell_0(f)$ instances of x_0 or x_0^{-1}.

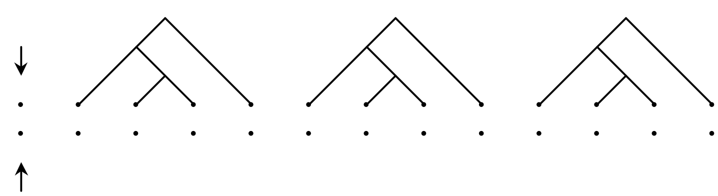
Proof. By the previous theorem, it is always possible to travel from f to the identity in such a way that each left-multiplication by x_1 deletes a bottom caret and each left-multiplication by x_1^{-1} deletes a top caret. □

Of course, not every minimum-length word for f is of the given form. We will discuss this phenomenon in the next section.

4.4 Minimum-Length Words

In principle, the results from the last section specify an algorithm for finding minimum-length words. (Given an element, find a generator which shortens it. Repeat.) In practice, though, no algorithm is necessary: one can usually guess a minimum-length word by staring at the forest diagram. Our goal in this section is to convey this intuition.

Example 4.4.1. Let f be the element of F with forest diagram:

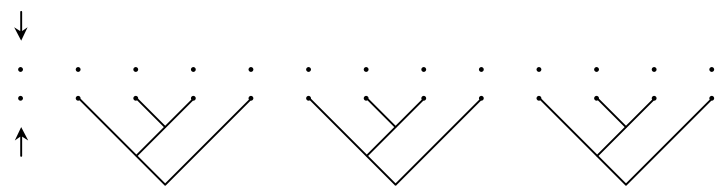


Then there is exactly one minimum-length word for f, namely:

$$x_0^{-3} u x_0 u x_0 u x_0$$

where $u = x_1^2 x_0^{-1} x_1 x_0$. Note that the trees of f are constructed from *left to right.*

Similarly, f^{-1} has forest diagram:

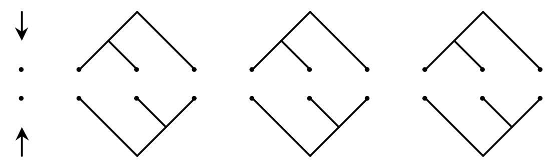


and the only minimum-length word for f^{-1} is:

$$x_0^{-1}u^{-1}x_0^{-1}u^{-1}x_0^{-1}u^{-1}x_0^3$$

Note that the trees of f^{-1} are constructed from *right to left*.

Example 4.4.2. Let f be the element of F with forest diagram:

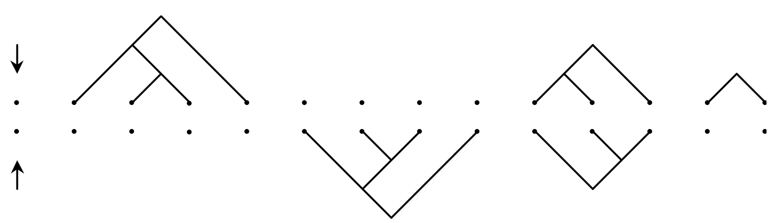


There are precisely four minimum-length words for f:

$$\begin{gathered} x_0^{-3}\,v\,x_0\,v\,x_0\,v\,x_0 \\ x_0^{-1}\,v\,x_0^{-2}\,v\,x_0\,v\,x_0^2 \\ x_0^{-2}\,v\,x_0^{-1}\,v\,x_0^2\,v\,x_0 \\ x_0^{-1}\,v\,x_0^{-1}\,v\,x_0^{-1}\,v\,x_0^3 \end{gathered}$$

where $v = x_1^2x_0^{-1}x_1^{-1}x_0x_1^{-1}$. In particular, each of the first two components can be constructed either when the pointer is moving right, or later when the pointer is moving back left.

Example 4.4.3. Let f be the element of F with forest diagram:

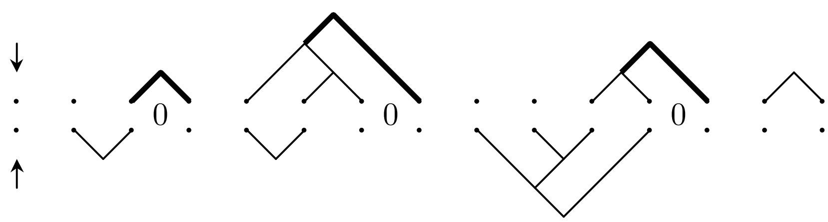


There are precisely two minimum-length words for f:

$$x_0^{-2}\,u^{-1}\,x_0^{-2}\,x_1\,x_0\,v\,x_0^2\,u\,x_0$$
$$x_0^{-2}\,u^{-1}\,x_0^{-1}\,v\,x_0^{-1}\,x_1\,x_0^3\,u\,x_0$$

where $u = x_1^2 x_0^{-1} x_1 x_0$ and $v = x_1^2 x_0^{-1} x_1^{-1} x_0 x_1^{-1}$. Note that the first component must always be constructed on the journey right, and the second component must always be constructed on the journey left. The only choice lies with the construction of the third component: should it be constructed when moving right, or should it be constructed while moving back left?

In general, certain components act like "top trees" while others act like "bottom trees", while still others are "balanced". For example, the forest diagram:



must be constructed from left to right (so all the components act like "top trees"). The reason is that the three marked spaces each have weight 0, so that each of the three highlighted carets must be constructed *before* the pointer can move farther to the right. Essentially, the highlighted carets are acting like *bridges* over these spaces.

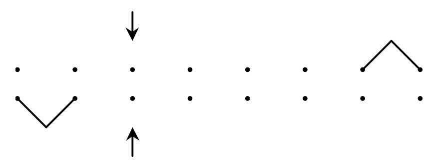
The idea of the "bridge" explains two phenomena we have already observed. First, consider the following contrapositive of proposition 4.3.6:

Proposition 4.4.4. *Let $f \in F$, and suppose that the top pointer of f points at a nontrivial tree. Then $\ell\left(x_1^{-1} f\right) < \ell(f)$ unless the resulting uncovered space has type* $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$. $\square$

This proposition states conditions under which the destruction of a top caret decreases the length of an element. Essentially, the content of the proposition is that it makes sense to delete a top caret *unless that caret is functioning as a bridge.* (Note that the deletion of any of the bridges in the example above would result in an $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$ space.) It makes no sense to delete a bridge, since the bridge is helping you access material further to the right.

Next, recall the statement of corollary 4.3.8: every $f \in F$ has a minimum-length word with $\ell_1(f)$ instances of x_1 or x_1^{-1} and $\ell_0(f)$ instances of x_0 or x_0^{-1}. After the corollary, we mentioned that not every minimum-length word for f is necessarily of this form. The reason is that it sometimes makes sense to build bridges during the creation of an element:

Example 4.4.5. Let f be the element of F with forest diagram:



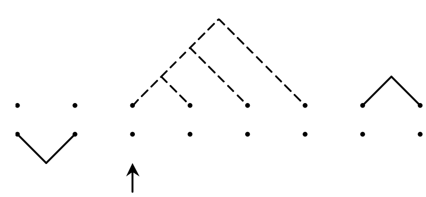
Then one minimum-length word for f is:

$$x_0^2 x_1^{-1} x_0^{-5} x_1 x_0^4$$

This word corresponds to the instructions "move right, create the top caret, move left, create the bottom caret, and then move back to the origin". However, here is another minimum-length word for f:

$$x_0^2 x_1^{-1}(x_0^{-1} x_1^{-3} x_0^{-1}) x_1 (x_0 x_1^3)$$

In this word, the "move right" is accomplished by building three temporary bridges:



These bridges are torn down during the "move left".

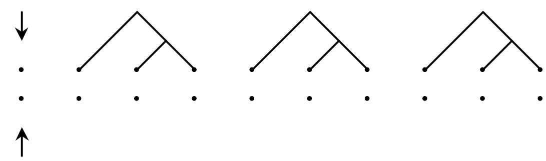
Finally, here is a third minimum-length word for f:

$$x_1^{-3} x_0^2 x_1^{-1} x_0^{-2} x_1 (x_0 x_1^3)$$

In this word, bridges are again built during the "move right", but they aren't torn down until the very end of the construction.

We now turn our attention to a few examples with some more complicated behavior.

Example 4.4.6. Let f be the element of F with forest diagram:

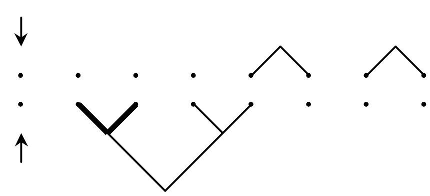


There are four different minimum-length words for f:

$$\begin{gathered}
x_0^{-3}x_1x_0^{-1}x_1x_0^2x_1x_0^{-1}x_1x_0^2x_1x_0^{-1}x_1x_0^2 \\
x_0^{-1}x_1x_0^{-3}x_1x_0^{-1}x_1x_0^2x_1x_0^{-1}x_1x_0^2x_1x_0^2 \\
x_0^{-2}x_1x_0^{-2}x_1x_0^{-1}x_1x_0^2x_1x_0^2x_1x_0^{-1}x_1x_0^2 \\
x_0^{-1}x_1x_0^{-2}x_1x_0^{-2}x_1x_0^{-1}x_1x_0^2x_1x_0^2x_1x_0^2
\end{gathered}$$

Note that each of the first two components may be either partially or fully constructed during the move to the right. This occurs because the trees in this example do not end with bridges. (Compare with example 4.4.1.)

Example 4.4.7. Let f be the element of F with forest diagram:



There is exactly one minimum-length word for f:

$$x_0^{-1}x_1^{-1}x_0^{-3}x_1x_0x_1x_0x_1^{-1}x_0x_1^{-1}x_0$$

Note that the highlighted caret must be constructed *last*, since the space it spans should not be crossed. However, we must begin by partially constructing the first component, because of the bridge on its right end.

5 Applications

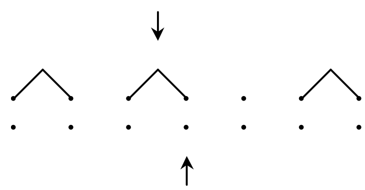
This section contains various applications of forest diagrams and the length formula.

5.1 Dead Ends and Deep Pockets

In [ClTa1], S. Cleary and J. Taback prove that F has "dead ends" but no "deep pockets". In this subsection, we show how forest diagrams can be used to understand these results.

Definition 5.1.1. A *dead end* is an element $f \in F$ such that $\ell(xf) < \ell(f)$ for all $x \in \left\{x_0, x_1, x_0^{-1}, x_1^{-1}\right\}$.

Example 5.1.2. Consider the element $f \in F$ with forest diagram:



Left-multiplying by x_0^{-1} decreases the length since the left space of f is of type $\begin{bmatrix}\mathbf{L}\\ \mathbf{L}\end{bmatrix}$. Left-multiplying by x_0 or x_1 decreases the length since the right space of f is of type $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$. Finally, left-multiplying by x_1^{-1} decreases the length since it deletes a top caret and the right space of $x_1^{-1}f$ is not of type $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$.

This example is typical:

Proposition 5.1.3. *Let $f \in F$. Then f is a dead end if and only if:*

1. *The current tree of f is nontrivial,*
2. *The left space of f has label $\begin{bmatrix}\mathbf{L}\\ \mathbf{L}\end{bmatrix}$,*
3. *The right space of f has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$, and*
4. *The right space of $x_1^{-1}f$ does not have label $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$.*

Proof. The "if" direction is trivial. To prove the "only if" direction, assume that f is a dead end. Then:

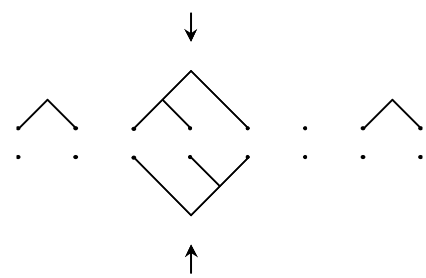
Condition (1) follows from the fact that $\ell\left(x_1^{-1}f\right) < \ell(f)$ (see proposition 4.3.5).

Condition (2) now follows from the fact that $\ell\left(x_0^{-1}f\right) < \ell(f)$ (see corollary 4.3.4).

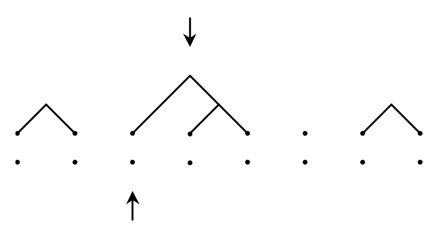
Condition (3) now follows from the fact that $\ell(x_1 f) < \ell(f)$ (see proposition 4.3.6).

Condition (4) now follows from the fact that $\ell\left(x_1^{-1}f\right) < \ell(x_1 f)$ (see proposition 4.3.6). $\square$

Note that there are several ways to meet condition (4): the right space of $x_1^{-1}f$ could be of type $\begin{bmatrix}\mathbf{R}\\ \mathbf{L}\end{bmatrix}$ (as in example 5.1.2), or it could be of type $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$:



or it could just have an **N** on top:



Notice, also, that the proof of proposition 5.1.3 never used the fact that $\ell\left(x_0 f\right) < \ell(f)$. In particular, if the length of f increases when you left-multiply by x_1, x_1^{-1}, and x_0^{-1}, then f must be a dead end.

Definition 5.1.4. Let $k \in \mathbb{N}$. A *k-pocket* of F is an element $f \in F$ such that:

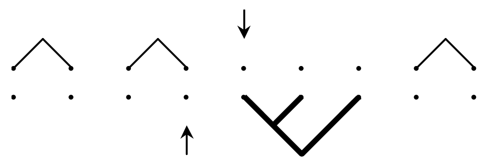
$$\ell(s_1 \cdots s_k f) \leq \ell(f)$$

for all $s_1, \ldots, s_k \in \left\{x_0, x_1, x_0^{-1}, x_1^{-1}, 1\right\}$.

A 2-pocket in F is just a dead end. S. Cleary and J. Taback demonstrated that F has no k-pockets for $k \geq 3$. We give an alternate proof:

Proposition 5.1.5. *F has no k-pockets for $k \geq 3$.*

Proof. Let $f \in F$ be a dead-end element. Then the right space of f has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{R}\end{bmatrix}$, so the tree to the right of the top pointer is trivial. Therefore, repeatedly left-multiplying $x_0 f$ by x_1^{-1} will create negative carets:



In particular, $x_1^{-1}x_1^{-1}x_0 f$ has length $\ell(f)+1$. □

5.2 Growth

We can use forest diagrams to calculate the growth function of the positive monoid with respect to the $\{x_0, x_1\}$-generating set. Burillo [Bur] recently arrived at the same result using tree diagrams and Fordham's length formula:

Theorem 5.2.1. *Let p_n denote the number of positive elements of length n, and let:*

$$p(x) = \sum_{n=0}^{\infty} p_n x^n$$

Then:

$$p(x) = \frac{1-x^2}{1-2x-x^2+x^3}$$

In particular, p_n satisfies the recurrence relation:

$$p_n = 2p_{n-1} + p_{n-2} - p_{n-3}$$

for all $n \geq 3$.

Proof. Let P_n be the set of all positive elements of length n. Define four subsets of P_n as follows:

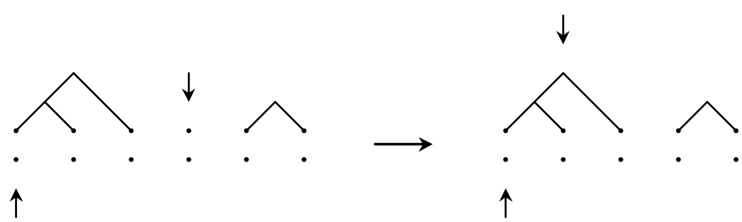
1. $A_n = \{f \in P_n :$ the current tree of f is trivial and is not the leftmost tree$\}$

2. $B_n = \{f \in P_n :$ the current tree of f is nontrivial, but its right subtree is trivial$\}$

3. $C_n = \{f \in P_n :$ the current tree of f is trivial and is the leftmost tree.$\}$

4. $D_n = \{f \in P_n :$ the current tree of f is nontrivial, and so is its right subtree.$\}$

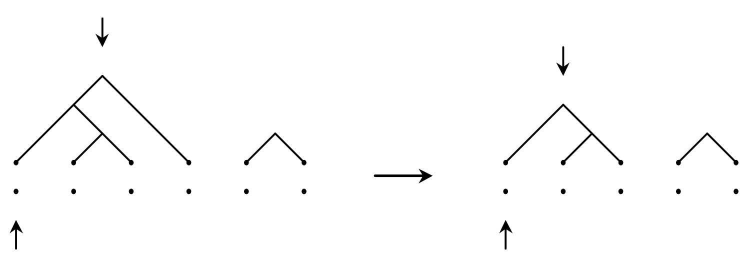
Given an element of A_n, we can remove the current tree and move the pointer left, like this:



This defines a bijection $A_n \to P_{n-1}$, so that:

$$|A_n| = |P_{n-1}|$$

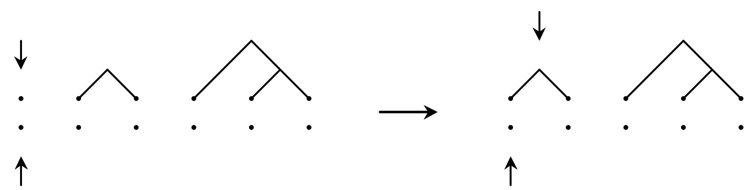
Given an element of B_n, we can remove the top caret together with the resulting trivial tree, like this:



This defines a bijection $B_n \to P_{n-1}$, so that:

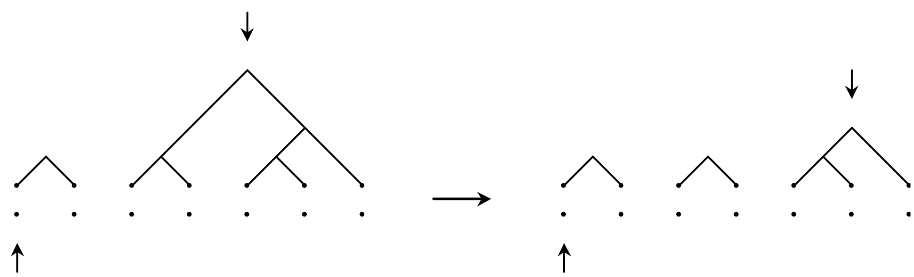
$$|B_n| = |P_{n-1}|$$

Given an element of C_n, we can move both the top and bottom arrows one space to the right, like this:



When $n \geq 2$, this defines an injection $\varphi\colon C_n \to P_{n-2}$. The image of φ is all elements of P_{n-2} whose current tree is the first tree.

Finally, given an element of D_n, we can remove the top caret and move the pointer to the right subtree, like this:



This defines an injection $\psi\colon D_n \to P_{n-2}$. The image of ψ is all elements of P_{n-2} whose current tree is nontrivial, and is not the first tree. In particular:

$$(\operatorname{im}\varphi) \cup (\operatorname{im}\psi) = P_{n-2} - A_{n-2}$$

so that:

$$|C_n| + |D_n| = |P_{n-2}| - |A_{n-2}| = |P_{n-2}| - |P_{n-3}|$$

This proves that p_n satisfies the given recurrence relation for $n \geq 3$. It is not much more work to verify the given expression for $p(x)$. $\square$

5.3 The Isoperimetric Constant

Let G be a group with finite generating set Σ, and let Γ denote the Cayley graph of G with respect to Σ. If $S \subset G$, define:

$$\delta S = \{\text{edges in } \Gamma \text{ between } S \text{ and } S^c\}$$

The *isoperimetric constant* of G is defined as follows:

$$\iota\,(G, \Sigma) = \inf \left\{ \frac{|\delta S|}{|S|} : S \subset G \text{ and } |S| < \infty \right\}$$

The group G is amenable if and only if $\iota(G, \Sigma) = 0$.

Guba [Guba] recently proved that $\iota(F, \{x_0, x_1\}) \leq 1$. We have obtained a slightly better estimate:

Proposition 5.3.1. $\iota(F, \{x_0, x_1\}) \leq 1/2$.

Sketch of Proof. Define the *height* of a binary tree to be length of the longest descending path starting at the root and ending at a leaf. Define the *width* of a binary forest to be the number of spaces in its support. For each $n, k \in \mathbb{N}$, let $S_{n,k}$ denote all positive elements whose forest diagram has width at most n and all of whose trees have height at most k. One can show that:

$$\lim_{k\to\infty} \lim_{n\to\infty} \frac{|\delta S_{n,k}|}{|S_{n,k}|} = \frac{1}{2} \qquad \square$$

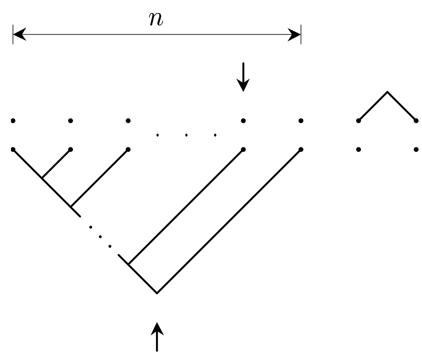
5.4 Convexity

A group G is *convex* (with respect to some generating set) if the n-ball $B^n(G)$ is a convex subset of the Cayley graph of G for each n. Very few groups are convex, but Cannon [Can] has introduced the following weaker property:

Definition 5.4.1. A group G is *almost convex* (with respect to some generating set) if there exists an integer L having the following property: given any $x, y \in B^n(G)$ a distance two apart, there exists a path from x to y in $B^n(G)$ of length at most L.

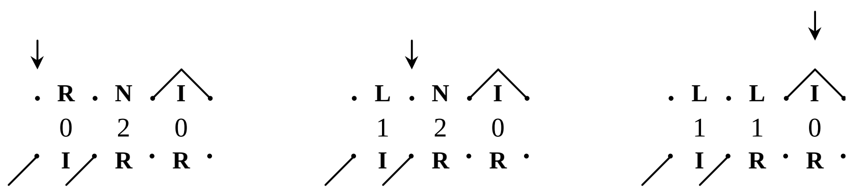
The convexity of F was first investigated by S. Cleary and J. Taback [ClTa2], who proved that F is not almost convex with respect to $\{x_0, x_1\}$. Recently, J. Belk and K. Bux [BeBu] have applied forest diagrams and the length formula to show that F is *maximally nonconvex*. Specifically:

Theorem 5.4.2. *For each $n \in \mathbb{N}$, let l_n be the element of F with forest diagram:*



and let $r_n = x_0^2\, l_n$. *Then* l_n *and* r_n *each have length* $2n+2$, *and the shortest path from* l_n *to* r_n *inside the* $(2n+2)$*-ball has length* $4n+4$.

Sketch of Proof. : Since the right space of l_n has label $\begin{bmatrix}\mathbf{R}\\ \mathbf{I}\end{bmatrix}$, $x_0 l_n$ has greater length than l_n:



l_n $\qquad$ $x_0 l_n$ $\qquad$ r_n

In particular, the path:

$$l_n — x_0 l_n — r_n$$

does not remain within the $(2n+2)$-ball.

Intuitively, if one wants to get from l_n to r_n while remaining inside the $(2n+2)$-ball, one must begin by moving all the way to the left and removing the accessible bottom caret. Taking this idea further, we might guess that the following path of length $4n+4$ is minimal:

1. Move left $n-1$ spaces, and delete the leftmost bottom caret.
2. Move right n spaces, and delete the top caret.
3. Move left n spaces, and re-create the leftmost bottom caret.
4. Move right $n+1$ spaces, and re-create the top caret.

This is in fact the case (see [BeBu]). □

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