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COMPLETE EULER CHARACTERISTICS AND FIXED-POINT THEORY

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Introduction

The purpose of this paper is to prove a fixed-point theorem for discrete group actions. In a separate paper we will apply this theorem to the theory of Euler characteristics of groups. We begin by briefly describing this application, since it provides the motivation for the fixed-point theorem.

Let Γ be a group such that $\mathbb{Q}$, regarded as $\mathbb{Q}\Gamma$-module with trivial Γ-action, admits a resolution $0 \to P_n \to \cdots \to P_0 \to \mathbb{Q} \to 0$, where each P_i is a finitely generated projective $\mathbb{Q}\Gamma$-module. (Γ is then said to be of type (FP) over $\mathbb{Q}$.) The *complete Euler characteristic* of Γ, first introduced by Stallings [13] and denoted here by $E(\Gamma)$, is then defined by $E(\Gamma) = \sum (-1)^i r(P_i)$, where $r(\ )$ denotes the Hattori–Stallings rank. Thus $E(\Gamma)$ is a finite linear combination (with $\mathbb{Q}$-coefficients) of Γ-conjugacy classes, and we denote by $E(\Gamma)(s)$ the coefficient of the conjugacy class $[s]$ of an element $s \in \Gamma$. We set $E(\Gamma)(1) = e(\Gamma)$. This is the *Euler characteristic* of Γ in the sense of Bass [2] and Chiswell [8]; under suitable hypotheses on Γ, it is known to agree with the Euler characteristics previously defined and studied by Wall [16], Serre [12], and Brown [6] (cf. [2, 10.6]).

A theorem of Bass [2, 9.2] implies that $E(\Gamma)(s) = 0$ for s of infinite order, provided Γ satisfies a certain 'non-divisibility' condition (which holds, for example, if Γ is a linear group). The coefficients $E(\Gamma)(s)$ for s of finite order have remained mysterious, however, and Serre [private communication] proposed the formula

$$E(\Gamma)(s) = \begin{cases} e(Z(s)) & \text{if } s \text{ has finite order,} \\ 0 & \text{if } s \text{ has infinite order,} \end{cases} \tag{1}$$

where $Z(s)$ (or $Z_\Gamma(s)$ if there is ambiguity) denotes the centralizer of s in Γ. It was in

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attempting to prove (1) that we needed the fixed-point theorem which is the subject of the present paper, and which we now describe.[1]

Suppose a group Γ acts cellularly on a CW-complex X. Under a suitable (FP) hypothesis, we will define an *equivariant Euler characteristic* $e_\Gamma(X)$ and a *complete equivariant Euler characteristic* $E_\Gamma(X)$, which reduce to $e(\Gamma)$ and $E(\Gamma)$ if X is a point (or, more generally, if X is $\mathbb{Q}$-acyclic). As before, $E_\Gamma(X)$ is a linear combination of conjugacy classes, and we denote by $E_\Gamma(X)(s)$ the coefficient of $[s]$. The fixed-point set X^s is invariant under the action of $Z(s)$, and our theorem will say, under suitable hypotheses on (Γ, X), that one has the following *fixed-point formula:*

$$E_\Gamma(X)(s) = \begin{cases} e_{Z(s)}(X^s) & \text{if } s \text{ has finite order,} \\ 0 & \text{if } s \text{ has infinite order.} \end{cases} \tag{2}$$

In particular, if X^s is $\mathbb{Q}$-acyclic for each element s of finite order (including $s=1$), then (2) yields (1). [Unfortunately, the precise hypotheses on Γ under which we can prove (1) by this method are too complicated to be stated here, but we mention an important family of examples: if G is an algebraic group over a number field k, then the arithmetic subgroups of $G(k)$ satisfy (1); if G is reductive, then (1) holds for the S-arithmetic subgroups also. The proof, which will be given in detail elsewhere, is based on Theorem 3.1 of the present paper, together with the work of Borel and Serre [3, 4].

The contents of this paper are as follows: Section 1 is preliminary; it contains some homological algebra that is needed later in the paper. In Section 2 we define $E_\Gamma(X)$ and give some elementary properties. Section 3 contains the first version of the fixed-point theorem. We impose very strong finiteness conditions on (Γ, X) (compact quotient, 'good' isotropy groups), and we then prove (2) by a completely elementary argument.

In Section 4 we weaken the finiteness hypotheses (in particular, we allow a non-compact quotient), and we show that the fixed-point formula (2) remains valid, provided we assume the truth of a conjecture of Bass [2]. (Conversely, the conjectured fixed-point formula (2) implies Bass's conjecture, cf. Remark 4.1.) In Section 5 we specialize to the case where Γ is finite (in which case Bass's conjecture is known to be true by a theorem of Swan). The content of the fixed-point theorem in this case is the following (see Theorem 5.1 and Remark 5.2):

Theorem. *Let X be a paracompact space of finite cohomological dimension and let s be a homeomorphism of X of finite order. For each element t of the cyclic group generated by s, assume that X^t has finitely-generated integral cohomology. Then the Lefschetz number of s is equal to the Euler characteristic of X^s.*

[1] It is, of course, not surprising that an attempt to understand $E(\Gamma)$ should lead to fixed-point theory, in view of the fact that the definition of $E(\Gamma)$ was originally motivated by classical fixed-point theory (cf. [13], introduction).

(Here 'cohomology' and 'cohomological dimension' are to be interpreted in the sense of sheaf theory [11].) This theorem is, in fact, an immediate consequence of our earlier work on finite group actions (cf. [6], p. 233, Theorem 2 and footnote; see also Verdier [15]), but it was not explicitly stated there. An analogous theorem in the context of étale cohomology theory has been proved by Deligne and Lusztig [9].

Finally, in Section 6 we return to the situation of Section 4 (with Γ allowed to be infinite) and we show that, without assuming Bass's conjecture, one can still prove a weak form of the fixed-point theorem.

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1. Chain complexes of type (FP)

All chain complexes in this section will be assumed to be non-negative. If C and C' are chain complexes (over an arbitrary ring A), we denote by $[C, C']$ the group of homotopy classes of chain maps from C to C'. A chain map $f: C \to C'$ is called a *weak equivalence* if $H_* f: H_* C \to H_* C'$ is an isomorphism. The following result is well known; see, for example, [10, Kor. 3.2].

1.1. Lemma. *If $f: C \to C'$ is a weak equivalence and P is a complex of projective modules, then the induced map $[P, C] \to [P, C']$ is an isomorphism.*

By a complex of *finite type* we will mean a complex $P = (P_i)$ such that each P_i is a finitely generated projective A-module. If, in addition, $P_i = 0$ for sufficiently large i, then we will say that P is *finite*. We will say that a complex C is of type (FP) (resp. $(\overline{\mathrm{FP}})$) if there is a weak equivalence $P \to C$ with P finite (resp. of finite type). If C is of type (FP) then we can associate to C a well-defined element $[C]$ of $K_0(A)$ by choosing a weak equivalence $P \to C$ with P finite and setting

$$[C] = \sum (-1)^i [P_i];$$

for if $P' \to C$ is any other such weak equivalence then Lemma 1.1 implies that P and P' are homotopy equivalent, hence $\sum (-1)^i [P_i] = \sum (-1)^i [P'_i]$ (cf. [1, Ch. VIII, § 4, Prop. 4.1(d)]).

As in [2], let $T(A) = A/[A, A]$, the target group for the Hattori–Stallings trace over A. We write $r(P)$ (or $r_A(P)$) for the Hattori–Stallings rank of a finitely generated projective A-module P. Since r induces a homomorphism $\bar{r}: K_0(A) \to T(A)$, we may define the *Hattori–Stallings rank* $r(C)$ (or $r_A(C)$) for a complex C of type (FP), by $r(C) = \bar{r}([C])$. Explicitly, if $P \to C$ is a weak equivalence with P finite, then

$$r(C) = \sum (-1)^i r(P_i).$$

In case C consists of single module M concentrated in dimension 0, C is of type (FP) (resp. $(\overline{\mathrm{FP}})$) if and only if M admits a finite (resp. finite type) projective resolution. In this case we say that the module M is of type (FP) (resp. $(\overline{\mathrm{FP}})$). In the (FP) case, $[C]=[M]$, the class of M in $K_0(A)$ in the usual sense, and $r(C)=r(M)$, the Hattori–Stallings rank of M as defined in [2, 2.8].

1.2. Proposition. *If P is a complex of projectives, then the following two conditions are equivalent:*

(i) *P is of type* (FP).

(ii) *P is of type* $(\overline{\mathrm{FP}})$ *and there is an integer n such that $H^i(\mathrm{Hom}(P,M))=0$ for all $i>n$ and all A-modules M.*

For a proof see [6, §1, proof of lemma].

We will denote by $M(f)$ the mapping cone of a chain map f.

1.3. Lemma. *Let $f: C' \to C$ be a chain map.*

(a) *Let $\pi': P' \to C'$ and $\pi: P \to C$ be weak equivalences such that P' is a complex of projectives. Then one can find a map $\tilde{f}: P' \to P$ and a weak equivalence $M(\tilde{f}) \to M(f)$ which fit into a diagram*

$$\begin{tikzcd} P' \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"] & P \arrow[r, hook, "\tilde{i}"] \arrow[d, "\pi"] & M(\tilde{f}) \arrow[d] \\ C' \arrow[r, "f"] & C \arrow[r, hook, "i"] & M(f), \end{tikzcd}$$

such that the right-hand square is commutative and the left-hand square is homotopy-commutative. (Here i and $\tilde{i}$ are the canonical inclusions.)

(b) *If C' and C are of type* (FP) (*resp.* $\overline{\mathrm{FP}}$)) *then the same is true of $M(f)$. In the* (FP) *case,* $[M(f)]=[C]-[C']$ *in* $K_0(A)$.

Proof. For (a) one uses Lemma 1.1 to find a map $\tilde{f}$ making the left-hand square above homotopy-commutative; the rest then follows easily (cf. [10, p. 283, proof of 2.10]). (b) is an immediate consequence of (a).

1.4. Proposition. *Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of chain complexes. If any two of the three complexes are of type* (FP) (*resp.* $(\overline{\mathrm{FP}})$) *then so is the third. In the* (FP) *case one has* $[C]=[C']+[C'']$, *and hence*

$$r(C)=r(C')+r(C'').$$

Proof. Let D be the mapping cone of the map $C' \to C$. There is a weak equivalence $D \to C''$, hence C'' if of type (FP) or $(\overline{\mathrm{FP}})$ if and only if D is, and $[C'']=[D]$ in the (FP) case. Also there is a short sequence

$$0 \to C \to D \to \Sigma C' \to 0, \qquad (*)$$

where $\Sigma C'$ is obtained from C' by re-indexing: $(\Sigma C')_n = C'_{n-1}$. It is easy to see that C' is of type (FP) or ($\overline{\text{FP}}$) if and only if $\Sigma C'$ is. We now consider the three possible cases. *Case 1:* If C and C' are of type (FP) (or ($\overline{\text{FP}}$)), then the desired result follows from 1.3(b). *Case 2:* If C and C'' are of type (FP) (or ($\overline{\text{FP}}$)), then the result follows from case 1, applied to the sequence (*). *Case 3:* If C' and C'' are of type (FP) (or ($\overline{\text{FP}}$)), then the result follows from case 2, applied to (*).

1.5. Lemma. *Let $C = (C_i)$ be a chain complex, and for each i let $(P_{ij})_{j \geq 0}$ be a projective resolution of C_i. Then one can find a complex Q with $Q_n = \bigoplus_{i+j=n} P_{ij}$, such that there is a weak equivalence $Q \to C$.*

Proof. For $k \geq 0$ let $C^{(k)}$ be the k-skeleton of C, i.e. the subcomplex of C defined by $C_i^{(k)} = C_i$ for $i \leq k$ and $C_i^{(k)} = 0$ for $i > k$. Let $\Sigma^{(k-1)}$ be the complex consisting of the module C_k concentrated in dimension $k-1$, and note that $C^{(k)}$ can be identified with the mapping cone of a chain map $\Sigma^{(k-1)} \to C^{(k-1)}$. We can therefore use Lemma 1.3(a) to construct, inductively, a compatible family of weak equivalences $Q^{(k)} \to C^{(k)}$, where $Q^{(k)}$ is a complex such that

$$Q_n^{(k)} = \bigoplus_{\substack{i+j=n \\ i \leq k}} P_{ij}.$$

Passing to the limit, we obtain the desired complex Q and weak equivalence $Q \to C$.

1.6. Proposition. *If C is a complex such that each module C_i is of type* ($\overline{\text{FP}}$), *then C is of type* ($\overline{\text{FP}}$). *If each C_i is of type* (FP) *and $C_i = 0$ for sufficiently large i, then C is of type* (FP) *and $[C] = \sum (-1)^i [C_i]$ in $K_0(A)$; consequently,*

$$r(C) = \sum (-1)^i r(C_i).$$

This is an immediate consequence of Lemma 1.5.

1.7. Lemma. *If C is a complex of type* ($\overline{\text{FP}}$) *and n is an integer such that $H_i C = 0$ for $i < n$, then $H_n C$ is a finitely-generated module.*

Proof. We may assume that C is a complex of projectives of finite type. Then the hypothesis implies that the module Z_i of i-cycles is a direct summand of C_i for $i \leq n$. In particular, Z_n is finitely generated, whence the lemma.

1.8. Lemma. *The following conditions on a chain complex C are equivalent:*

(i) *C is of type* ($\overline{\text{FP}}$).

(ii) *For any complex C' of type* ($\overline{\text{FP}}$) *and any map $C' \to C$, the first non-vanishing homology module of the mapping cone is finitely generated.*

Proof. If C is of type ($\overline{\text{FP}}$) and $C' \to C$ is as in (ii), then the mapping cone is of type

($\overline{\mathrm{FP}}$) by Lemma 1.3(b), hence its first non-vanishing homology module is finitely generated by Lemma 1.7. Conversely, if (ii) holds then it is easy to construct, step-by-step, a finite type complex F of free modules together with a weak equivalence $F \to C$. In fact, suppose inductively that we have constructed the k-skeleton $F^{(k)}$ of the desired F, together with a map $f^{(k)}: F^{(k)} \to C$ such that $H_i(M(f^{(k)})) = 0$ for $i \leq k$. Then $H_{k+1}(M(f^{(k)}))$ is finitely generated by hypothesis, hence we can attach to $F^{(k)}$ a finitely generated free module in dimension $k+1$ to get a complex $F^{(k+1)}$ such that $f^{(k)}$ extends to a map $f^{(k+1)}: F^{(k+1} \to C$ with $H_i(M(f^{(k+1)})) = 0$ for $i \leq k+1$ (cf. [7, Lemma 3]). This completes the inductive step and hence the proof of the lemma.

1.9. Proposition. *If the ring A is (left) noetherian, then a chain complex C of A-modules is of type* ($\overline{\mathrm{FP}}$) *if and only if H_iC is finitely generated for each i.*

Proof. It is trivial that a complex of type ($\overline{\mathrm{FP}}$) over a noetherian ring has finitely generated homology. Conversely, if C has finitely generated homology then it is clear from a consideration of long exact homology sequences that condition (ii) of Lemma 1.8 is satisfied, so C is of type ($\overline{\mathrm{FP}}$).

Recall (cf. [2, 2.10]) that if $A \to B$ is a ring homomorphism such that B is of type (FP) as (left) A-module, then there is a trace map $\mathrm{Tr}_{B/A}: T(B) \to T(A)$ such that

$$r_A(P) = \mathrm{Tr}_{B/A}(r_B(P))$$

for any finitely generated projective B-module P. (This formula makes sense because P is of type (FP) as A-module.)

1.10. Proposition. *Let $A \to B$ be a ring homomorphism such that B is of type* ($\overline{\mathrm{FP}}$) *as A-module, and let C be a complex of B-modules. Then:*

(a) *C is of type* ($\overline{\mathrm{FP}}$) *as complex of B-modules if and only if it is of type* ($\overline{\mathrm{FP}}$) *as complex of A-modules.*

(b) *If B is of type* (FP) *as A-module and C is of type* (FP) *as complex of B-modules, then C is of type* (FP) *as complex of A-modules and*

$$r_A(C) = \mathrm{Tr}_{B/A}(r_B(C)).$$

Proof. Assume first that the hypotheses of (b) hold, and let P be a finite complex of projective B-modules which admits a weak equivalence $P \to C$. Then Proposition 1.6 implies that P, and hence also C, is of type (FP) over A, and that

$$r_A(C) = r_A(P) = \sum (-1)^i r_A(P_i) = \sum (-1)^i \mathrm{Tr}_{B/A}(r_B(P_i)) = \mathrm{Tr}_{B/A}(r_B(C)).$$

This proves (b), and a similar argument proves the 'only if' part of (a). The 'if' part of (a) now follows easily from Lemma 1.8. For suppose C is of type ($\overline{\mathrm{FP}}$) over A and let $C' \to C$ be a map of B-complexes, where C' is of type ($\overline{\mathrm{FP}}$) over B; then C' is also of type ($\overline{\mathrm{FP}}$) over A by what we have just proved, so the first non-vanishing homology group of the mapping cone is finitely generated over A and hence also over B.

We close this section by remarking that the rank $r(C)$ which we have defined in this section for a complex C of type (FP) is really a special case of a *Hattori–Stallings trace* $\mathrm{Tr}(f)$ which can be defined for an endomorphism f of such a complex C. In fact, if $P \to C$ is weak equivalence with P finite, then we can lift f (up to homotopy) to an endomorphism $\tilde{f} = (\tilde{f}_i)$ of $P = (P_i)$, and we then set $\mathrm{Tr}(f) = \Sigma(-1)^i \mathrm{Tr}(\tilde{f}_i)$. It is easy to check that $\mathrm{Tr}(f)$ is well defined. For future reference we record the following obvious fact (cf. [2, 3.1]):

1.11. Proposition. *Let C be a complex of type* (FP) *over A, and suppose that A is an algebra over a commutative ring R. Then the trace map* $\mathrm{Tr}: \mathrm{End}(C) \to T(A)$ *is R-linear.*

2. Equivariant Euler characteristics

Let Γ be a group and X a CW-complex on which Γ acts. We will assume that the Γ-action satisfies the following two conditions, in which case we will say that X is an *admissible* Γ-complex:

(i) The action of Γ permutes the cells of X.

(ii) For each cell σ of X, the isotropy group Γ_σ fixes every point of σ.

Note that (i) and (ii) imply that the fixed-point set X^Γ and the orbit space X/Γ inherit CW-structures.

For any commutative ring k, the cellular chain complex $C(X, k)$ is a complex of modules over the group algebra $k\Gamma$, and we will say that (Γ, X) is of type (FP) (resp. $(\overline{\mathrm{FP}})$) over k if this chain complex is of type (FP) (resp. $(\overline{\mathrm{FP}})$) in the sense of Section 1. If it is of type (FP) then we define the *complete equivariant Euler characteristic* $E_\Gamma(X, k)$ by

$$E_\Gamma(X, k) = r_{k\Gamma}(C(X, k)).$$

If there is no ambiguity then we will suppress the coefficient ring k from the notation and simply write $E_\Gamma(X)$. As usual we identify $T(k\Gamma)$ with the free k-module generated by the Γ-conjugacy classes; thus $E_\Gamma(X)$ is a k-linear combination of Γ-conjugacy classes, and, as in the introduction, we denote by $E_\Gamma(X)(s)$ the coefficient of $[s]$. We set

$$E_\Gamma(X)(1) = e_\Gamma(X)$$

and call this the *equivariant Euler characteristic* of (Γ, X) (over k).

All of these definitions extend in the obvious way to the relative situation. Thus if $X' \subseteq X$ is a subcomplex invariant under the action of Γ, then we will say that $(\Gamma; X, X')$ is of type (FP) or $(\overline{\mathrm{FP}})$ (over k) if this is true of $C(X, X'; k)$, and in the (FP) situation we can define $E_\Gamma(X, X')$, $E_\Gamma(X, X')(s)$, and $e_\Gamma(X, X')$ as above.

In case X is a point, $C(X, k)$ is simply k, with trivial Γ-action, concentrated in

dimension 0. Hence $(\Gamma, \text{pt.})$ is of type (FP) (resp. $(\overline{\text{FP}})$) over k if and only if k is a $k\Gamma$-module of type (FP) (resp. $(\overline{\text{FP}})$), in which case one says that Γ is of type (FP) (resp. $(\overline{\text{FP}})$) over k. If Γ is of type (FP) over k then we set

$$E(\Gamma) = E_\Gamma(\text{pt.}) = r_{k\Gamma}(k);$$

this is the Stallings complete Euler characteristic of Γ over k (cf. [13], [2]). As before, we write $E(\Gamma)(s)$ for the coefficient of $[s]$ and we set

$$e(\Gamma) = E(\Gamma)(1) = e_\Gamma(\text{pt.});$$

this is the Euler characteristic of Γ (over k) in the sense of Bass [2] and Chiswell [8].

We now record some properties of the (complete) equivariant Euler characteristic. To simplify the notation we will state the results only in the absolute case, but it will be obvious from the proofs that everything extends to pairs (X, X').

2.1. Proposition. (i) *Let X and Y be admissible Γ-complexes such that there exists a cellular Γ-map $X \to Y$ which induces an isomorphism $H_*(X, k) \to H_*(Y, k)$. Then (Γ, X) is of type* (FP) *or* $(\overline{\text{FP}})$ *over k if and only if (Γ, Y) is, and in the* (FP) *case $E_\Gamma(X) = E_\Gamma(Y)$.*

(ii) *If X is a k-acyclic admissible Γ-complex (i.e. $\tilde{H}_*(X, k) = 0$), then (Γ, X) is of type* (FP) *(resp.* $(\overline{\text{FP}})$*) over k if and only if Γ is of type* (FP) *(resp.* $(\overline{\text{FP}})$*) over k, and in the* (FP) *case $E_\Gamma(X) = E(\Gamma)$.*

(iii) *If (Γ, X) is of type* (FP) *over k and $\Gamma' \subset \Gamma$ is a subgroup of finite index, then (Γ', X) is of type* (FP) *over k and*

$$E_{\Gamma'}(X)(s) = (Z_\Gamma(s) : Z_{\Gamma'}(s)) \cdot E_\Gamma(X)(s)$$

for $s \in \Gamma'$. In particular,

$$e_{\Gamma'}(X) = (\Gamma : \Gamma') \cdot e_\Gamma(X).$$

(iv) *Let X be an admissible Γ-complex such that X/Γ is compact and each isotropy group Γ_σ is of type* (FP) *over k. Then (Γ, X) is of type* (FP) *over k and*

$$E_\Gamma(X) = \sum (-1)^{\dim \sigma} i_{\Gamma_\sigma}^{\Gamma}(E(\Gamma_\sigma)),$$

where σ ranges over a set of representatives for the cells of X mod Γ *and $i_{\Gamma_\sigma}^{\Gamma}$: $T(k\Gamma_\sigma) \to T(k\Gamma)$ is the map induced by the inclusion $\Gamma_\sigma \hookrightarrow \Gamma$.*

(v) *If (Γ, X) is of type* (FP) *over k and Γ has a central subgroup of infinite order which acts trivially on X, then $e_\Gamma(X) = 0$.*

(Note that (v) reduces to the Gottlieb–Stallings theorem [13, 3.5] in case X is a point.)

Proof. (i) follows from the fact that the given map induces a weak equivalence $C(X, k) \to C(Y, k)$. (ii) is the special case of (i) where Y is a point. (iii) follows from

the restriction formula for the Hattori–Stallings rank [2, 6.3]. For (iv) we note that

$$C_p(X,k) \approx \bigoplus_{\sigma \in \Sigma_p} (k\Gamma \otimes_{k\Gamma_\sigma} k),$$

where Σ_p is a set of representatives for the p-cells of X mod Γ; the result now follows from Proposition 1.6. Finally, (v) is an immediate consequence of Proposition 1.11, exactly as in the proof of the Gottlieb–Stallings theorem (cf. [2, 10.4]).

2.2. Remark. If X is an admissible Γ-complex, then one can show that the cellular complex $C(X,k)$ is weakly equivalent over $k\Gamma$ to the singular chain complex of X. It follows that the definitions of this section ((FP), $E_\Gamma(X)$, etc.) are independent of the CW-structure on X. It also follows that we can drop in Proposition 2.1(i) the assumption that the map be cellular.

Let X be an admissible Γ-complex. We will say that (Γ, X) is *proper* (resp. *free*) if each isotropy group Γ_σ is finite (resp. trivial). We will say that (Γ, X) is *virtually free* if (Γ', X) is free for some subgroup $\Gamma' \subseteq \Gamma$ of finite index. Any virtually free Γ-complex is proper, and the converse holds if Γ is virtually torsion-free.

2.3. Proposition. *Assume that (Γ, X) is proper and that the order of each isotropy group Γ_σ is invertible in k.*

(a) *If X is finite dimensional, then (Γ, X) is of type* (FP) *over k if and only if it is of type* $(\overline{\text{FP}})$ *over k.*

(b) *If (Γ, X) is of type* (FP) *over k, then so is $(\Gamma/\Gamma_0, X/\Gamma_0)$ for any normal subgroup Γ_0 of Γ, and $E_{\Gamma/\Gamma_0}(X/\Gamma_0)$ is equal to the image of $E_\Gamma(X)$ under the canonical map $T(k\Gamma) \to T(k[\Gamma/\Gamma_0])$.*

Proof. The hypothesis implies that $C(X,k)$ is a complex of projective $k\Gamma$-modules; (a) is therefore a consequence of Proposition 1.2. Now suppose (Γ, X) is of type (FP) over k and let $f: P \to C(X,k)$ be a weak equivalence with P a finite chain complex over $k\Gamma$. Then f is a homotopy equivalence, hence it induces a homotopy equivalence

$$k[\Gamma/\Gamma_0] \otimes_{k\Gamma} P \to k[\Gamma/\Gamma_0] \otimes_{k\Gamma} C(X,k) \approx C(X/\Gamma_0, k)$$

over $k[\Gamma/\Gamma_0]$, and (b) follows at once.

We close this section by briefly describing a homologically defined 'equivariant Euler characteristic' $\chi_\Gamma(X)$, which is closely related to $e_\Gamma(X)$. This will be needed in Section 6. Let X be a finite dimensional, virtually free, admissible Γ-complex. We will say that (Γ, X) is of *finite homological type* if X/Γ' has finitely generated integral homology for each subgroup Γ' of finite index which acts freely on X. In this case we choose such a subgroup Γ' and set

$$\chi_\Gamma(X) = \frac{\chi(X/\Gamma')}{(\Gamma : \Gamma')};$$

one can show, exactly as in Section 4 of [6], that this is a well-defined rational number, independent of the choice of Γ'. (Here $\chi(X/\Gamma')$ is defined, as usual, as the alternating sum of the ranks of the homology groups of X/Γ'.) Note that if X is $\mathbb{Q}$-acyclic then $\chi_\Gamma(X)$ is simply $\chi(\Gamma)$, as defined in [6] for groups Γ of 'finite homological type'.

In case (Γ, X) is of finite homological type and also of type (FP) over $\mathbb{Q}$, one can ask whether $\chi_\Gamma(X)$ is equal to $e_\Gamma(X)$ $(=e_\Gamma(X, \mathbb{Q}))$. The following proposition gives a sufficient condition for this equality.

2.4. Proposition. *Let X be a finite dimensional, virtually free, admissible Γ-complex of type* $(\overline{\mathrm{FP}})$ *over $\mathbb{Z}$. Then:*

(a) *(Γ, X) is of finite homological type and also of type* (FP) *over $\mathbb{Q}$.*

(b) *If Γ is residually finite then $\chi_\Gamma(X) = e_\Gamma(X)$.*

Proof. (Γ, X) is of type (FP) over $\mathbb{Q}$ by Proposition 2.3(a), applied with $k = \mathbb{Q}$. If Γ' is a subgroup of finite index which acts freely on X, then another application of Proposition 2.3(a) (this time with $k = \mathbb{Z}$) shows that (Γ', X) is of type (FP) over $\mathbb{Z}$. We now apply Proposition 2.3(b) to (Γ', X) (with $k = \mathbb{Z}$ and $\Gamma_0 = \Gamma'$) to deduce that X/Γ' has finitely generated integral homology, whence (a), and that

$$\chi(X/\Gamma') = \sum E_{\Gamma'}(X)(t), \tag{$*$}$$

where t ranges over a set of representatives for the Γ'-conjugacy classes. Assuming now that Γ is residually finite, the restriction formula of Proposition 2.1(iii) implies that we can choose Γ' so that $E_{\Gamma'}(X)(t) = 0$ for $t \neq 1$ $(t \in \Gamma')$. Then $(*)$ says that $\chi(X/\Gamma') = e_{\Gamma'}(X)$, so

$$\chi_\Gamma(X) = \frac{\chi(X/\Gamma')}{(\Gamma : \Gamma')} = \frac{e_{\Gamma'}(X)}{(\Gamma : \Gamma')} = e_\Gamma(X),$$

whence (b).

2.5. Remark. It follows from the proof that $\chi_\Gamma(X) = e_\Gamma(X)$ if Γ has a subgroup of finite index for which Bass's 'weak conjecture' [2, p. 156] is true. Thus (b) could have been deduced from Bass's observation [2, 6.10] that his weak conjecture is true for residually-finite groups.

3. The fixed-point theorem in the case of a compact quotient

We continue to denote by k an arbitrary commutative ring, fixed throughout this section; all Euler characteristics $(E_\Gamma(X), e_\Gamma(X), E(\Gamma), e(\Gamma))$ will be understood to be those defined over k. We will say that a group Γ is *good* (relative to k) if the following three conditions are satisfied:

(i) Up to conjugacy Γ has only finitely many elements of finite order.

(ii) For each $s \in \Gamma$ of finite order, the centralizer $Z(s)$ is of type (FP) over k.

(iii) $E(\Gamma)$ (which is defined, in view of (ii)) satisfies:

$$E(\Gamma)(s) = \begin{cases} e(Z(s)) & \text{if } s \text{ has finite order,} \\ 0 & \text{if } s \text{ has infinite order.} \end{cases}$$

For example, if $k = \mathbb{Q}$ then every finite group is good (cf. [2, 10.1, remark 3]).

3.1. Theorem. *Let Γ be an arbitrary group and X an admissible Γ-complex such that X/Γ is compact and each isotropy group Γ_σ is good. Then:*

(i) *Up to conjugacy Γ has only finitely many elements of finite order which have fixed-points in X.*

(ii) *For each $s \in \Gamma$ of finite order, $(Z(s), X^s)$ is of type* (FP) *over k.*

(iii) *One has the formula*

$$E_\Gamma(X)(s) = \begin{cases} e_{Z(s)}(X^s) & \textit{if } s \textit{ has finite order,} \\ 0 & \textit{if } s \textit{ has infinite order.} \end{cases}$$

Proof. Let Σ be a set of representatives for the cells of X mod Γ. If s has a fixed-point in X then s is conjugate to an element of Γ_σ for some $\sigma \in \Sigma$, whence (i). To prove (ii) and (iii) we use Proposition 2.1(iv); in particular, since $E(\Gamma_\sigma)(t) = 0$ for $t \in \Gamma_\sigma$ of infinite order, (iii) follows at once for s of infinite order. Now let s be of finite order and let $T_\sigma = \{\gamma \in \Gamma : \gamma^{-1} s \gamma \in \Gamma_\sigma\}$. T_σ is closed under left multiplication by $Z(s)$ and right multiplication by Γ_σ, and we choose a set U_σ of representatives for the double cosets $Z(s) \backslash T_\sigma / \Gamma_\sigma$. Then $(\gamma^{-1} s \gamma)_{\gamma \in U_\sigma}$ is a set of representatives for the Γ_σ-conjugacy classes of elements of Γ_σ which are conjugate to s in Γ. Since Γ_σ is good, it follows that U_σ is finite and that $Z_{\Gamma_\sigma}(\gamma^{-1} s \gamma)$ is of type (FP) over k for $\gamma \in U_\sigma$, and the formula of Proposition 2.1(iv) yields

$$E_\Gamma(X)(s) = \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} \sum_{\gamma \in U_\sigma} e(Z_{\Gamma_\sigma}(\gamma^{-1} s \gamma)).$$

Now $Z_{\Gamma_\sigma}(\gamma^{-1} s \gamma)$ is conjugate to $Z_{\Gamma_{\gamma\sigma}}(s) = Z(s) \cap \Gamma_{\gamma\sigma} = Z(s)_{\gamma\sigma}$, hence the latter is of type (FP) over k and we have

$$E_\Gamma(X)(s) = \sum_{\substack{\sigma \in \Sigma \\ \gamma \in U_\sigma}} (-1)^{\dim \sigma} e(Z(s)_{\gamma\sigma}). \tag{3.2}$$

Finally, we note that the cells $\gamma\sigma$ ($\sigma \in \Sigma$, $\gamma \in U_\sigma$) form a set of representatives for the cells of X^s mod $Z(s)$, so we may apply 2.1(iv) to $(Z(s), X^s)$ to deduce that the latter is of type (FP) over k and that the right-hand side of (3.2) is equal to $e_{Z(s)}(X^s)$. This proves (ii) and (iii).

3.3. Remark. One might expect to have, instead of (iii), the uniform formula

$$E_\Gamma(X)(s) = e_{Z(s)}(X^s) \tag{3.4}$$

for all $s \in \Gamma$. Note, however, that the generalized Gottlieb–Stallings theorem

(Proposition 2.1(v)) implies that the right-hand side of (3.4) must vanish, if it is defined, for s of infinite order. Thus (iii) is consistent with (3.4) and has the advantage that it makes sense even for elements s of infinite order such that $(Z(s), X^s)$ is not of type (FP). A similar remark applies to condition (iii) in the definition of 'good'.

4. The case of non-compact quotient

In this section we will specialize to the case $k = \mathbb{Q}$ for simplicity, and we will attempt to prove that the formula of Theorem 3.1(iii) remains valid with the compactness assumption replaced by milder finiteness assumptions on (Γ, X). We will use the techniques of [6, §6] in which we studied finite group actions and made crucial use of a theorem of Swan [14] concerning projective modules over the integral group ring of a finite group. Bass has conjectured that Swan's theorem generalizes to infinite groups, and we will have to assume that Γ and certain of its subquotients satisfy this conjecture in order to prove our fixed-point theorem. We begin, therefore, by recalling Swan's theorem and Bass's conjecture.

Swan's theorem can be stated in terms of the Hattori–Stallings rank, as follows (cf. [2]):

Theorem. *Let Γ be a finite group and P a finitely generated projective $\mathbb{Z}\Gamma$-module. Then $r(P) = n \cdot [1]$ for some integer n.*

We will say that a group Γ has *property* (S) if the conclusion of Swan's theorem holds for all finitely-generated projective $\mathbb{Z}\Gamma$-modules P. Bass [2] proved that many torsion-free groups (including all torsion-free linear groups) have property (S), and he conjectured that *every* group has property (S).

4.1. Remark. It is possible to restate property (S) topologically, as a 'fixed-point property' for free actions: Γ has property (S) if and only if for every free, finite-dimensional Γ-complex X of type (FP) over $\mathbb{Z}$ one has $E_\Gamma(X)(s) = 0$ for $s \neq 1$. [The point is that every element of $K_0(\mathbb{Z}\Gamma)$ can be realized as $[C(X)]$ for some such Γ-complex X. In case Γ is finitely presented this follows from [17, Thm. F], and the general case is easily reduced to the finitely presented case.]

Now let Γ be an arbitrary group and X an admissible Γ-complex. For any subgroup $H \subseteq \Gamma$, we define a subcomplex $X^{>H}$ of X by

$$X^{>H} = \bigcup_{H' > H} X^{H'}.$$

Let $N(H)$ (or $N_\Gamma(H)$ if there is ambiguity) be the normalizer of H in Γ, and note that $N(H)/H$ acts on the pair $(X^H, X^{>H})$. This action freely permutes the cells of $X^H - X^{>H}$, which is simply the set X_H of points whose isotropy group is H.

We can now state the first version of our fixed-point theorem. For the moment we will assume that the action of Γ is *proper* (cf. Section 2) and that (Γ, X) has only finitely many orbit types (i.e. the isotropy groups Γ_σ form only finitely many conjugacy classes of subgroups of Γ). Both of these assumptions are easily removed (see Theorem 4.10 below) in the important special case where vcd $\Gamma < \infty$. (Here 'vcd Γ' is the virtual cohomological dimension of Γ in the sense of Serre [12].)

4.2. Theorem. *Let X be a finite-dimensional, proper, admissible Γ-complex with only finitely many orbit types. Assume that $(N(H), X^H)$ is of type* $(\overline{\mathrm{FP}})$ *over $\mathbb{Z}$ for each subgroup $H \subseteq \Gamma$ which occurs as an isotropy group in X. Then:*

(i) $(N(H); X^H, X^{>H})$ *is of type* (FP) *over $\mathbb{Z}$ for each H.*

(ii) $(Z(s), X^s)$ *is of type* (FP) *over $\mathbb{Q}$ for each $s \in \Gamma$.*

(iii) *If $E_{N(H)/H}(X^H, X^{>H})(t) = 0$ for all H and all $t \neq 1$ in $N(H)/H$ (e.g. if each $N(H)/H$ has property* (S)*), then*

$$E_\Gamma(X)(s) = e_{Z(s)}(X^s)$$

for all $s \in \Gamma$.

Proof. We begin by showing that $(N(H), X^{>H})$ and $(N(H), X^H)$ are of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$ for every subgroup $H \subseteq \Gamma$. Fix H and let T be the set of subgroups $F > H$ such that F occurs as an isotropy group in X. Then $N(H)$ acts on T by conjugation, and I claim that T mod $N(H)$ is finite. In fact, let Φ be a (finite) set of representatives for the conjugacy classes of isotropy groups, and for each $F \in \Phi$ let I_F be a finite subset of Γ such that the groups $\gamma^{-1}H\gamma$ $(\gamma \in I_F)$ exhaust all the Γ-conjugates of H contained in F; one then sees easily that every element of T is $N(H)$-conjugate to some $\gamma F \gamma^{-1}$ $(F \in \Phi,\ \gamma \in I_F)$, whence the claim.

We now analyze $X^{>H} = \bigcup_{F \in T} X^F$ as in [6, Appendix B]. Let K be the simplicial complex associated to the ordered set T; thus the n-simplices of K correspond to the chains $F_0 < \cdots < F_n$ $(F_i \in T)$. Note that $N(H)$ acts on K and that, in view of the above claim, K has only finitely many simplices mod $N(H)$ (cf. [6, §5, Lemma]). As in [6, Appendix B], we have an exact sequence of chain complexes (with $N(H)$-action)

$$0 \to \bigoplus_{\sigma \in K_d} C(X^\sigma) \to \cdots \to \bigoplus_{\sigma \in K_0} C(X^\sigma) \to C(X^{>H}) \to 0,$$

where $d = \dim K$, K_n $(0 \leq n \leq d)$ is the set of n-simplices of K, and $X^\sigma = X^{F_n}$ if $\sigma = (F_0 < \cdots < F_n)$. Now each complex $\bigoplus_{\sigma \in K_n} C(X^\sigma)$ is a finite direct sum of chain complexes of the form $\mathbb{Z}[N(H)] \otimes_{\mathbb{Z}N} C(X^F)$, where $F \in T$, $N \subseteq N(F) \cap N(H)$, and $(N(F) : N) < \infty$; this complex is therefore of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}[N(H)]$. In view of the above exact sequence and Proposition 1.4, it follows that $(N(H), X^{>H})$ is of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$. The same proof, but with T replaced by $T \cup \{H\}$, shows that $(N(H), X^H)$ is of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$.

Using Proposition 1.4 again we see that $(N(H); X^H, X^{>H})$ is of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$, hence so is $(N(H)/H; X^H, X^{>H})$ by Proposition 1.10(a). Since $N(H)/H$ acts freely in $X^H - X^{>H}$, (i) now follows from the relative version of Proposition 2.3(a).

For (ii) we may assume s has finite order. Then $Z(s)$ has finite index in $N(H)$, where $H=\langle s\rangle$, the cyclic subgroup generated by s. Since we proved above that $(N(H), X^s)$ is of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$, the same is true of $(Z(s), X^s)$. In particular, $(Z(s), X^s)$ is of type $(\overline{\mathrm{FP}})$ over $\mathbb{Q}$, so another application of Proposition 2.3(a) now yields (ii).

To prove (iii), we begin by using the technique of 'stratification by orbit type', exactly as in [6, §2, proof of Theorem 2], to obtain a formula for $E_\Gamma(X)$. Thus we filter X by Γ-invariant subcomplexes $\emptyset=A_0\subset A_1\subset\cdots\subset A_n=X$ such that $A_j-A_{j-1}=\Gamma\cdot X_{H_j}$, where $H_1,\ldots,H_n$ are the elements of Φ, suitably ordered. (Here Φ is as in the first paragraph of the proof.) We then observe that

$$C(A_j, A_{j-1})\approx \mathbb{Z}\Gamma\otimes_{\mathbb{Z}[N(H)]} C(X^H, X^{>H}),$$

where $H=H_j$; using Propositions 1.4 and 1.10(b), we conclude

$$\begin{aligned} E_\Gamma(X) &= \sum_{H\in\Phi} i^\Gamma_{N(H)} E_{N(H)}(X^H, X^{>H}) \\ &= \sum_{H\in\Phi} i^\Gamma_{N(H)} t^{N(H)}_{N(H)/H} E_{N(H)/H}(X^H, X^{>H}), \end{aligned} \tag{4.3}$$

where $t^{N(H)}_{N(H)/H}=\mathrm{Tr}_{\mathbb{Q}[N(H)/H]/\mathbb{Q}[N(H)]}$ and $i^\Gamma_{N(H)}$ is induced by the inclusion $N(H)\hookrightarrow\Gamma$, as in Proposition 2.1(iv). By hypothesis we have

$$E_{N(H)/H}(X^H, X^{>H})=e_{N(H)/H}(X^H, X^{>H})\cdot[1],$$

and it is easy to see that $t^{N(H)}_{N(H)/H}[1]=\sum_{h\in H}[h]/|H|$. It follows that

$$t^{N(H)}_{N(H)/H} E_{N(H)/H}(X^H, X^{>H})=e_{N(H)}(X^H, X^{>H})\sum_{h\in H}[h],$$

and (4.3) thus takes the form

$$E_\Gamma(X)=\sum_{H\in\Phi} e_{N(H)}(X^H, X^{>H})\sum_{h\in H}[h]. \tag{4.4}$$

Now fix $s\in\Gamma$ and let $T_H=\{\gamma\in\Gamma\colon \gamma^{-1}s\gamma\in H\}$. Note that T_H is closed under left multiplication by $Z(s)$ and right multiplication by $N(H)$ and that $Z(s)\backslash T_H$ can be identified with the set of Γ-conjugates of s which are contained in H. Hence $Z(s)\backslash T_H$ is finite and (4.4) yields

$$E_\Gamma(X)(s)=\sum_{H\in\Phi} e_{N(H)}(X^H, X^{>H})\cdot\mathrm{card}(Z(s)\backslash T_H). \tag{4.5}$$

Before proceeding further, we remark that (4.5) immediately yields the fixed-point formula (iii) if s has no fixed-points in X (in which case each $T_H=\emptyset$), or if the subgroup $H_0=\langle s\rangle$ is an isotropy group and is maximal among the isotropy groups (in which case $T_{H_0}=N(H_0)$ and $T_H=\emptyset$ for $H\neq H_0$). To prove (iii) in general, however, we must work a little harder.

Let U_H be a set of representatives for the double cosets $Z(s)\backslash T_H/N(H)$. Decomposing the set $Z(s)\backslash T_H$ according to the right-action of $N(H)$, we find

$$\mathrm{card}(Z(s)\backslash T_H)=\sum_{\gamma\in U_H}(N(H):N(H)\cap\gamma^{-1}Z(s)\gamma).$$

Hence (4.5) can be rewritten

$$E_\Gamma(X)(s) = \sum_{H\in\Phi} \sum_{\gamma\in U_H} (N(H) : N(H)\cap \gamma^{-1}Z(s)\gamma)\cdot e_{N(H)}(X^H, X^{>H})$$
$$= \sum_{H\in\Phi} \sum_{\gamma\in U_H} e_{N(H)\cap\gamma^{-1}Z(s)\gamma}(X^H, X^{>H}). \tag{4.6}$$

We now compute $e_{Z(s)}(X^s)$ using the same filtration (A_j) that we used above to prove (4.3). Thus we set $B_j = (A_j \cap X^s)$ and we observe that we have an isomorphism (over $\mathbb{Z}$)

$$C(B_j, B_{j+1}) \approx \bigoplus_{F\in\mathscr{F}_j} C(X^F, X^{>F}), \tag{4.7}$$

where $\mathscr{F}_j$ is the set of conjugates of H_j which contain s. The action of $Z(s)$ permutes the summands of (4.7) according to the conjugation-action of $Z(s)$ on $\mathscr{F}_j$, hence one obtains a $Z(s)$-isomorphism

$$C(B_j, B_{j-1}) \approx \bigoplus \mathbb{Z}[Z(s)] \otimes_{\mathbb{Z}[Z(s)\cap N(F)]} C(X^F, X^{>F}), \tag{4.8}$$

where F ranges over a set of representatives for $\mathscr{F}_j$ mod $Z(s)$-conjugation. Finally, note that $(\gamma H\gamma^{-1})_{\gamma\in U_H}$ is such a set of representatives, where $H = H_j$, so (4.8) yields

$$e_{Z(s)}(B_j, B_{j-1}) = \sum_{\gamma\in U_H} e_{Z(s)\cap N(\gamma H\gamma^{-1})}(X^{\gamma H\gamma^{-1}}, X^{>\gamma H\gamma^{-1}})$$
$$= \sum_{\gamma\in U_H} e_{\gamma^{-1}Z(s)\gamma\cap N(H)}(X^H, X^{>H});$$

summing over j and comparing with (4.6), we obtain the desired equality, $E_\Gamma(X)(s) = e_{Z(s)}(X^s)$.

4.9. Remark. Even without the hypothesis of (iii), the proof of the theorem yields information about $E_\Gamma(X)$. For example, it follows from formula (4.3) that $m\cdot E_\Gamma(X)$ has integral coefficients, where m is the least common multiple of the orders of the subgroups of Γ which have fixed-points in X.

We now give a variant of Theorem 4.2, for which the finiteness hypotheses on (Γ, X) are more agreeable.

4.10. Theorem. *Let X be a finite-dimensional admissible Γ-complex such that $(Z(s), X^s)$ is of type* $(\overline{\mathrm{FP}})$ *over $\mathbb{Z}$ for each $s\in\Gamma$ of finite order. Assume either that* $\operatorname{vcd}\Gamma<\infty$ *or that the action of Γ on X is virtually free. Then:*

(i) *Up to conjugacy, Γ has only finitely many elements t of finite order such that $X^t\neq\emptyset$, and for each such t, $(Z(t), X^t)$ is of type* (FP) *over $\mathbb{Q}$.*

(ii) *For each t as in* (i), *assume that every subgroup of $Z(t)/\langle t\rangle$ of finite index has property* (S). *Then one has, for any $s\in\Gamma$,*

$$E_\Gamma(X)(s) = \begin{cases} e_{Z(s)}(X^s) & \text{if } s \text{ has finite order,} \\ 0 & \text{if } s \text{ has infinite order.}\end{cases}$$

The proof will require the following lemma:

4.11. Lemma. *Let X be a finite-dimensional, virtually free, admissible Γ-complex.*

(a) *Let p be a prime number. Suppose that Γ has a normal subgroup Γ' of finite index such that Γ' acts freely on X and $H_*(X/\Gamma', \mathbb{Z}/p)$ is finite. Then Γ has, up to conjugacy, only finitely many elements s of order a power of p such that $X^s \neq \emptyset$.*

(b) *For each $s \in \Gamma$, suppose $Z(s)$ has a normal subgroup $Z'(s)$ of finite index such that $Z'(s)$ acts freely on X^s and $H_*(X^s/Z'(s))$ is finitely generated. Then Γ has, up to conjugacy, only finitely many elements s such that $X^s \neq \emptyset$.*

Proof of lemma. (a) Let $G = \Gamma/\Gamma'$ and let Y be the G-complex X/Γ'. For any $t \in G$ it is easy to see that

$$Y^t = \coprod X^s/Z(s) \cap \Gamma', \tag{4.12}$$

where s ranges over a set of representatives for the Γ'-conjugacy classes of elements $s \in \Gamma$ whose image in G is t (cf. [6, top of p. 246]). Suppose now that the order of t is a power of p. Since $H_*(Y, \mathbb{Z}/p)$ is finite, it follows from Smith theory (cf. [5, Ch. III]) that $H_*(Y^t, \mathbb{Z}/p)$ is finite. Therefore only finitely many of the complexes on the right-hand side of (4.12) can be non-empty, and (a) follows at once.

(b) It suffices to prove for a fixed positive integer n that Γ has (up to conjugacy) only finitely many elements s of order n such that $X^s \neq \emptyset$. We argue by induction on n. Let p be a prime dividing n and write $n = n_1 n_2$, where $p \nmid n_1$ and n_2 is a power of p. Then any element s of order n can be written uniquely in the form $s = s_1 s_2$, where s_i has order n_i and $s_2 \in Z(s_1)$. Since s_1 and s_2 are in the cyclic subgroup generated by s, it is clear that $X^s = (X^{s_1})^{s_2}$. Now we may assume inductively that (up to Γ-conjugacy) there are only finitely many s_1 of order n_1 such that $X^{s_1} \neq \emptyset$; and for each fixed s_1 we know from (a) applied to $(Z(s_1), X^{s_1})$ that (up to $Z(s_1)$-conjugacy) there are only finitely many $s_2 \in Z(s_1)$ of order n_2 such that $(X^{s_1})^{s_2} \neq \emptyset$. (b) follows at once.

Proof of Theorem 4.10. Assume first that (Γ, X) is virtually free. Then we may apply Proposition 2.4(a) to each $(Z(s), X^s)$ to deduce that the hypotheses of Lemma 4.11(b) are satisfied, whence (i). To prove (ii), fix an element $s_0 \in \Gamma$, let $\Gamma' \subseteq \Gamma$ be a normal subgroup of finite index which acts freely on X, and let Γ_0 be the subgroup generated by Γ' and s_0. Then every isotropy group $(\Gamma_0)_\sigma$ is cyclic, so (i) implies that (Γ_0, X) has only finitely many orbit types. Moreover, since Γ_0/Γ' is abelian, it is clear that $N_{\Gamma_0}(H) = Z_{\Gamma_0}(H)$ for any subgroup $H \subseteq \Gamma_0$ which has fixed-points in X. Thus (Γ_0, X) satisfies all the hypotheses of Theorem 4.2(iii), so we may apply the latter to compute $E_{\Gamma_0}(X)$. In particular, $E_{\Gamma_0}(X)(s_0) = e_{Z_{\Gamma_0}(s_0)}(X^{s_0})$. [Note that all we need here is the easy case of Theorem 4.2 mentioned after formula (4.5).] In view of the restriction formula of Proposition 2.1(iii), it follows that $E_\Gamma(X)(s_0) = e_{Z(s_0)}(X^{s_0})$, as required.

Now suppose $\operatorname{vcd} \Gamma < \infty$. A construction due to Serre (cf. [12, No. 1.7] or [6, §6, lemma]) shows that there exists a finite-dimensional, contractible, proper, admissible Γ-complex Z such that Z^H is contractible for each finite subgroup $H \subseteq \Gamma$. Let Y

be the Γ-complex $X \times Z$ (with the diagonal Γ-action). Note that the action of Γ on Y is proper, hence virtually free. The projection $Y \to X$ is a cellular Γ-map and a homotopy equivalence, and it induces a homotopy equivalence $Y^s \to X^s$ for every $s \in \Gamma$ of finite order. In view of Proposition 2.1(i), the theorem now follows from the virtually-free case, applied to (Γ, Y).

5. Finite group actions

Theorem 4.10 is particularly easy to interpret if the group Γ is finite. The hypothesis that $(Z(s), X^s)$ be of type $(\overline{\mathrm{FP}})$ over $\mathbb{Z}$ simply means in this case that $H_*(X^s)$ is finitely generated (cf. Proposition 1.9). The property (S) hypothesis holds automatically by Swan's theorem. And the conclusion about $E_\Gamma(X)$ can be restated in terms of Lefschetz numbers; in fact, we clearly have

$$E_\Gamma(X) = \sum (-1)^i r_{\mathbb{Q}\Gamma}(H_i(X, \mathbb{Q})),$$

hence (cf. [2, 5.8])

$$E_\Gamma(X)(s) = \frac{L(s^{-1})}{|Z(s)|},$$

where the Lefschetz number $L(s)$ is defined as usual by

$$L(s) = \sum (-1)^i \operatorname{tr}_{\mathbb{Q}}(s \text{ acting on } H_i(X, \mathbb{Q})).$$

Similarly,

$$e_{Z(s)}(X^s) = \frac{\chi(X^s)}{|Z(s)|} = \frac{\chi(X^{s^{-1}})}{|Z(s)|}.$$

Thus Theorem 4.10, yields:

5.1. Theorem. *Let Γ be a finite group and X a finite-dimensional admissible Γ-complex such that X^s has finitely generated integral homology for each $s \in \Gamma$. Then $L(s) = \chi(X^s)$ for any $s \in \Gamma$.*

5.2. Remark. As we stated in the introduction, this theorem remains valid if we drop the assumption that X be a CW-complex and simply require that it be a paracompact space of finite cohomological dimension in the sense of sheaf theory [11]. (Homology is then replaced by sheaf-theoretic (or Čech) cohomology in the statement of the theorem.) In fact, the proof of Theorem 5.1 goes through with no difficulty, the only essential change being the following: In the proof of Theorem 5.1 (for which all the work was done in the proof of Theorem 4.2 or, equivalently, in the proof of Theorem 2 of [6]), we considered the (relatively) free action of $N(H)/H$ on $(X^H, X^{>H})$, and we used Swan's theorem to conclude that the non-trivial elements of $N(H)/H$ have vanishing Lefschetz number; in the sheaf-theoretic version, one must instead use a theorem of Zarelua [18, Thm. 3].

5.3. Remark. It would be of interest to similarly have a sheaf-theoretic version of the results of Sections 3 and 4 in case Γ is infinite.

6. A weak form of the fixed-point theorem

We continue to work over the ground ring $k=\mathbb{Q}$. The result of this section is, roughly speaking, that if we drop the property (S) assumption in Theorem 4.10(ii) then we can still prove that the fixed-point formula holds 'modulo any subgroup of finite index'. More precisely:

6.1. Theorem. *Let X be a finite-dimensional admissible Γ-complex such that $(Z(s), X^s)$ is of type* $(\overline{\mathrm{FP}})$ *over $\mathbb{Z}$ for each $s\in\Gamma$ of finite order. Assume either that* $\operatorname{vcd}\Gamma<\infty$ *or that the action of Γ on X is virtually free. Let $\tilde{E}_\Gamma(X)\in T(\mathbb{Q}\Gamma)$ be defined by*

$$\tilde{E}_\Gamma(X)=\sum_{s\in\Psi} e_{Z(s)}(X^s)\cdot[s],$$

where Ψ is a set of representatives for the conjugacy classes of elements of Γ of finite order. If Γ is residually finite, then $E_\Gamma(X)$ and $\tilde{E}_\Gamma(X)$ have the same image in $T(\mathbb{Q}G)$ for every finite quotient G of Γ.

(Note that the definition of $\tilde{E}_\Gamma(X)$ makes sense because of Theorem 4.10(i).)

As in the last paragraph of the proof of Theorem 4.10, one reduces easily to the case where the action is virtually free. In view of Propositions 2.3(b) and 2.4, the theorem now follows from:

6.2. Theorem. *Let Γ be an arbitrary group and X a finite-dimensional, virtually free, admissible Γ-complex. Assume that $(Z(s), X^s)$ is of finite homological type for each $s\in\Gamma$, and let*

$$\tilde{E}_\Gamma(X)=\sum_{s\in\Psi} \chi_{Z(s)}(X^s)\cdot[s],$$

where Ψ is a set of representatives for the Γ-conjugacy classes. Let $G=\Gamma/\Gamma'$ be a finite quotient of Γ and let Y be the G-complex X/Γ'. Then (G, Y) is of type (FP) *over $\mathbb{Q}$ and $E_G(Y)$ is equal to the image in $T(\mathbb{Q}G)$ of $\tilde{E}_\Gamma(X)$.*

(Note that the definition of $\tilde{E}_\Gamma(X)$ makes sense because of Lemma 4.11(b).)

Proof. Using Proposition 2.3(b), it is easy to see that we may replace Γ' by a smaller normal subgroup of finite index if necessary, and thereby reduce to the case where Γ' acts freely on X. By hypothesis $H_*(X^s/Z'(s))$ is finitely generated for each $s\in\Gamma$, where $Z'(s)=Z(s)\cap\Gamma'$; hence Lemma 4.11(b) and (4.12) imply that $H_*(Y^t)$ is finitely generated for each $t\in G$ and that

$$\chi(Y^t)=\sum_{s\in\Psi_t}\chi(X^s/Z'(s))=\sum_{s\in\Psi_t}(Z(s):Z'(s))\cdot\chi_{Z(s)}(X^s),$$

where Ψ_t is a set of representatives for the Γ'-conjugacy classes of elements of Γ whose image in G is t. We may therefore apply Theorem 5.1 (see also the discussion preceding it) to obtain

$$E_G(Y)(t) = \frac{\chi(Y^t)}{|Z(t)|} = \frac{1}{|Z(t)|} \sum_{s \in \Psi_t'} (Z(s) : Z'(s)) \cdot \chi_{Z(s)}(X^s). \tag{6.3}$$

Note that the term $(Z(s) : Z'(s)) \cdot \chi_{Z(s)}(X^s)$ above depends only on the Γ-conjugacy class of s. Moreover, I claim that the Γ-conjugacy class of s contains exactly $|Z(t)|/(Z(s) : Z'(s))$ Γ'-conjugacy classes of elements of Γ whose image in G is t; for if $\tilde{Z}(t)$ is the inverse image of $Z(t)$ in Γ and $\bar{Z}(s)$ is the image of $Z(s)$ in G, then the number of such Γ'-classes is given by

$$\operatorname{card}(\Gamma' \backslash \tilde{Z}(t)/Z(s)) = \operatorname{card}(Z(t)/\bar{Z}(s)) = \frac{|Z(t)|}{(Z(s) : Z'(s))}.$$

Thus if we group together in (6.3) those terms corresponding to a given Γ-conjugacy class, we obtain

$$E_G(Y)(t) = \sum \chi_{Z(s)}(X^s),$$

where s ranges over a set of representatives for the Γ-conjugacy classes of elements whose image in G is t. The right-hand side of this equation is clearly equal to the coefficient of t in the image of $\tilde{E}_\Gamma(X)$ in $T(\mathbb{Q}G)$, whence the theorem.

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